Supergravity

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Abstract

Supergravity theory is an effective theory of M-theory in 11 dimensions (11D), hence the study of supergravity helps us understanding the fundamental theory. In this project, we worked on the construction of supergravity theories in certain conditions and studied the supersymmetry transformations of field contents in the theories. Furthermore, we studied a bosonic sector of matter coupled $\mathcal{N} = 2$ 4D supergravity using coset manifold. In conclusion, we obtained $\mathcal{N} = 1$, $\mathcal{N} = 2$ supergravity theories in 4D, 11D supergravity and the special Kahler manifold from the bosonic sector of matter coupled $\mathcal{N} = 2$ 4D supergravity written in the coset form G/H.

Contents

1 Introduction

The minimal supergravity in 4 dimension was constructed by [1] in 1976. After that it was generalized in various dimensional spacetimes and numbers of supersymmetries. Then, Cremmer, Julia and Scherk [2] found the mother theory of supergravity, which was the theory in 11 spacetime dimensions. Since $D > 11$ dimensional theories consist of particles with helicities greater than 2. Therefore, the greatest supergravity theory is the one that lives in 11 dimensions (some physicists might say that supergravity is in its most beautiful form in 11 dimensions). In 1995, Edward Witten [3] found the link between five candidates of string theory which are type I, type IIA, type IIB, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic string theories. These five theories also correspond to a theory in eleven-dimension called M-Theory. The correspondences between the string theories are called dualities, i.e., T-duality and S-duality. The following discovery had shown that M-Theory is a background theory of string theories and its effective theory turned out to be Supergravity in eleven-dimension.

Supergravity theory in $D < 11$ dimension can be obtained by compactifying the $D = 11$ theory on a certain geometry. The compactification of supergravity is called Kaluza-Klein dimensional reduction [4, 5]. This really helps us in the study of supergravity in various dimensions. In addition, supergravity theories with $\mathcal{N} > 1$ are called extended supergravity. An extended theory of supergravity contains many scalar fields which span a manifold called a scalar manifold $\mathcal{M}_{scalar}$. This scalar manifold can be seen also in extend supersymmetry theories. For instance, a manifold spaned by chiral multiplets is called Kahler manifold, which is a complex scalar manifold with dimensions according to total number of scalar fields in the multiplets. The same logic used to describe any kinds of scalar manifolds that emerge in supersymmetry (supergravity) theories. However, since supergravity is based on supersymmetry, namely local supersymmetry. Then it only allows its vacuum to be Minkowski spacetime. In order to have different kinds of spacetime in the theory, we need to introduce a scalar potential. This can be done by gauging a theory.

2 $\mathcal{N} = 1$ $D = 4$ supergravity

$\mathcal{N} = 1$ $D = 4$ Supergravity is the simplest theory of supergravity, in which we have a graviton e^a_μ and gravitino ψ_μ with spin 2 and $\frac{3}{2}$, respectively. From General Relativity, an action of a theory without matter field is

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R. \tag{1}$$

or in vielbein formution

$$S = \frac{1}{2\kappa^2}\int d^4xe[e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}]. \tag{2}$$

where $\kappa^2 = 8\pi G$, $g = det(g_{\mu\nu}), e = det(e^a_\mu)$ and $\sqrt{-g} = e$.

From supersymmetry, we know that a boson has a superpartner, i.e., a fermion. Therefore we have to include a fermion to the action 2. The extended part of action 2 is Rarita-Schwinger (RS) field [6], by which we obtain spin $\frac{3}{2}$ particle action. The RS field for massless spin $\frac{3}{2}$ gravitino is

$$\mathcal{L}_{RS} = \bar{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho. \tag{3}$$

Here $\nabla_\nu \psi_\rho = \partial_\nu \psi_\rho - \Gamma^\lambda_{\nu\rho}\psi_\lambda + \frac{1}{4}\omega_\nu{}^{ab}\gamma_{ab}\psi_\rho = D_\nu\psi_\rho - \Gamma^\lambda_{\nu\rho}\psi_\lambda$ but the fact that $\gamma^{\mu\nu\rho} = \gamma^{[\mu\nu\rho]}$ makes $\gamma^{\mu\nu\rho}\Gamma^\lambda_{\nu\rho}\psi_\lambda = 0$. Thus $\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho = \gamma^{\mu\nu\rho}D_\nu\psi_\rho$. When combining equation 3 with 2, we get

$$S = \frac{1}{2\kappa^2}\int d^4x e[e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho]. \tag{4}$$

Equation 4 is the simplest action of $\mathcal{N} = 1$ Supergravity, which is a theory in four dimensional spacetime.

To check whether the action 4 is invariant under supersymmetry transformation or not, we use the method called 1.5 order formalism (see [7, 8] for more details), by which we make a variation of S in the form

$$\delta S = \frac{\delta S}{\delta e} + \frac{\delta S}{\delta \psi} + \frac{\delta S}{\delta \omega}(\frac{\delta \omega}{\delta e} + \frac{\delta \omega}{\delta \psi}). \tag{5}$$

From 4 we can see that $\frac{\delta S}{\delta \omega} = 0$. Therefore we have

$$\delta S = \frac{\delta S}{\delta e} + \frac{\delta S}{\delta \psi}. \tag{6}$$

After that, we operate the supersymmetry transformation to e^a_μ and ψ_μ, we get

$$\begin{aligned} \delta e^a_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \\ \delta\psi_\mu &= D_\mu\epsilon \end{aligned} \tag{7}$$

and for inverse vielbein

$$\begin{aligned} \delta e^\mu_a &= -\tfrac{1}{2}\bar{\epsilon}\gamma^\mu\psi_a \\ \delta e &= \tfrac{1}{2}e(\bar{\epsilon}\gamma^\rho\psi_\rho). \end{aligned} \tag{8}$$

With 6 we can see that the action 4 is invariant under transformations 7 and 8. Hence we obtain $\mathcal{N} = 1$ $D = 4$ supergravity.

3 $D = 11$ supergravity

Supergravity in 11 dimensions was first constructed by [2] (see also [7]). The 11D supergravity theory, which is the $\mathcal{N} = 1$ SUSY theory, contains the graviton and gravitino. When checking for a number of degrees of freedom of each field, we find that the graviton has 44 degrees of freedom and the gravitino has 128 degrees of freedom. The problem is that in a SUSY theory we need to match the numbers of degrees of freedom of bosonic and fermionic fields. Since $128 - 44 = 84$, we need a field that contains 84 degrees of freedom. Fortunately, there is a field in 11D that contains exactly 84 degrees of freedom. That field is an antisymmetric rank 3 tensor $A_{\mu\nu\rho}$.

We already obtained the $\mathcal{N} = 1$ $D = 4$ supergravity theory from the last section. By modifying the action 4, we get the complete action for 11D supergravity. First of all, we define the kinetic term for $A_{\mu\nu\rho}$

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]} \tag{9}$$

Then we have the 11D action written as

$$\begin{aligned} S = \tfrac{1}{2\kappa^2}\int d^{11}xe[R(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\tfrac{\omega+\hat{\omega}}{2})\psi_\rho - \tfrac{1}{24}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \\ - \tfrac{\sqrt{2}}{192}\bar{\psi}_\nu(\gamma^{\alpha\beta\gamma\delta\nu\rho} + 12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho})\psi_\rho(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}) \\ - \tfrac{2\sqrt{2}}{144^2}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8\mu_9\mu_{10}\mu_{11}}F_{\mu_1\mu_2\mu_3\mu_4}F_{\mu_5\mu_6\mu_7\mu_8}A_{\mu_9\mu_{10}\mu_{11}}] \end{aligned} \tag{10}$$

where

$$\begin{aligned} \hat{\omega}_{\mu ab} &= \omega_{\mu ab}(e) - \tfrac{1}{4}(\bar{\psi}_\mu\gamma_a\psi_b - \bar{\psi}_a\gamma_\mu\psi_b + \bar{\psi}_b\gamma_a\psi_\mu) \\ \hat{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma} + \tfrac{3}{2}\sqrt{2}\bar{\psi}_{[\mu}\gamma_{\nu\rho}\psi_{\sigma]} \end{aligned} \tag{11}$$

are the supercovariant extensions of ω and $F_{(4)}$. Then, we construct transformation rules of all the fields using reference from the last section, we get

$$\begin{aligned} \delta e^a_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \\ \delta\psi_\mu &= D_\mu(\hat{\omega})\epsilon + \tfrac{\sqrt{2}}{288}(\gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\mu)\hat{F}_{\alpha\beta\gamma\delta}\epsilon \\ \delta A_{\mu\nu\rho} &= -\tfrac{3\sqrt{2}}{4}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}. \end{aligned} \tag{12}$$

The results above give us the complete theory of supergravity in 11 dimensions. Notice that the first 3 terms in the action 10 are kinetic terms for the graviton, gravitino and antisymmetric tensor, respectively. The forth term in 10 is a coupling between the gravitino and antisymmetric tensor. Finally, the last term is a Chern-Simons term, which is there to make the action invariant under supersymmetry transformation.

4 $\mathcal{N} = 2$ $D = 4$ Supergravity

$\mathcal{N} = 2$ $D = 4$ supergravity is an extended theory of $\mathcal{N} = 1$ $D = 4$ theory. This theory contains a graviton field e^a_μ, 2 gravitini fields ψ^i_μ, where $i = 1, 2$, and a (abelian) vector field A_μ. Notice that we can think of $\mathcal{N} = 2$ theory as $\mathcal{N} = 1$ gravity multiplet coupling with $\mathcal{N} = 1$ gravitino multiplet. Thus we can write down the theory as

$$S = \frac{1}{2\kappa^2}\int d^4xe[e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}^i_\mu\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}\epsilon_{ij}\bar{\psi}^i_\mu\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi^j_\nu F_{\alpha\beta}\hat{F}^{\alpha\beta}] \tag{13}$$

where $\epsilon_{12} = \epsilon^{12} = 1$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{14}$$

is an abelian field strength tensor and

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \epsilon_{ij}\bar{\psi}^i_\mu \psi^j_\nu \tag{15}$$

is its supercovariant extension. The last term of the action 17 is the interaction between the gravitini and the vector field. To construct the transformation rules for the theory, we modify those of section 2, we get

$$\begin{aligned} \delta e^a_\mu &= \tfrac{1}{2}\bar{\epsilon}^i \gamma^a \psi^i_\mu \\ \delta\psi_\mu &= D_\mu(\hat{\omega})\epsilon - \tfrac{1}{8}\epsilon^{ij}\gamma^{\rho\sigma}\gamma_\mu \epsilon^j \hat{F}_{\rho\sigma} \\ \delta A_\mu &= \epsilon_{ij}\bar{\epsilon}^i \psi^j_\mu. \end{aligned} \tag{16}$$

To check whether 17 is invariant under 16 or not, we use the same method as in section 2, i.e. 1.5 order formalism.

Notice that if we set $\psi^{i=2}_\mu = A_\mu = 0$, we get

$$S = \frac{1}{2\kappa^2}\int d^4x e[e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}^1_\mu \gamma^{\mu\nu\rho} D_\nu \psi^1_\rho], \tag{17}$$

which is the same theory with $\mathcal{N} = 1$ supergravity, where $\psi_\mu = \psi^1_\mu$. Thus, this fact shows that we can obtain $\mathcal{N}$ theory from $\mathcal{N}' > \mathcal{N}$ theory that lives in the same dimensional spacetime. This method of reducing supersymmetry is called truncation.

5 The bosonic sector of matter coupled $\mathcal{N} = 2$ $D = 4$ supergravity

We couple $\mathcal{N} = 2$ $D = 4$ Supergravity with n vector multiplets and n_H hypermultiplets (see [9, 10]). Thus, in this theory, we have a graviton, 2 gravitini, $n+1$ vectors, n scalar complex field from n vector multiplets, $2n$ spin-$\frac{1}{2}$ fields from n vector multiplets, $2n_H$ spin-$\frac{1}{2}$ from n_H hypermultiplets, and $4n_H$ real scalar fields from n_H hypermultiplets. We can write the bosonic of matter coupled $\mathcal{N} = 2$ $D = 4$ supergravity as

$$\begin{aligned} S = \textstyle\int d^4x e[&R + g_{i\bar{j}}(z,\bar{z})\nabla^\mu z^i \nabla_\mu z^{\bar{j}} - 2\lambda h_{uv}(q)\nabla^\mu q^u \nabla_\mu q^v \\ &+ i(\bar{\mathcal{N}}_{AB}\mathcal{F}^{-A\mu\nu}\mathcal{F}^{-B}_{\mu\nu} - \mathcal{N}_{AB}\mathcal{F}^{+A\mu\nu}\mathcal{F}^{+B}_{\mu\nu}) - V] \end{aligned} \tag{18}$$

where $I, i = 1, 2, ..., n$ and $u, v = 1, 2, ..., 4n_H$. As we can see, the second and third terms in the equation 18 are the terms for scalar fields z from vector multiplets and q hypermultiplets, respectively. The complex scalar fields z span the special Kahler manifold (see [11] for detail) $\mathcal{M}_{SK}$ and the real scalar fields q span the quaternionic Kahler manifold $\mathcal{M}_{QK}$. Therefore the scalar manifold is written as

$$\mathcal{M}_{scalar} = \mathcal{M}_{SK} \times \mathcal{M}_{QK}. \tag{19}$$

Since $\mathcal{M}_{scalar}$ is in the form of a coset manifold G/H, where G is an isometry group and H is a holonomy group. In particular, H has a general form as

$$H = H_R \times H_{matt} \tag{20}$$

where H_R is an automorphism of supersymmetry algebra (R-symmetry) and H_{matt} is a compact group acting on matter fields. R-symmetry group for $\mathcal{N} = 2$ supergravity in 4 dimensions is $U(2)$ and we know the relation that $U(2) \sim U(1) \times SU(2)$ and from 19 we have $H = H^{(SK)} \times H^{(QK)}$. Thus, we find that

$$\begin{aligned} H^{(SK)} &= U(1) \times H^{(SK)}_{matt} \\ H^{(QK)} &= SU(2) \times H^{(QK)}_{matt}. \end{aligned} \tag{21}$$

The metric $g_{i\bar{j}}$ in the second term of 18 is the metric of the special Kahler manifold and is defined as

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K} \tag{22}$$

where $\mathcal{K}$ is the Kahler potential. Moreover, we can define Kahler 2-form as

$$K = i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}. \tag{23}$$

This geometry will be called the Kahler space if the form K is closed, i.e. $dK = 0$. The special Kahler space has isometry which is described by Killing vectors k^i_I. We have a symmetry of the metric an infinitesimal transformation

$$z'^i \to z^i + \theta^A k^i_A \tag{24}$$

and we have the condition for isometry transformation (see [8] for the proof) as

$$\begin{aligned} \nabla_i k_{Aj} + \nabla_j k_{Ai} &= 0 \\ \nabla_i k_{A\bar{j}} + \nabla_{\bar{j}} k_{Ai} &= 0 \end{aligned} \tag{25}$$

where $k_{Aj} = g_{ji} k^j_A$ and $\nabla_i k_{Aj} = \partial_i k_{Aj} - \Gamma^m_{ij} k_{Am}$. In addition, we can also write an algebra for the Killing vectors as

$$[k_A, k_B] = f_{AB}^{\ \ C} k_C \tag{26}$$

where $k_A = k^i_A \partial_i + k^{\bar{i}}_A \partial_{\bar{i}}$ is a Killing vector on a tangent space. Then we define the Killing vector in terms of a momentum map as

$$k^i_A = i g^{i\bar{j}} \partial_{\bar{j}} \mathscr{P}_A \tag{27}$$

and since $\mathcal{K}$ is invariant under the transformation of isometry group G, we have

$$k^i_A \partial_i \mathcal{K} + k^{\bar{i}}_A \partial_{\bar{i}} \mathcal{K} = 0. \tag{28}$$

Then we have the condition

$$i\mathscr{P}_A = \frac{1}{2}(k^i_A \partial_i \mathcal{K} - k^{\bar{j}}_A \partial_{\bar{j}} \mathcal{K}) = k^i_A \partial_i \mathcal{K} = -k^{\bar{j}}_A \partial_{\bar{j}} \mathcal{K}. \tag{29}$$

Furthermore, the Killing vectors contain both isometry indices i, j and symplectic indices A, B. Then we define a vector V^M in terms of X^A and F_A where $A = 1, 2, ..., n+1$ according to the numbers of vector fields in the theory. Thus we have

$$V^M = \begin{pmatrix} X^A \\ F_A \end{pmatrix}. \tag{30}$$

When acting the transformation of group G on V^M, we have

$$(k^i_A \partial_i + k^{\bar{i}}_A \partial_{\bar{i}}) \begin{pmatrix} X^A \\ F_A \end{pmatrix} = \mathcal{G} \begin{pmatrix} X^A \\ F_A \end{pmatrix} \tag{31}$$

then we see that $\mathcal{G} \in Sp(2n+2, \mathbb{R})$. Hence we can conclude that the isometry group G is embedding in the symplectic group $Sp(2n+2, \mathbb{R})$. In addition, [9] proved that the holonomy group is $U(n+1)$. Therefore we have the special Kahler manifold written in the form

$$\mathcal{M}_{SK} = \frac{Sp(2n+2, \mathbb{R})}{U(n+1)}. \tag{32}$$

6 Future work

Our next work is to study $\mathcal{M}_{QK}$ side of the relation 19. Then study the fermionic sector of matter coupled $\mathcal{N} = 2$ $D = 4$ supergravity and the gauged theory.

References

[1] Progress Toward A Theory Of Supergravity *Daniel Freedman, Sergio Ferrara and Peter van Nieuwenhuizen.* Physical Review D. 13(12): 32143218, 1976.

[2] Supergravity in theory in 11 dimensions. *Eugene Cremmer, Bernard Julia and Joel Scherk.* Physics Letters B, 76(4): 409-412, 1978.

[3] String theory dynamics in various dimensions. *Edward Witten.* Nuclear Physics B, 443(1): 85-126, 1995.

[4] Zum Unittsproblem in der Physik. *Theodor Kaluza.* Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.): 966972, 1921.

[5] Quantentheorie und fnfdimensionale Relativittstheorie. *Oscar Klein.* Zeitschrift fr Physik A. 37(12): 895906, 1926.

[6] On a theory of particles with half-integral spin. *William Rarita and Julian Schwinger.* Phys. Rev., 60: 6161, Jul 1941.

[7] Supergravity. *Daniel Freedman and Antoine Van Proeyen.* Cambridge University Press, New York, USA, 2012.

[8] Gauged Supergravity and AdS/CFT Holography. *Parinya Karndumri.* Danex Intercorporation, Bangkok, Thailand, 2018.

[9] Gauged Supergravities. *Mario Trigiante.* arXiv:1609.09745v4, 2017.

[10] Introduction to Supergravity. *Horatiu Nastase.* arXiv:1112.3502v3, 2012.

[11] Special Kahler Manifolds. *Daniel Freed.* Communications in Mathematical Physics, 203: 31-52, 1999.