Phenomenological applications of the Weak Gravity Conjecture

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Abstract

In the present report the Weak Gravity Conjecture, its sharpened statements, and its application to phenomenology are discussed. In literature circular, spherical, and toroidal compactifications of the Standard Model were studied. We perform analitical calculation of the radion potential in S_1/Z_2 orbifold type of compactification for scalar lagrangian with kinetic and potential boundary terms that lead to non-trivial boundary conditions. Also we discuss calculation procedures in curved spacetime.

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1. Introduction

The weak gravity conjecture (WGC) is a bright example of how theory and phenomenology can yield interesting results when considered together. Firstly, the conjecture was first formulated in [1]. In its simplest form it states that for gauge field coupled to gravity, there exists a state of mass m and charge Q, satisfying the inequality

$$\frac{m}{M_{\rm Pl}} \leq |Q|, \tag{1}$$

where the exact equality has been yet found only for a SUSY theory with BPS states (state that preserve some of the supersymmetry turn out to saturate the relation). In other words, gravity is always the weakest force. There are several motivations for WGC but essentially they stem out of Gell-Mann principle *"Everything that is not forbidden is compulsory"* and the observation that no consistent compactification of string theory is known to violate the conjecture. One of the arguments is that black holes should be forced to decay after they get big enough, because otherwise there would be a large number of remnants in the Universe.

This statement was used to test effective theories and setting constraints (see [8-10] references in [2]). They can be imposed by a number of reasons, one of them is reading that gravitational correction to the running constants should be less than the contributions from other fields. Also, one gets untrivial result concerning the trusted cut-off scale in theories with monopoles, which takes the value of

$$\Lambda \sim g M_{\rm Pl}, \tag{2}$$

where g is a coupling constant. It presents the reason for the source of problems in the limit of $g \to 0$.

Recently, WGC was sharpened in [6], where authors speculated that the equality will only be reached in SUSY theories with BPS states and nowhere else. They support this conclusion by mentioning that equality, if present, should not be sensitive to small perturbations, which is not the case for a generic theory, except for SUSY. Moreover, the strict inequality in the relation has been obtained by a number of cases, which are enlisted in [6]. In addition, the authors proposed another conjecture, which states that "non-supersymmetric AdS is not renormalisable as a consistent quantum theory with low energy description in term of the Einstein gravity coupled to a finite number of matter fields". In practice it means that a stable AdS vacuum under compactification of some effective field theories (EFT) renders it inconsistent at UV scale.

Authors of [2] made use of the new conjecture and obtained constraints on the masses of light particles in the SM via compactifying 1 and 2 spacelike dimensions. The research tool is the radion potential. If it exhibits an AdS minimum for some SM configuration in any type of compactification, it means that one can rule out this configuration as inconsistent with UV completion. The obtained results are very interesting as they contain two important statements. Firstly, neutrinos with Majorana masses are excluded by the observed value of cosmological constant

$$\Lambda_0 \approx 3.25 \times 10^{-11}\ (\mathrm{eV})^4$$

(with assumption that no other light particles are present in spectrum). Therefore, a potential measurement of ν-less double β decay would imply additional BSM light particles. The other statement is that the mere presence of neutrinos give support for a non-vanishing cosmological constant. This is the first particle physics argument implying a non-zero value of Λ without any cosmological input.

In the present report we are intent to discuss how AdS vacua might appear in compactified SM, techniques that one might use to calculate effective potentials, and then we will consider the case of orbifold compactification with localized kinetic and potential boundary terms. The calculation of AdS space Casimir energy is presented in Appendix as an example of dealing with curved spacetime.

1.1. Compactification on a circle

Papers [2, 5] consider 1D compactification on S_1. The 4D metric reads

$$ds^2 = \frac{r^2}{R^2} dx_\mu^2 + R^2 d\varphi^2, \tag{3}$$

where r is some constant, R is the radius of compactification, and $\varphi \in [0, 2\pi)$ is the 4th coordinate. We consider the mostly positive type of metric throughout this report. The 3D action then yields the effective potential, associated with R "field", which is usually referred to as "radion", consists of two terms – cosmological constant contribution and the overall Casimir energy. The potential reads as

$$V(R) = \frac{2\pi r^3 \Lambda}{R^2} - \sum_{\text{all particles}} (2\pi R) \frac{r^3}{R^3} (-1)^{s_i} n_i \rho_i(R), \tag{4}$$

where $s_i = 0(1)$ for bosons and fermions respectively, n_i denotes the number of degrees of freedom and

$$\rho_i(R) = \sum_{j=1}^{\infty} \frac{2m_i^4}{(2\pi)^2} \frac{K_2(2\pi R m_i j)}{(2\pi R m_i j)^2} \xrightarrow{m_i R \ll 1}$$
$$\xrightarrow{m_i R \ll 1} \left[\frac{\pi^2}{90(2\pi R)^4} - \frac{\pi^2}{6(2\pi R)^4}(m_i R)^2 + \frac{\pi^2}{48(2\pi R)^4}(m_i R)^4 + \mathcal{O}(m_i R)^6 \right],$$

where m_i is the mass of the i$^{\text{th}}$ particle. This expression can be derived by a variety of techniques, one of them being Green function method, which is used by [5], seeks for the periodic Green function and then acts on it by the energy-stress operator.

The authors of [2] applied the radion potential (4) to SM and seeked for a $V(R) < 0$ minimum. If there is one, the parameters are excluded by the WGC. Depending on the assumed hierarchy structure and the value of the mass of the lightest neutrino flavor, they obtained potentials such as the one in Figure 1.1.

The radion potential (4) with only free massless particles (photon and graviton) has a maximum at the point

$$R_{max} = \left(\frac{1}{120\pi^2\Lambda}\right)^{1/4} = 7.55 \cdot 10^{10}\ \text{GeV}^{-1}\,. \tag{5}$$

The associated mass scale here is

$$M = \frac{1}{2\pi R_{max}} = 2.11 \cdot 10^{-3}\ \text{eV}, \tag{6}$$

which, taking into account the exponential suppression of the Casimir contribution e^{-mR}, means that only neutrinos and other light unobserved particles might give any sugnificant contribution to the radion potential.

The Casimir contribution is exponentially suppressed by the factor of e^{-mR}, and taken the value of Λ, only particles roughly of order 10^{-3}eV might influence the behaviour of the radion potential [2]. The contribution of electron and other particles can be totally neglected.

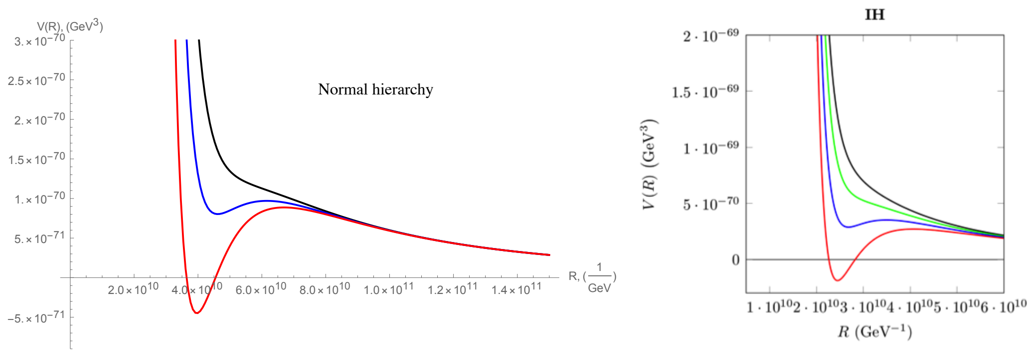

Figure 1: Radion potential for different neutrino hierarchies and masses. On the left side we reproduced the plot for normal hierarchy (6-8 eV from black to red). On the right side the plot for inverted hierarchy (1.5-3.0 eV from black to red) is taken from [2]

.

2. Orbifold compactification

Another way of compactifying 4D manifold is on a so-called orbifold, which is basically an interval, or topologically S_1/Z_2. This type of finite dimension has advantages over circle in 5D spacetime as it solves the chirality problem as mentioned in [8]. In addition, on this interval one is able to impose different boundary conditions from that on a circle.

2.1. Classical contribution

The action compactified on S_1 has no Casimir energy contribution at the tree level. It is not, however, obvious, that this is the case for any other compactification. Here we will show, that indeed for orbifold case it is also true.

We consider a generic scalar lagrangian in 4D spacetime with local kinetic and potential boundary terms at one side in curved spacetime.

$$S = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left((\partial_M\phi)^2 + m^2\phi^2\right) + \widetilde{S}, \tag{7}$$

$$\widetilde{S} = -\frac{1}{2}\int d^3x\,\sqrt{-g}\cdot\frac{1}{\mu}\left(a_1\mu^2\phi^2 + b_1 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \tag{8}$$

with μ as mass-dimension parameter and a_1, b_1 as some numerical constants. The equation of motion reads

$$(\Box - m^2)\phi - 2\mu a_1\delta(y)\phi + 2\frac{b_1}{\mu}\delta(y)\Box_3\phi = 0, \tag{9}$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)$ is a D'Alambertian operator, and $\Box_3$ is the same but with summation indices running in 3D space.

Thus, if we generally assume Neuman boundary condition of the outer side of the brane, the delta terms transfer it to

$$\partial_y\phi - \mu a_1\phi + \frac{b_1}{\mu}\Box_3\phi = 0, \tag{10}$$

which is easily obtained by integrating (9) in the vicinity of the boundary.

Partial integration of the action leads to

$$S = \frac{1}{2}\int d^3x\,\sqrt{-g}\,\phi\partial_y\phi\big|_{y=0} - \frac{1}{2\mu}\int d^3x\,\sqrt{-g}\,\left(a_1\mu^2\phi^2 - b_1\phi\cdot\Box_3\phi\right)\big|_{y=0}, \tag{11}$$

which is equal to zero regardless of whether Neuman or Dirichlet boundary conditions are imposed on the outer surface of the boundary. The same is true for the local boundary term at the other end on the orbifold.

2.2. Radion potential

To apply the WGC sharpened hypothesis one should construct the radion potential in spacetime of our choice and take a look at its minima. Since cosmological constant has a vanishing value, we describe our 4D spacetime as flat.

The metric takes the form of (3), which is the Weyl rescaled metric of

$$ds^2 = dx_\mu^2 + \gamma^2\cdot dy^2, \tag{12}$$

where $y \in [0, L]$ represents the finite dimension ($L = \text{const}$) and γ is a scaling parameter. The reason for one to rescale it is because after integrating our the 4th coordinate, one obtains

$$S_{grav} = \int d^3x\, dy\, \sqrt{-g}\mathcal{R} = \int d^3x \sqrt{-g_3}\gamma\mathcal{R}, \tag{13}$$

where g_3 indicates the 3D metric and R is the Ricci scalar. To make the gravitational term look canonical, one can implement conformal transformations [14]:

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{14}$$

$$\mathcal{R} \to \mathcal{R}' = \Omega^{-2}\left(\mathcal{R} + \frac{\Box\Omega}{\Omega}\right). \tag{15}$$

The case of $\Omega^2 = \gamma^{-1}$ leads to the canonical form. One gets the metric

$$ds^2 = \frac{1}{\gamma}dx_\mu^2 + \gamma dy^2. \tag{16}$$

It is straightforward to calculate the lambda term contribution to the radion potential now

$$-\int dy\, \sqrt{-g}\,(-\Lambda) = \frac{\Lambda L}{\gamma} \tag{17}$$

2.3. Casimir energy

Another contribution to the radion potential is due to light particles in the theory. In SM there are 2 massless free bosons (photon and graviton), each of them having 2 degrees of freedom; and three flavours of neutrino. Their Casimir contribution is highly dependent on the exact mass values, as shown in [2]. In this section we derive Casimir energy for scalar fields with special boundary conditions. We will calculate the effective action, which can be easily translated to effective potential by sign flipping

$$\Gamma_{eff} = -V_{eff}. \tag{18}$$

Casimir energy is a 1-loop effect in our spacetime. In pursue of 1-loop effective action one follows Peskin & Shreder textbook

$$Z[J] = e^{-iE[J]} = \int \mathcal{D}\phi \exp\left[i\int d^4x(\mathcal{L}[\phi] + J\phi)\right] \tag{19}$$

$$\Gamma[\phi_{cl}] \equiv -E[J] - \int d^4y\ J(y)\phi_{cl}(y) \tag{20}$$

To the quadratic order one finds that

$$\Gamma[\phi_{cl}] = \int d^4x\mathcal{L}[\phi_{cl}] + \frac{i}{2}\log\det\left[-\frac{\delta^2\mathcal{L}}{\delta\phi\delta\phi}\right] \tag{21}$$

We consider 4D space with metric (3) with local boundary terms

$$S = -\frac{1}{2}\int d^3x\,dy\,\left((\partial_\mu\phi)^2 + \gamma^{-2}(\partial_y\phi)^2 + m^2\gamma^{-1}\phi^2\right) + \widetilde{S}, \tag{22}$$

$$\widetilde{S} = -\int d^3x\,dy\,\cdot\frac{\delta(y)}{\mu}\gamma^{-1/2}\left(a_1\mu^2\gamma^{-1}\phi^2 + b_1\partial_\mu\phi\partial_\nu\phi\right) - \tag{23}$$

$$-\int d^3x\,dy\,\cdot\frac{\delta(y-L)}{\mu}\gamma^{-1/2}\left(a_2\mu^2\gamma^{-1}\phi^2 + b_2\partial_\mu\phi\partial_\nu\phi\right), \tag{24}$$

where μ is a mass-dimension parameter and a_i, b_i are numerical constants. The equations of motion and boundary conditions read

$$\begin{cases} (\partial_\mu^2 + \gamma^{-2}\partial_y^2 - \gamma^{-1}m^2)\phi = 0, \\ \partial_y\phi - \frac{1}{\mu}\gamma^{-1/2}\left[a_1\mu^2\gamma^{-1}\phi - \gamma^{-2}b_1(m^2\gamma - \partial_y^2)\,\phi\right]\Big|_{y=0} = 0, \\ \partial_y\phi + \frac{1}{\mu}\gamma^{-1/2}\left[a_2\mu^2\gamma^{-1}\phi - \gamma^{-2}b_2(m^2\gamma - \partial_y^2)\,\phi\right]\Big|_{y=L} = 0; \end{cases} \tag{25}$$

where boundary terms on the plane $y = L$ are introduced analogously (with the same sign), and we have employed the equation of motion for $\partial_\mu^2\phi$ in boundary conditions. Introducing variable substitutions such as

$$M^2 = \gamma m^2; \quad \tilde{\mu}^2 = \gamma\mu^2; \quad A_i = a_i\gamma^{-2}; \quad B_i = b_i\gamma^{-2}, \tag{26}$$

one makes the equations look more neat.

In flat space general solution can be decomposed as follows:

$$\phi(x^\mu, y) = \int \frac{d^3p}{(2\pi)^3}\,e^{ip_\mu x^\mu}\varphi(p_\mu)\cdot\sum_n(A\sin(\lambda_n y) + B\cos(\lambda_n y)), \tag{27}$$

where λ_n satisfy the boundary conditions in (25).

One loop effective action then reads

$$\Gamma_{1-loop} = \frac{i}{2}\int\frac{dp_0}{2\pi}\int\frac{d^2p_i}{(2\pi)^2}\sum_n \log\left[p_\mu^2 + \gamma^{-2}\left(\lambda_n^2 + M^2\right)\right], \tag{28}$$

Firstly, we perform Wick rotation, integrate out p_0, and to ourselves to dimensional regularization $\epsilon \to 0$. One obtains

$$-\frac{1}{2}\int\frac{d^{2-\epsilon}p}{(2\pi)^{2-\epsilon}}\sum_n\sqrt{p_i^2 + \gamma^{-2}\left(\lambda_n^2 + M^2\right)} \tag{29}$$

For convenience we denote $\Lambda_n^2 = M^2 + \lambda^2$. The integral over p can be carried out in the following manner:

$$-\frac{1}{2}\int\frac{d^{2-\epsilon}p}{(2\pi)^{2-\epsilon}}\sqrt{\gamma^{-2}\Lambda_n^2 + p^2} = -\frac{2^{\epsilon-3}\pi^{\frac{\epsilon-2}{2}}}{\Gamma\left(\frac{2-\epsilon}{2}\right)}\cdot\int_0^\infty p^{1-\epsilon}\cdot(p^2 + \gamma^{-2}\Lambda_n^2)^{1/2}dp = \tag{30}$$

$$-\frac{2^{\epsilon-2}\pi^{\frac{\epsilon-2}{2}}}{\Gamma\left(\frac{2-\epsilon}{2}\right)}\gamma^{-3+\epsilon}\Lambda_n^{3-\epsilon}\cdot\int_0^\infty p^{-\epsilon/2}(p+1)^{1/2}dp = 2^{\epsilon-4}\pi^{\frac{\epsilon-3}{2}}\cdot\Gamma\left(\frac{\epsilon-3}{2}\right)\gamma^{-3+\epsilon}\Lambda_n^{3-\epsilon}, \tag{31}$$

where the integral was performed using beta-function properties. Thus, one-loop effective action is reduced to the sum over Λ_n

$$\Gamma_{1-loop} = 2^{\epsilon-4}\pi^{\frac{\epsilon-3}{2}}\gamma^{-3+\epsilon}\cdot\Gamma\left(\frac{\epsilon-3}{2}\right)\cdot\sum_n \Lambda_n^{3-\epsilon}. \tag{32}$$

Before we start dealing with the sum let us write the eigenvalue problem for λ_n explicitly. Boundary conditions (25) with substitution (27) lead to

$$\tan\lambda L = -\frac{\lambda(C_2+C_1)}{\tilde{\mu}\lambda^2 - C_1C_2}, \quad \text{where} \quad C_i = B_i(\lambda^2+M^2) - A_i\tilde{\mu}^2. \tag{33}$$

In other words, Λ_n are defined as zeros of the function

$$F(\Lambda) = \left[-C_1C_2 + \tilde{\mu}(\Lambda^2 - M^2)\right]\cdot\tan\left(\sqrt{\Lambda^2-M^2}L\right) + (C_1+C_2)\cdot\sqrt{\Lambda^2-M^2}, \tag{34}$$

where $C_i = B_i\Lambda^2 - A_i\tilde{\mu}^2$.

We apply analytic continuation to calculate the sum in (32), for which we define $s = \epsilon - 3$ but initially calculate the sum for positive values of s, and then take the limit $s \to -3$. For sufficiently large s the sum can be cast into form

$$\sum_n \Lambda_n^{-s} - \sum_n Z_n^{-s} = \frac{1}{2\pi i}\int_C dz\, z^{-s}\frac{F'(z)}{F(z)} = \frac{s}{2\pi i}\int_C dz\, z^{-s-1}\log\left[F(z)\right], \tag{35}$$

where

$$Z_n = \sqrt{M^2 + \left(\frac{\pi}{L}\right)^2\left(n+\frac{1}{2}\right)^2} \tag{36}$$

are the poles of $F(z)$; $\mathcal{C}$ contour goes along Re z axis and circumscribes all poles and zeros of $F(z)$, which are more than M, in counterclockwise manner. More precisely,

$$\mathcal{C} = (\infty + i\delta, M + i\delta)\cup(M+i\delta, M-i\delta)\cup(M-i\delta, \infty - i\delta), \tag{37}$$

where the middle interval is in the form of a semicircle, placed on the right of the point $z = M$.

To deal with the integral in (35) we perform analytical continuation and transform the contour into a more convenient form. According to the residual theorem it is possible if the contour transformation does not change the poles that the contour circumscribes. In principle, one should prove this fact precisely but since we do not expect poles to exist anywhere except for real axis, we just present a plot on the complex plane for some arbitrary values of constants of the order of one.

We tried different constant values but poles and zeros are always of the real axis. For Fig.(2.3),(2.3) we took $M = 5.0,\ \tilde{\mu} = 1.0,\ A_1 = 0.9,\ A_2 = 0.2,\ B_1 = 0.5,\ B_2 = 0.9,\ L = 1.0$.

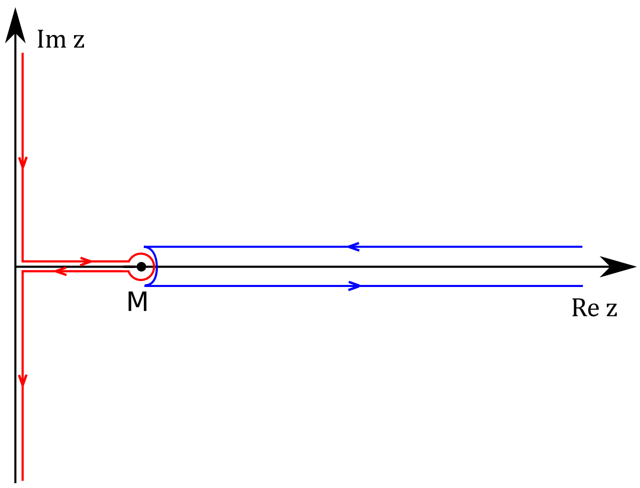

Figure 2: The initial (blue) integration contour gets transformed (red).

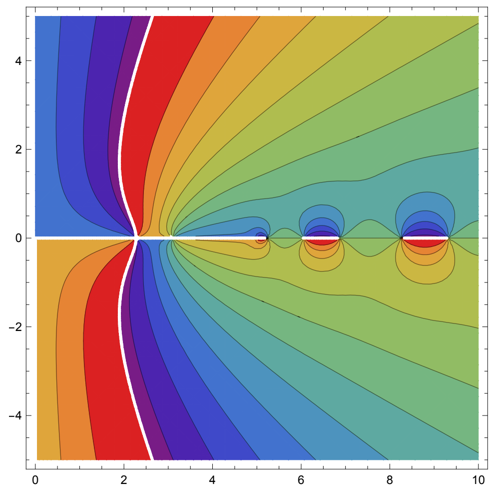

Figure 3: Arg $F(z)$ is shown by means of color regions. Red and violet regions correspond to π and $-\pi$ respectively. Poles and zeros are nodes for the contour lines.

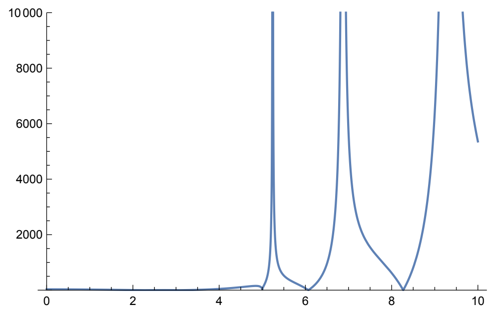

Figure 4: The white region at the Arg $F(z)$ plot between points $z = 0, z = M$ does not have poles, as shown here.

Thus, we transform the contour into the sum of two separate ones. The first one takes the form of a straight line and runs down along the imaginary axis, whereas the other one is a circular counterclockwise contour around $z = M$. The rest two integrals between the points $z = 0$ and $z = M$ cancel each other. The function (34) for $\Lambda = ix$

transfers to

$$
\begin{aligned}
i \cdot F(ix) \equiv \tilde{F}(x) = \\
&= \left[(B_1x^2 + A_1\tilde{\mu}^2)(B_2x^2 + A_2\tilde{\mu}^2) + \tilde{\mu}(x^2+M^2)\right] \cdot \tanh\left(\sqrt{x^2+M^2}L\right) \\
&\quad + \left[(B_2+B_1)x^2 + (A_2+A_1)\tilde{\mu}^2\right] \cdot \sqrt{x^2+M^2}. \qquad (38)
\end{aligned}
$$

The integral over imaginary axis is simplified via variable substitution $z = ix$:

$$
\frac{s}{2\pi i}\int\limits_{i\infty}^{-i\infty} dz\, z^{-s-1}\log\left[F(z)\right] = \frac{s}{2\pi i}\int\limits_{\infty}^{\infty} dx\, i^{-s}x^{-s-1}\log\left[F(ix)\right] = \qquad (39)
$$

$$
= \frac{s}{2\pi i}\int\limits_{\infty}^{0} dx\, i^{-s}x^{-s-1}\log\left[\tilde{F}(x)\right] + \frac{s}{2\pi i}\int\limits_{0}^{\infty} dx\, i^{s}x^{-s-1}\log\left[\tilde{F}(x)\right] = \qquad (40)
$$

$$
= \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\cdot\int\limits_{0}^{\infty} dx\, x^{-s-1}\log\left[\tilde{F}(x)\right]. \qquad (41)
$$

The only problem is that $\tilde{F}(x) \not\to 1$ when $x \to \infty$. To solve it we add and subtract the asymptotics of $\tilde{F}(x)$ which we denote as $F_0(x)$:

$$
\begin{aligned}
F_0(x) = &\left[(B_1x^2 + A_1\tilde{\mu}^2)(B_2x^2 + A_2\tilde{\mu}^2) + \tilde{\mu}(x^2+M^2)\right] \\
&+ \left[(B_2+B_1)x^2 + (A_2+A_1)\tilde{\mu}^2\right] \cdot \sqrt{x^2+M^2} \qquad (42)
\end{aligned}
$$

Thus, apart from the convergent integral, we also have two divergent integrals: over $\mathrm{Im}\, z$ and around the point $z = M$. The last one vanishes when $s < 0$ because

$$
\lim_{\delta\to 0}\delta^{-s}\log\delta = 0. \qquad (43)
$$

For the same reason the integral

$$
\int\limits_{\delta}^{\infty} dx\, x^{-s-1}\log\left[F_0(x)\right] = \frac{1}{s}\int\limits_{\delta}^{\infty} dx\, x^{-s}\frac{F_0'(x)}{F_0(x)} \qquad (44)
$$

diverges as δ to some power which vanishes then in the limit $s \to -3$.

Overall, the sum in (32) becomes

$$
\sum_n \Lambda_n^{-s} = \sum_n Z_n^{-s} + \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\cdot\int\limits_{0}^{\infty} dx\, x^{-s-1}\log\left[\frac{\tilde{F}(x)}{F_0(x)}\right]. \qquad (45)
$$

One can take the sum (36) in the limit $ML \ll 1$. Expanding the expression in Taylor series

$$\left(\sqrt{1+x}\right)^{-s} = 1 - \frac{sx}{2} + \frac{1}{8}(2s+s^2)x^2 - \frac{1}{48}(8s+6s^2+s^3)x^3 + \\ + \frac{1}{384}(48s+44s^2+12s^3+s^4)x^4 + \mathcal{O}(x^5) \quad (46)$$

and using the identity $\sum\limits_{n=0}^{\infty}(n+1/2)^{-s} = (2^s-1)\zeta(s)$, one obtains for the case $s=-3$:

$$\sum_n Z_n^3 = \left(\frac{\pi}{L}\right)^3 \cdot \left[-\frac{7}{960} + \frac{1}{16}\frac{(ML)^2}{\pi^2} + \frac{3}{8}\tilde{\gamma}\frac{(ML)^4}{\pi^4} - \frac{7\zeta(3)}{16} \cdot \frac{(ML)^6}{\pi^6} + \mathcal{O}(ML)^8 \right], \quad (47)$$

where $\tilde{\gamma}$ is Euler-Mascheroni constant and we omited the divergent term $\frac{1}{s+3}\frac{(ML)^4}{\pi^4}$. The divergence of this term is not unexpected, as constant vacuum density always appears to be infinite.

Overall, the effective action has the form of (32), where $s \to -3$ limit is assumed and the sum over Λ is defined by (42), (38), (45), and (47).

3. Conclusions

In this report we presented the ideas of the Weak Gravity Conjecture and what has been done recently concerning its phenomenological applications. We took another compactification manifold which we believe might give better constraints on Standard Model parameters due to its localized boundary terms. The analytical calculations were performed. Although the phenomenological application of these results has not been yet discussed, the present calculation represents a ready setup.

Fermion case should also be studied in future, since its Casimir energy might behave differently from that of scalar contribution with opposite sign.

Appendices

A. AdS-space Casimir energy.

Although compactification of AdS space is not directly related to WGC, it is quite important in modern theoretical applications. For this reason we find description of this case relevant anyway.

Here we consider the case of scalar field action without local terms with Dirichle boundary conditions. We mostly follow [4], where the case of 5D space was assumed.

The action reads

$$S = \int d^4x\sqrt{-g}\ \frac{1}{2}\left[-g^{MN}\partial_M\phi\partial_N\phi - m^2\phi^2\right]. \tag{48}$$

The metric in this space can be cast into the form

$$g_{\mu\nu} = e^{-2ky}\eta_{\mu\nu}dx^\mu dx_\nu + dy^2, \qquad \eta_{\mu\nu} = \mathrm{diag}\{-,+,+,+\}, \tag{49}$$

where $y \in [0, \pi R]$ is the x_3 coordinate. The 1-loop contribution to the effective action looks like

$$\Gamma_{1-loop} = \frac{i}{2}\log\det\left[\sqrt{-g}(-\Box + m^2)\right], \tag{50}$$

where $\Box$ operator is defined by

$$\Box = \frac{1}{\sqrt{-g}}\partial_M(\sqrt{-g}g^{MN}\partial_N) = e^{2ky}\partial_\mu^2 + \partial_y^2 - 3k\partial_y. \tag{51}$$

It is straightforward to check that the equation

$$(\partial_y^2 - 3k\partial_y + M^2e^{2ky} - m^2)f(y) = 0 \tag{52}$$

has solution of the form

$$f(y) = e^{\frac{3}{2}ky}\left[A\cdot J_\nu\left(\frac{M\cdot e^{ky}}{k}\right) + B\cdot Y_\nu\left(\frac{M\cdot e^{ky}}{k}\right)\right], \tag{53}$$

where $\nu = \sqrt{\frac{9}{4} + \frac{m^2}{k^2}}$ and coefficients A and B are defined by normalisation and boundary conditions.

Thus, to solve the eigenvalue problem (50) we perform a decomposition:

$$\phi = \varphi(x^\mu)\cdot f(y) \tag{54}$$

and obtain

$$\sqrt{-g}(-\Box + m^2)\phi = e^{-3ky+2ky}(M_N^2 + p_\mu^2)\phi, \tag{55}$$

where M_n are effective masses that appear due to the bulk action, and $p_\mu^2 = \eta_{\mu\nu}p^\mu p^\nu$.

Overall, the (50) transforms to

$$\Gamma_{1-loop} = \frac{i}{2}\int\frac{dp_0}{2\pi}\int\frac{d^2p_i}{(2\pi)^2}\sum_n[-ky + \log(M_n^2 + p_\mu^2)] \tag{56}$$

Wick rotation, dimensional regularization, and integration over spatial impulses in the same manner as in the section 2.3 leads to the result

$$\Gamma_{1-loop} = 2^{\epsilon-4}\pi^{\frac{\epsilon-3}{2}}\cdot\Gamma\left(\frac{\epsilon-3}{2}\right)\cdot\sum_n M_n^{3-\epsilon}. \tag{57}$$

Recall that M_n are defined by the Dirichlet boundary condition:

$$J_\nu\left(\frac{M_n}{k}\right)\cdot Y_\nu\left(\frac{M_n e^{\pi kR}}{k}\right)-Y_\nu\left(\frac{M_n}{k}\right)\cdot J_\nu\left(\frac{M_n e^{\pi kR}}{k}\right)=0. \tag{58}$$

For convenience, we denote $a=e^{-\pi kR}, x_n=M_n/(ka)$. Then, it takes the form

$$F(x_n)\equiv J_\nu(x_n)\cdot Y_\nu(ax_n)-Y_\nu(x_n)\cdot J_\nu(ax_n)=0. \tag{59}$$

We compute the sum via complex integral methods:

$$\sum_n x_n^{-s}=\frac{1}{2\pi i}\int\limits_C dz\, z^{-s}\frac{F'(z)}{F(z)}=\frac{s}{2\pi i}\int\limits_C dz\, z^{-s-1}\log\left[J_\nu(z)\cdot Y_\nu(az)-Y_\nu(z)\cdot J_\nu(az)\right], \tag{60}$$

where C contour goes along $z\in\mathbb{R}, z>0$ axis and circumscribes all the positive $F(z)$ poles (except for possible pole at $z=0$). The assymtotics of the function under logarithm is the following:

$$J_\nu(z)\cdot Y_\nu(az)-Y_\nu(z)\cdot J_\nu(az)\xrightarrow{z\to\infty}\frac{2}{\pi z\sqrt{a}}\sin\left((a-1)z\right)=\frac{-i}{\pi z\sqrt{a}}\left(e^{i(a-1)z}-e^{-i(a-1)z}\right) \tag{61}$$

Contour C consists of two parts C_+ and C_- which run parallel to the $\mathrm{Re}(z)$ axis in the 1st and in the 4th quadrant respectively. For them the asymptotics read

$$\mp\frac{i}{\pi z\sqrt{a}}e^{\pm i(a-1)z}. \tag{62}$$

Now we add and subtract the assymptotics from the expression for the sum

$$\sum_n x_n^{-s}=\frac{s}{2\pi i}\int\limits_{C_\pm} dz\, z^{-s-1}\log\left[\pm i\pi z\sqrt{a}\cdot e^{\pm iz(1-a)F(z)}\right]-\frac{s}{2\pi i}\int\limits_{C_\pm} dz\, z^{-s-1}\log\left[\pm i\pi z\sqrt{a}\cdot e^{\pm iz(1-a)}\right]. \tag{63}$$

The last integral has the only pole $z=0$ which is not present in the integration contour. Thus, it is possible to integrate along the imaginary axis from $z=i\infty+\delta$ to $z=i\infty+\delta$.

Let us take the second integral. We divide the logarithm into sum of two. One of the terms read

$$\int\limits_{i\infty+\delta}^{\delta} dz\, z^{-s-1}iz(1-a)-\int\limits_{\delta}^{\delta-i\infty} dz\, z^{-s-1}iz(1-a)=\frac{2i(1-a)}{(1-s)}\cdot\delta^{1-s}, \tag{64}$$

while the other term has the form

$$\int_{i\infty+\delta}^{\delta} dz\, z^{-s-1} \log\left(i\pi z\sqrt{a}\right) + \int_{\delta}^{-i\infty+\delta} dz\, z^{-s-1} \log\left(-i\pi z\sqrt{a}\right) = \quad (65)$$

$$= \frac{1}{-s}\ z^{-s} \log\left(i\pi z\sqrt{a}\right)\Big|_{i\infty+\delta}^{\delta} + \frac{1}{-s}\ z^{-s} \log\left(-i\pi z\sqrt{a}\right)\Big|_{\delta}^{-i\infty+\delta} + \frac{1}{s}\ z^{-s}\cdot\frac{1}{z}\Big|_{i\infty+\delta}^{-i\infty+\delta} = \quad (66)$$

$$= -\frac{1}{s}\,\delta^{-s}\log(-1) = -\frac{i\pi}{s}\,\delta^{-s} \quad (67)$$

Overall, the asymptotics term contribution to the sum is

$$-\frac{s}{\pi}\cdot\left[\frac{(1-a)}{(1-s)}\,\delta^{1-s} - \frac{\pi}{2s}\delta^{-s}\right]. \quad (68)$$

This contribution is negligent in the limit of our interest ($s \to -3$). Thus, the sum (63) has the form

$$\sum x_n^{-s} = \frac{s}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_0^{\infty} dt\, t^{-s-1}\log\left[2t\sqrt{a}\,e^{-t(1-a)}\left(K_\nu(t)\cdot I_\nu(at) - K_\nu(at)\cdot I_\nu(t)\right)\right], \quad (69)$$

where we used

$$J_\nu(iz) = i^\nu I_\nu(z), \quad (70)$$

$$Y_\nu(iz) = i^{\nu+1}I_\nu(z) - \frac{2}{\pi}i^{-\nu}K_\nu(z), \quad (71)$$

$$\sin\frac{\pi s}{2} = (-i)^{-s} - (-i)^s. \quad (72)$$

The sum is still ill-defined for $s = -3+\epsilon$. Thus, we transform it as follows:

$$\sum x_n^{-s} = \frac{-3}{\pi}\left\{\int_0^{\infty} dt\, t^{2+\epsilon}\log\left[1 - \frac{K_\nu(t)\cdot I_\nu(at)}{K_\nu(at)\cdot I_\nu(t)}\right]\right. \quad (73)$$

$$\left.+\int_0^{\infty} dt\, t^{2+\epsilon}\log\left[\sqrt{\frac{2}{t}}e^{-t}I_\nu(t)\right] + \frac{1}{a^{3-\epsilon}}\int_0^{\infty} dt\, t^{2+\epsilon}\log\left[-\sqrt{\frac{2}{t}}e^{t}K_\nu(t)\right]\right\}, \quad (74)$$

Going back to (30) we get

$$\Gamma_{1-loop} = \frac{2}{3\pi}(ka)^{3-\epsilon}\cdot\sum_n x_n^{3-\epsilon} = \Gamma_1 + \Gamma_2\,a^3 - \frac{2(ka)^3}{\pi^2}\int_0^{\infty} dt\, t^2\log\left[1 - \frac{K_\nu(t)\cdot I_\nu(at)}{K_\nu(at)\cdot I_\nu(t)}\right], \quad (75)$$

where Γ_1 and Γ_2 are not dependent on a. The numerical value of the integral appears to be negative.

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