University of Cyprus

# The Double Pentaladder integral in the strong coupling limit

by

Adam Lantos

Supervisor: Dr. Georgios Papathanasiou

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*"My work always tried to unite the truth with the beautiful, but when I had to choose one or the other, I usually chose the beautiful."*

-Hermann Weyl

# Contents

# 1 Introduction

Physicists have long been accustomed to the idea of the perturbative study of a quantum field theory. The booming advancements in the field of holography, and more precisely the gauge/gravity duality, have opened new paths to explore these theories. Recent work has shown that through the study of symmetries and the use of modern and more elegant formalism, some amplitudes can be calculated to all orders in perturbation theory.

In the present work, we expand on previous work ([1]) which was concerned with a particular class of dual conformal invariant integrals that arise from a specific class of scattering processes, in the context of the planar maximally supersymmetric Yang-Mills (SYM) theory. The integrals we are concerned with correspond to Feynman diagrams with a so-called double pentaladder topology; what this means is that they are essentially ladder integrals which are closed on each end by a pentagon. Each pentagon has 3 incoming/outgoing massless particles. A representative example is shown in the figure below:

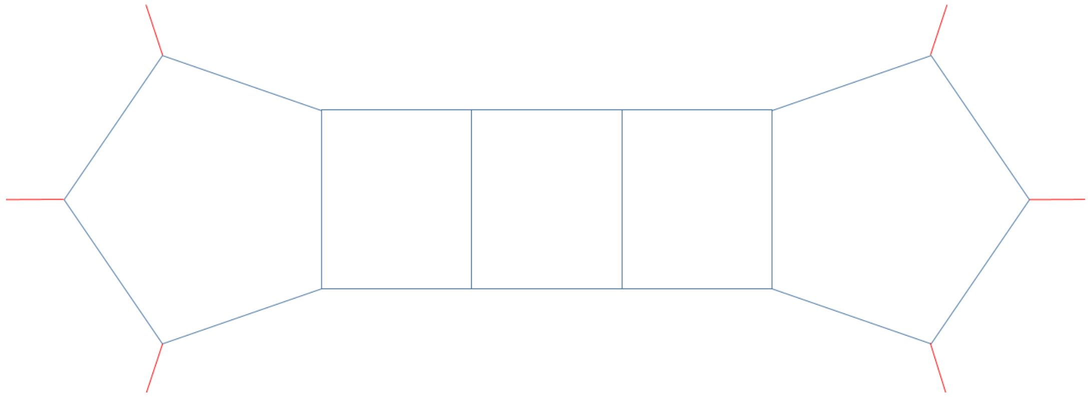

FIGURE 1: A representative example of a (5-loop) pentaladder Feynman diagram

More specifically, the present work expands on the work done in [1] by focusing on the strong coupling limit of these pentaladder integrals. The present report is structured as follows: first, we will present some basic background. Then, we will give a very brief review of what has been done in [1] in order to put our own work in a physical context and, lastly, we will then proceed to our own work.

# 2 Basic background

In this section, we will present some of the background needed for understanding the rest of the report. Unfortunately, in order to keep the size of the presentation relatively small, we will not include introductions, or even brief reviews to some of the more vast subjects, but we will touch upon the points that we need. The reader can read through the relevant chapters of [2], [3] and [4]. The latter is more useful for what is covered in section 2.2.

## 2.1 Spinor-helicity formalism

We are now going to introduce a new notation, which will allow us to re-write null 4-vectors in terms of spinors. This is because the Lorentz group $SO(3,1)$ is locally equivalent to $SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$, where $SL(2,\mathbb{C})$ is a "complexified" version of our more familiar $SU(2)$. We do this by adopting the following convention for the $(\frac{1}{2},0)$ and the $(0,\frac{1}{2})$ representations, with respect to the spin labels of each of the $SL(2,\mathbb{C})$ factors, respectively:

$$\chi_\alpha \text{ for left-handed spinors} \tag{2.1}$$
$$\bar{\eta}^{\dot{\alpha}} \text{ for right-handed spinors} \tag{2.2}$$

To lower and raise spinor indices, we use

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta,\ \chi_\alpha = \epsilon_{\alpha\beta}\chi^\beta,\ \bar{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\eta}_{\dot{\beta}},\ \bar{\eta}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\eta}^{\dot{\beta}} \tag{2.3}$$

with $\epsilon$ can be intuitively thought of as a kind of spinor metric; it raises and lowers spinor indices in the same way that the spacetime metric raises and lowers spacetime indices. That being said, this "metric" has some very different properties than those of the spacetime metric. It is defined as

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = \epsilon^{\dot{1}\dot{2}} = -\epsilon^{\dot{2}\dot{1}} = -\epsilon_{\dot{1}\dot{2}} = \epsilon_{\dot{2}\dot{1}} = 1 \tag{2.4}$$

This means that $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$, and analogously for the dotted indices. We also define the following inner products:

$$\eta\chi \equiv \eta^\alpha\chi_\alpha \tag{2.5}$$
$$\bar{\eta}\bar{\chi} \equiv \bar{\eta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{2.6}$$

Since $\left(\frac{1}{2},\frac{1}{2}\right)$ representations of the Lorentz group are vector quantities[1], we must find a way to go from a vector to its spinorial form. To do this, we introduce the following matrices:

$$(\sigma^\mu)_{\alpha\dot{\beta}} = \sigma^\mu_{\alpha\dot{\beta}} = (\mathbb{1},\vec{\sigma})_{\alpha\dot{\beta}} \tag{2.7}$$
$$(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} = \bar{\sigma}^{\mu\dot{\beta}\alpha} = (\mathbb{1},-\vec{\sigma})^{\dot{\beta}\alpha} \tag{2.8}$$

[1]The reader can just perform a Lorentz transform and see that it does transform like a vector.

The reason for this particular placement of the spinor indices comes from the fact that

$$\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}$$

must give back something with the same spinor indices as the initial $\begin{pmatrix} \chi_\alpha & \bar{\eta}^{\dot{\alpha}} \end{pmatrix}^{\mathrm{T}}$. Before moving on, we state some very useful identities.

$$(\sigma^\nu)_{\alpha\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha} = 2g^{\mu\nu} \tag{2.9}$$

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma_\mu)_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{2.10}$$

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{2.11}$$

We are now ready to write 4-vectors in a spinor form. For a 4-vector $A^\mu$, we define

$$A_{\alpha\dot{\beta}} = A_\mu(\sigma^\mu)_{\alpha\dot{\beta}} \tag{2.12}$$

Combining (2.10) with (2.9) we get $A^\mu = \frac{1}{2}A_{\alpha\dot{\beta}}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}$. Through (2.10), we can write down the spinor form of the 4-momentum of a particle as

$$p_{\alpha\dot{\alpha}} = p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} = (p_o\mathbb{1} - p_i\sigma_i)_{\alpha\dot{\alpha}} = \begin{pmatrix} p_o - p_3 & -(p_1 - ip_2) \\ -(p_1 + ip_2) & p_o + p_3 \end{pmatrix}_{\alpha\dot{\alpha}} \tag{2.13}$$

Using (2.11), we can define the inner product between the 4-momentum $p^\mu$ with the 4-momentum $q^\mu$ as follows:

$$p \cdot q = p^\mu q_\mu = \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\alpha\beta}q_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}p_{\alpha\dot{\alpha}}q^{\alpha\dot{\alpha}} \tag{2.14}$$

For massless particles, where we have light-like momenta, $p \cdot p = 0$, or $\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}p_{\alpha\beta}p_{\dot{\alpha}\dot{\beta}} = \det(p) = 0$. This means that at least one of the eigenvalues of (2.13) is equal to zero. We can thus write

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}} \qquad \text{(light-like momenta)} \tag{2.15}$$

with $p_{\alpha\dot{\alpha}}$ having an eigenspinor that corresponds to a zero eigenvalue equal to some spinor whose inner product with $\bar{\lambda}$ is equal to zero.

Using the above result, we can write the following for light-like momenta $p_{\alpha\dot{\alpha}} = \lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$ and $q_{\alpha\dot{\alpha}} = \chi_\alpha\bar{\chi}_{\dot{\alpha}}$:

$$p \cdot q = \frac{1}{2}\lambda^\alpha\chi_\alpha\left(-\bar{\lambda}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}\right) = \frac{1}{2}\langle\lambda\chi\rangle\,[\chi\lambda] \tag{2.16}$$

where we have defined

$$\langle\lambda\chi\rangle = \lambda^\alpha\chi_\alpha \tag{2.17}$$

$$[\lambda\chi] = -\bar{\lambda}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \bar{\lambda}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \tag{2.18}$$

This definitions will prove themselves to be very useful in the next sections.

## 2.2 Dual coordinates and Dual Conformal Symmetry

For this section, we will introduce a simple motivating example for a specific set of variables that will be of particular use to us later on. For this part of the report, the references [4] and [1] are followed.

Let us examine the amplitude for the 2-loop pentaladder. We can convert our momenta (internal and external) into other coordinates, called dual coordinates, by the transformation

$$p_i^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}, \qquad x_{N+1}^{\alpha\dot{\alpha}} \equiv x_1^{\alpha\dot{\alpha}} \tag{2.19}$$

where $i$ corresponds to the $i-$th scattering particle and $N$ is the total number of external particles. Given this identification, all scalar products that appear in the integrals can be written in terms of the following squared differences:

$$x_{ij}^2 \equiv \left(x_i - x_j\right)^2 \tag{2.20}$$

The 2-loop pentaladder integral is given to us by the Feynman rules for scalar particles for the following scattering process: Note that $x_s$ and $x_r$ replace the internal momenta

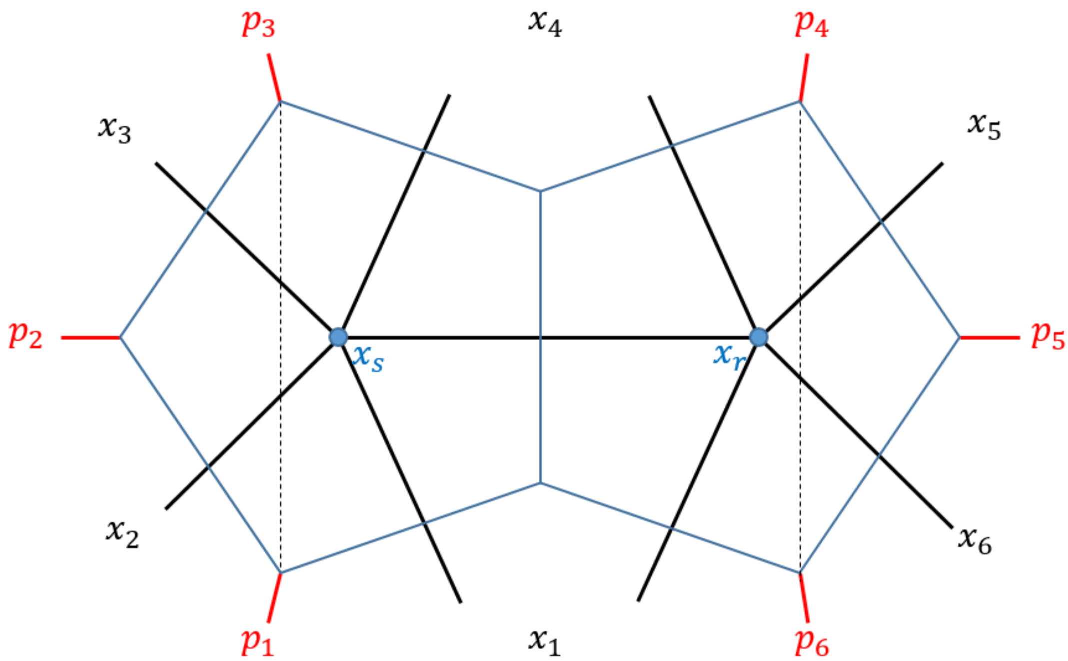


FIGURE 2: Feynman diagram for 2-loop pentaladder process for scalar particles. The momenta of the particles are shown in red lines. The propagators are represented by scalar propagators in the dual space, which are shown in solid black lines. $x_s$ and $x_r$ replace the internal momenta as integration variables. The dashed lines represent the factors in the numerator of (2.21) which make it dual conformal invariant.

as integration variables. Applying the Feynman rules, then converting to the dual coordinates, and finally using (2.20), we find an integral

$$I^{(2)} \propto \int \frac{d^4x_r}{i\pi^2}\frac{d^4x_s}{i\pi^2}\frac{x_{Ar}^2 x_{Bs}^2}{\left(x_{1r}^2 x_{2r}^2 x_{3r}^2 x_{4r}^2\right) x_{rs}^2 \left(x_{1s}^2 x_{4s}^2 x_{5s}^2 x_{6s}^2\right)} \tag{2.21}$$

where we have introduced $x_A^{\alpha\dot{\alpha}}$ and $x_B^{\alpha\dot{\alpha}}$ in order to keep the integral finite. They satisfy the null-separation conditions $x_{A1}^2 = x_{A2}^2 = x_{A3}^2 = x_{A4}^2 = 0$ and $x_{B1}^2 = x_{B2}^2 = x_{B3}^2 = x_{B4}^2 = 0$.

Let us now inspect how the integral (2.21) transforms under conformal transformation of the dual coordinates, $x_i^{\alpha\dot{\alpha}}$. We perform a dual conformal inversion to get

$$x_i^{\alpha\dot{\alpha}} \to \frac{x_i^{\alpha\dot{\alpha}}}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad d^4x_{r/s} \to \frac{d^4x_{r/s}}{x_{r/s}^2} \tag{2.22}$$

Under this transformation, the extra $x_s^2$ and $x_s^2$ factors in (2.21) cancel each other out. On the other hand, the extra $x_i^2$ that correspond to the external momenta, as well as the extra $x_A^2$ and $x_B^2$, do not. This way, (2.21) transforms as

$$I^{(2)} \to \frac{x_1^4 x_2^2 x_3^2 x_4^4 x_5^2 x_6^2}{x_A^2 x_B^2} I^{(2)} \tag{2.23}$$

As the extra factor in front of the transformed $I^{(2)}$ is just a number that depends on the external momenta, we see that the amplitude is conformally invariant, i.e. invariant up to an overall factor[2].

From the way that the integral (2.21) transforms under (2.22), we are motivated to define

$$\Omega^{(2)} = \frac{x_{26}^2 x_{35}^2 x_{14}^2}{x_{AB}^2} I^{(2)} \tag{2.24}$$

It is evident that $\Omega^{(2)}$ is invariant under the conformal transformation 2.22. This appropriately normalized integral is the main object that we examine[3].

Since we have symmetries and conditions to be satisfied by the momenta of the problem, we have a smaller number of independent variables than the initial 24 which correspond to the components of the external momenta. Let us be more precise. For each momentum, we have the restriction $p_i^2 = 0$ because it is associated with a massless particle. This gives 6 conditions. Moreover, since we have (dual) conformal symmetry, we must also subtract from the initial degrees of freedom the 15 generators of the conformal group. This means that we are left with just 3 independent variables. There is a systematic way to find these and one can read the relevant sections of [5] to learn more about them, but

[2] For the reader who finds the above incomprehensible, there is a much easier example with 1-loop integral in [4]

[3] To be more precise, it is its L-loop generalization that is of interest to us.

we will not present them here. They are given by

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad v = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2}, \quad w = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} \tag{2.25}$$

Notice that under the transformation (2.22), these coordinates are invariant, which,in a certain sense, showcases that they are *the* variables to work with. Using these variables will prove to be indispensable for the solution of the all-loop problem.

## 3 Lightning fast review of previous results

Let us denote by $\Omega^{(L)}$ the integral associated with the scattering process of an L-loop double pentaladder scattering process; it is obviously a generalization of (2.24) to L-loops. Such process is illustrated in fig.3.

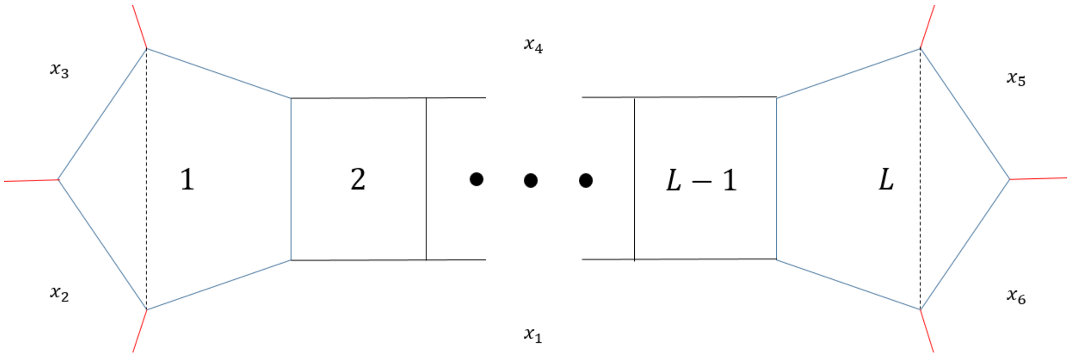


FIGURE 3: The L-loop double pentaladder scattering process.

The main quantity that we want to calculate is

$$\Omega = \sum_L \left(-g^2\right)^L \; \Omega^{(L)} \tag{3.1}$$

Luckily and amazingly, there is a property that connects adjacent integrals; there exists a differential operator, which we denote by $\mathcal{D}$, which acts as some kind of lowering operator in sense that $\mathcal{D}\Omega^{(L)} = \Omega^{(L-1)}$. Obviously, $\mathcal{D}^{-1}$ acts as a raising operator, in the same sense, although we will not need the inverse differential operator here. The good news is that people have already found out what this differential operator is! In particular, in [6], they have found the following

$$w\partial_w \left[-u(1-u)\partial_u - v(1-v)\partial_v + (1-u-v)(1-w)\partial_w\right] \Omega^{(L)} = \Omega^{(L-1)} \tag{3.2}$$

By exploiting the symmetries of the problem, the authors of [1] have motivated the use of a new set of variables, which also turned out to give a much more simplified version of the final differential equation that must be solved in order to get (3.1). Without repeating their arguments, we simply state the transformation from the set of variables $(u, v, w)$ to the new set of variables $(x, y, z)$:

$$x = 1 + \frac{1 - u - v - w + \sqrt{\Delta}}{2uv}, \quad y = 1 + \frac{1 - u - v - w - \sqrt{\Delta}}{2uv} \quad z = \frac{u(1-v)}{v(1-u)} \tag{3.3}$$

with[4] $\Delta = (1 - u - v - w)^2 - 4uvw$. One can find the details of this conversion in the original reference.

With these coordinates on hand, we can turn (3.2) into the simpler

$$\frac{y-1}{y}\left[\left(y\partial_y\right)^2 - \left(z\partial_z\right)^2\right]\Omega^{(L)} = \Omega^{(L-1)} \tag{3.4}$$

Using (3.1), we obtain

$$\frac{1-y}{y}\left[\left(y\partial_y\right)^2 - \left(z\partial_z\right)^2 - g^2\right]\Omega(x, y, z, g^2) = 0 \tag{3.5}$$

Using the fact that $x$ and $y$ are related by a parity transformation and that $\Omega(x, y, z, g^2)$ is even under that transformation, we also obtain

$$\frac{1-x}{x}\left[\left(x\partial_x\right)^2 - \left(z\partial_z\right)^2 - g^2\right]\Omega(x, y, z, g^2) = 0 \tag{3.6}$$

We thus expect $\Omega(x, y, z, g^2)$ to have the same behaviour for $x$ and $y$. The reader should observe the usefulness of using the $(x, y, z)$ variables: equations (3.5) and (3.6) imply that $\Omega(x, y, z, g^2)$ can be separated into functions of $(y, z)$ and $(x, z)$ respectively, which makes things tremendously easier to deal with. This originates from the exploitation of the underlying symmetries of the problem that we mentioned earlier. We will shortly see that in these variables, the differential equations that we have to solve have known functions as solutions.

All that is left now is to solve (3.4) and (3.5). We begin by using the ansatz

$$z^{\frac{i\nu}{2}} f_\nu(x, y) \tag{3.7}$$

which turns (3.5) and (3.6) into

$$\left[(1-y)\left(y\partial_y\right)^2 + \frac{1}{4}(1-y)\nu^2 - yg^2\right] f_\nu(x, y) = 0 \tag{3.8}$$

$$\left[(1-x)\left(x\partial_x\right)^2 + \frac{1}{4}(1-x)\nu^2 - xg^2\right] f_\nu(x, y) = 0 \tag{3.9}$$

[4] The sign on front of $\sqrt{\Delta}$ is arbitrarily chosen because parity changes its sign and also interchanges $x$ with $y$.

respectively. As expected, we can use separation of variables to get four independent solutions for (3.8) and (3.9); these are given by

$$F^{j(\nu)}_{\pm\nu}(x)F^{j(\nu)}_{\pm\nu}(y), \quad \text{with} \;\; j(\nu) \equiv \sqrt{\nu^2+(2g)^2} \tag{3.10}$$

$$F^{j(\nu)}_{\nu}(x) \equiv \frac{\Gamma(1+\frac{i\nu+j}{2})\Gamma(1+\frac{i\nu-j}{2})}{\Gamma(1+i\nu)} x^{\frac{i\nu}{2}} {}_2F_1\left(\frac{i\nu+j}{2}, \frac{i\nu-j}{2}, 1+i\nu, x\right) \tag{3.11}$$

where $f_\nu(x,y)$ is some linear combination of the functions (3.10). Here, ${}_2F_1$ is a hypergeometric function. Obviously, since ${}_2F_1$ is a hypergeometric function, $F^j_\nu(x)$ is also one. Furthermore, the $\Gamma-$factors in front of (3.11) were chosen so that $F^j_\nu(x)$ is normalized according to $F^j_\nu(1)=1$.

The full solution for $\Omega(x,y,z,g^2)$ is given by some linear combination of (3.10). We will only briefly explain here what the authors in [1] did[5] in order to find a meaningful combination, since this is not the main subject of this report. The authors of [1], then, first require that $\Omega(x,y,z,g^2)$ be smooth in the octant $u,v,w>0$ and by examining the neighborhood of $(u,v,w)=(1,1,1)$, they conclude that only combinations of $F^{j(\nu)}_{+\nu}(x)F^{j(\nu)}_{+\nu}(y)$ and $F^{j(\nu)}_{-\nu}(x)F^{j(\nu)}_{-\nu}(y)$ satisfy this condition. For the same condition, by examining the sign flips of $x$ and $y$ in the positive $(u,v,w)$ octant and relating them to the singularities of the logarithms found in the hypergeometric functions, they where able to pinpoint the exact and unique linear combination for $\Omega(x,y,z,g^2)$ but only up to some factor. To find the specific factor, they study the boundary condition of large $\nu$. The final result is given by

$$\boxed{\Omega(x,y,z,g^2) = \int_{-\infty}^{+\infty} \frac{d\nu}{2i} z^{\frac{i\nu}{2}} \frac{F^{j(\nu)}_{+\nu}(x)F^{j(\nu)}_{+\nu}(y) - F^{j(\nu)}_{-\nu}(x)F^{j(\nu)}_{-\nu}(y)}{\sinh \pi\nu}} \tag{3.12}$$

This is also the point where our own study begins.

# 4 Double Pentaladder integral in the strong coupling limit

Our interest lies in the study of (3.12) in the strong coupling limit where

$$g \gg |\nu|, \quad g \to \infty \tag{4.1}$$

To do this, we perform an asymptotic expansion of the hypergeometric function ${}_2F_1$. The expansion for the case $0<x<1$ which we will focus on, is given in theorem 3.2 of [7]. To use it, we perform the replacement

$$a = \frac{i\nu}{2}, \quad \lambda = \frac{j}{2}, \quad c = 1+i\nu, \quad z = 1-2x, \quad \xi(x) = -i\phi(x) \tag{4.2}$$

[5]The details of what I am going to describe can be found in the original reference.

where $\xi(z) = \ln\Big(-z - i\sqrt{1-z^2}\Big)$. We choose a branch so that $\phi(x = o^+) = \pi$ and $\phi(x = 1^-) = 0$. Combining this with the fact that at strong coupling we have $j \approx 2ig$, we can write their expansion as

$$
{}_2F_1\left(\frac{i\nu}{2} + ig, \frac{i\nu}{2} - ig, 1 + i\nu, x\right) \approx \frac{\Gamma(1+i\nu)\Gamma(ig - \frac{i\nu}{2})}{i\pi\Gamma(ig + \frac{i\nu}{2})}\times
\left[\left((e^{\pi(g-\frac{i\nu}{2})}K_0(g\phi) + e^{-\pi(g-\frac{i\nu}{2})}K_0(-g\phi)\right)\sum_{j=1}\frac{d_j}{(ig)^j}
-i\frac{\phi}{2}\left(e^{\pi(g-\frac{i\nu}{2})}K_1(g\phi) + e^{-\pi(g-\frac{i\nu}{2})}K_1(-g\phi)\right)\sum_{j=1}\frac{c_j}{(ig)^j}\right] \tag{4.3}
$$

where $K_0$ and $K_1$ are modified Bessel functions of the second kind. We also emphasize that $\phi$ can be written as $\phi(x) = \arccos(2x-1)$, as we will use it later on. Now, to go to $F^j_\nu(x)$, we use (4.3) along with the identities $\Gamma(1 \pm z) = \pm z\Gamma(\pm z)$ and $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$; doing this, we obtain

$$
F^j_\nu(x) \approx \frac{x^{i\frac{\nu}{2}}(g + \frac{\nu}{2})}{i\sinh\big(\pi g - \frac{\pi\nu}{2}\big)}\left[\left((e^{\pi(g-\frac{i\nu}{2})}K_0(g\phi) + e^{-\pi(g-\frac{i\nu}{2})}K_0(-g\phi)\right)\sum_{j=1}\frac{d_j}{(ig)^j}
-i\frac{\phi}{2}\left(e^{\pi(g-\frac{i\nu}{2})}K_1(g\phi) + e^{-\pi(g-\frac{i\nu}{2})}K_1(-g\phi)\right)\sum_{j=1}\frac{c_j}{(ig)^j}\right] \tag{4.4}
$$

The leading order coefficients $c_0$ and $d_0$ are also given in [7]. We present them here:

$$
c_0 = -\sqrt{\frac{2}{\phi}}\left(\frac{1}{x} - 1\right)^{\frac{1}{4}} x^{-i\frac{\nu}{2}}, \quad d_0 = 0 \tag{4.5}
$$

These were used in [1] to get the leading order expansion of $F^j_\nu(x)$ (i.e. of order $\mathcal{O}(\frac{1}{g})$ ), which we state here for completeness purposes.

$$
F^j_\nu(x) \approx \sqrt{\pi g}\left(\frac{1}{x} - 1\right)^{\frac{1}{4}} e^{-g\phi(x)} \tag{4.6}
$$

We have successfully expanded the work in [7] to include the next to leading order expansion. Before proceeding to our findings, we want to further simplify (4.4) and write it in a form that is easier to manipulate. We start by writing

$$
\frac{1}{\sinh\big(\pi g - \frac{\pi\nu}{2}\big)} \approx 2e^{-\pi g}\sum_{n=0}^{\infty}\left(e^{-2\pi g + \pi\nu}\right)^n
$$

This way, we can write (4.4) in the following form

$$F_\nu^j(x) \approx A_+(x,\nu,g^2) + \Big[A_+(x,\nu,g^2) + A_-(x,\nu,g^2)\Big] \sum_{n=1}^{\infty} \left(e^{-2\pi g+\pi\nu}\right)^n \tag{4.7}$$

where

$$A_\pm(x,\nu,g^2) = -x^{\frac{i\nu}{2}}(2g+\nu) \sum_{j=1} \frac{1}{(ig)^j} \left[\frac{\phi(x)}{2} K_1(\pm g\phi(x))c_j + iK_o(\pm g\phi(x))d_j\right] \tag{4.8}$$

In order to perform the next to leading order expansion, we have to keep in mind that

$$K_0(\pm g\phi) \approx \sqrt{\frac{\pi}{\pm 2g\phi}}\left(1 \mp \frac{1}{8g\phi}\right) e^{\mp g\phi}, \quad K_1(\pm g\phi) \approx \sqrt{\frac{\pi}{\pm 2g\phi}}\left(1 \pm \frac{3}{8g\phi}\right) e^{\mp g\phi} \tag{4.9}$$

Since $\phi(x) \in (0,\pi)$ and we work in the limit (4.1), we find that $K_{1/2}(g\phi) \gg K_{1/2}(-g\phi)$, but we cannot neglect the latter for reasons that we will shortly explain. In order to go further, we will state the coefficients $c_1$ and $d_1$ which we have calculated through the procedure proposed in [7][6]. The calculation of these coefficients, as well as the subsequent simplifications of the central results presented here, where the main goal of the project for which this report is written. The coefficients are then given by:

$$c_1 = \frac{x^{-i\frac{\nu}{2}}}{8\phi^2}\left(\Phi(z,\phi(z),\nu) + \Phi(z,-\phi(z),\nu)\right) \tag{4.10}$$

$$d_1 = -i\frac{\phi}{2}\left[\frac{x^{-i\frac{\nu}{2}}}{8\phi^2}\left(\Phi(z,\phi(z),\nu) - \Phi(z,-\phi(z),\nu)\right)\right] \tag{4.11}$$

with $z$ as given in (4.2) and with

$$\Phi(z,\phi(z),\nu) = \frac{3(1+ze^{-i\phi}) + i\phi\left((z-2)e^{-i\phi} - 4i\nu(1+ze^{-i\phi}) - 2\nu^2(1+z)e^{-i\phi}\right)}{(1+e^{-i\phi})\sqrt{\frac{1+ze^{-i\phi}}{i\phi}}} \tag{4.12}$$

We now use the fact that $z = -\cos\phi$ to write the above only in terms of $\phi$ and $\nu$:

$$\begin{aligned}
\Phi(\phi(x),\nu) =& \frac{3i\sin\phi e^{-i\phi} + i\phi\left(-(2+\cos\phi)e^{-i\phi} + 4\nu\sin\phi\, e^{-i\phi} - 2\nu^2(1-\cos\phi)e^{-i\phi}\right)}{2\cos\left(\frac{\phi}{2}\right)\sqrt{\frac{\sin\phi}{\phi}}e^{-i\phi}} \\
=& \frac{i\sqrt{\phi}}{2\cos\left(\frac{\phi}{2}\right)\sqrt{\sin\phi}}\left[3\sin\phi - \phi\left(2\nu^2(1-\cos\phi) - 4\nu\sin\phi + (2+\cos\phi)\right)\right] \\
=& \frac{i\sqrt{\phi}}{2\cos\left(\frac{\phi}{2}\right)\sqrt{\sin\phi}}\left[3\sin\phi - \phi\left(2\left(\nu\sqrt{1-\cos\phi} - \sqrt{1+\cos\phi}\right)^2 - \cos\phi\right)\right]
\end{aligned} \tag{4.13}$$

[6]with some tricks added in the mix

What we can immediately see is that $\Phi(\phi(x),\nu)$ is an imaginary quantity. By sending $\phi \to -\phi$ in the second line of (4.13), we find that it is too an imaginary quantity, and it is given by

$$\Phi(-\phi(x),\nu) = -\frac{i\sqrt{\phi}}{2\cos\left(\frac{\phi}{2}\right)\sqrt{\sin\phi}}\left[3\sin\phi+\phi\left(2\left(\nu\sqrt{1-\cos\phi}+\sqrt{1+\cos\phi}\right)^2-\cos\phi\right)\right] \tag{4.14}$$

One thing that the reader might have noticed is the fact that $c_0$, $c_1$ and $d_1$ all have a factor of $x^{-i\frac{\nu}{2}}$ in front of them, while $A_\pm$ has a factor of $x^{i\frac{\nu}{2}}$ We make the following re-definition of these coefficients to make them real:

$$\widetilde{c}_0 = -x^{i\frac{\nu}{2}}\frac{\phi}{2}c_0 \tag{4.15}$$

$$\widetilde{c}_1 = -x^{i\frac{\nu}{2}}\frac{\phi}{2i}c_1 \tag{4.16}$$

$$\widetilde{d}_1 = -x^{i\frac{\nu}{2}}d_1 \tag{4.17}$$

This turns (4.8) into

$$A_\pm(x,\nu,g^2) \approx (2g+\nu)\left[K_1(\pm g\phi(x))\left(\widetilde{c}_o+\frac{\widetilde{c}_1}{g}\right)+K_0(\pm g\phi(x))\frac{\widetilde{d}_1}{g}\right] \tag{4.18}$$

which is a much nicer form than (4.8) with real coefficients.

Now, from the exact function $F^j_\nu(x)$, we know from numerical calculations that it satisfies the property $F^j_{-\nu}(x) = \left(F^j_\nu(x)\right)^*$. Hence, $F^j_\nu(x)F^j_\nu(y) - F^j_{-\nu}(x)F^j_{-\nu}(y) \propto \text{Im}\left(F^j_\nu(x)F^j_\nu(y)\right)$. This means that the only way to get $\Omega(x,y,z,g^2)$ to be non-zero with our approximate $F^j_\nu(x)$ is if we approximate it up to the point that it gains an imaginary part. The same is true for either or both of $A_\pm$. Through (4.18), it is clear that the only way for (4.18) to have an imaginary part is through the Bessel functions. A look at (4.9) reveals that it is $K_0(-g\phi)$ and $K_1(-g\phi)$ that will contribute to this due to being imaginary.

We can now write:

$$\begin{aligned} F^j_\nu(x)F^j_\nu(y) \approx & A_+(x)A_+(y)+ \\ & \sum_{n=1}\left(e^{-2\pi g+\pi\nu}\right)^n\left(A_+(x)A_-(y)+A_+(y)A_-(x)+2A_+(x)A_+(y)\right) \\ \approx & \underbrace{A_+(x)A_+(y)}_{\text{real}}+\left(\underbrace{A_+(x)A_-(y)+A_+(y)A_-(x)}_{\text{imaginary}}\right)e^{-2\pi g+\pi\nu} \end{aligned} \tag{4.19}$$

This is the point at which our project has come to an end. What is left now is to do a systematic expansion in exponentials in $g$ and their polynomial corrections in order to obtain a non-zero $\Omega(x,y,z,g^2)$. The coefficients $\widetilde{c}_1$ and $\widetilde{d}_1$ provide more accuracy to the approximation.

## 5 Conclusions and outlook

We have made an asymptotic expansion of the hypergeometric function ${}_2F_1$ to obtain the coefficients $\widetilde{c}_1$ and $\widetilde{d}_1$. We have also made significant simplifications of their forms, as seen from (4.10), (4.11), (4.15), (4.16) and (4.17). These, in turn, allowed us to make important statements about the imaginary part of our approximation of $F_\nu^j(x)$, which in turn allowed us to figure out the order of approximation that is needed for a non-zero $\Omega(x, y, z, g^2)$. In the next stage, we will focus on the systematic expansion of (4.19) in terms of powers of $e^{-\pi g}$ and their subsequent polynomial corrections.

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