Resummation in Higgs Production with Soft Collinear Effective Field Theory

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Abstract

Big logarithms invalidate perturbation theory in QCD in the low $p_\perp$ spectrum. These arise because the physical process involve different energy scales and is usually solved by effective theories, factorization and resummation. However rapidity logarithms appear in the corners of phase space that also need resummation. This report deals with resumming the cross-section for the low $p_\perp$ spectrum of Higgs production up to next to leading logarithm through the use of renormalization group equations in the formalism of Soft Collinear Effective Field Theory.

Contents

This Report

This report is a short summary of my work at the DESY Summer School 2014. The main topic of my project deals with RGEs in SCET which is presented below. Much of my time was also dedicated to studying the necessary background like QCD, factorization, resummation and SCET. However to present everything I learned would be quite hard and without a real meaning and therefore this report deals with mainly some of the calculations I have checked and done.

I would like to thank DESY for the opportunity to come here and also my supervisors Frank Tackmann and Maximilian Stahlhofen for helping out in the times of confusion!

1. Introduction

Some observables in QCD involve large logarithms that makes perturbation theory lose its predictive power. These large logarithms can arise when the physical process involves physics on different scales which results in logarithms of large or small energy ratios. To deal with this problem one factorizes the scattering process into several factors which each account for the physics at a specific scale.

Resummation refers to the process of summing up all the relevant logarithmic terms in perturbation theory. This can be done with renormalization group techniques, where one solves a corresponding Renormalization Group Equation(RGE) for each factor in the cross section. However large logarithms does not always come from different fixed energy scales, such as masses. They can also come from corners in phase space involving ratios of momenta. These are called rapidity divergences and must also be resummed.

To separate the energy scales one need an effective field theory to integrate out the irrelevant degrees of freedom. Soft Collinear Effective Theory(SCET) is designed for processes where the hard scale is much greater than the hadronic scale. It also accounts for emissions of soft and collinear gluons.

In [1] the authors showed how one can do the resummation for Higgs production in SCET with a "Rapidity Renormalization Group Equation"(RRGE). This project deals with how one could extend this to next to leading order(NLL). I will also present some numerical calculations for the cross section of Higgs production to see how sensitive the computation is on energy scale parameters..

In this report I will first in part 2 give a glimpse of the relevant parts of SCET and present the relevant RGEs. After that I will show how these are solved to do resummation in part 3. The evolution of the factorized cross-section is explained in part 4 and the results are presented in part 4.3. Some formulas and gathered up in appendix A and the necessary details about plus distributions that was used to solve the RRGEs are presented in appendix B.

2. Higgs production in Soft Collinear Effective Field Theory

For an effective field theory to correctly describe the infrared physics of QCD, it needs to include the emissions of soft and collinear gluons. This is done in the formalism of SCET [2], more specifically SCET_{II} for Higgs production. The characteristic low energy scale is $\lambda = \frac{p_T}{m_h}$, but it is a little bit more complicated since the momenta of the on-shell constituents of the process has two scales, λ and λ^2.

To get a physical picture of the process it is useful to look at the constituents characteristic momenta. For low $p_\perp$ there is a natural preferred axis which makes light cone coordinates particularly useful. In light cone coordinates the different momenta scale as

$$\begin{aligned} p_h &\sim m_h(1, 1, \lambda) \\ p_n &\sim m_h(1, \lambda^2, \lambda) \\ p_{\bar{n}} &\sim m_h(\lambda^2, 1, \lambda) \\ p_s &\sim m_h(\lambda, \lambda, \lambda) \end{aligned}$$

Where p_n and $p_{\bar{n}}$ stands for momenta along the x_+ and x_- lightcone directions. And p_s is the momenta of soft radiation.

The rapidity divergences are not traditional UV or IR divergences, instead they manifests themselves in ratios of momenta, like $\frac{p_+}{p_-}$[1]. The phase space can be split up through factorization, which separates the radiation with different momenta scaling. The factorization of the cross section can be sketched up as

$$\sigma \sim H(m_h) S(\vec{p}_s) f_{g/P}(\vec{p}_1) f_{g/P}(\vec{p}_2). \tag{1}$$

The hard function H is the matrix element of the scattering at the scale $\sim m_h$ and the soft function S takes care of the soft radiation. The $f_{g/P}$ are called transverse momentum dependent parton distribution functions or simply beam functions.

Traditionally the resummation can be done through a RGE procedure where one renormalize the specific factors in the cross section at a specific scale. This gives rise to a differential equation that can be solved for each function. The solution can then be used to “run” the functions between energy scales, which corresponds to summing all the big logarithms into a prefactor.

2.1. Renormalization Group Equations

To deal with the rapidity divergences, one can introduce a dimensionful parameter ν that acts as a momentum cutoff for the beam and soft functions. This introduces an differential equation that acts as a rapidity renormalization group equation.

[1] Rapidity can be defined as $y = \frac{1}{2} \log \frac{p_+}{p_-}$.

In addition to the rapidity renormalization, there is the usual renormalization with the momentum parameter μ. The bare functions are independent of both μ and ν,

$$\frac{d}{d\log\mu}\{S, f_{g/P}\} = \frac{d}{d\log\nu}\{S, f_{g/P}\} = 0. \tag{2}$$

In our case the renormalized beam and soft functions can be expressed as convolutions with some renormalization factors as

$$f^{R}_{g/P}{}^{\mu\nu}(z,\vec{p}_i,\mu,\nu) = Z^f(\vec{p},\mu,\nu)^{-1} \otimes_\perp f^{B}_{g/P}{}^{\mu\nu}(z,\vec{p}_i), \tag{3}$$
$$S^R(\vec{p}_s,\mu,\nu) = Z^S(\vec{p},\mu,\nu)^{-1} \otimes_\perp S(\vec{p}_s). \tag{4}$$

Where the convolution is defined as

$$g \otimes_\perp f(\vec{p}) = \int \frac{d^2\vec{q}}{(2\pi)^2} g(\vec{q}) f(\vec{p}-\vec{q}), \tag{5}$$

with the identity

$$\mathbb{I}_{\vec{p}} = (2\pi)^2\delta^{(2)}(\vec{p}) = \int \frac{d^2\vec{q}}{(2\pi)^2} Z^{-1}(\vec{q}) Z(\vec{p}-\vec{q}). \tag{6}$$

The hard function is independent on ν and is renormalized as

$$H^R(m_h,\mu) = Z^H(\mu, mh)^{-1} H^B(m_h) \tag{7}$$

and the μ part of eq. 2 also holds for the hard function. Combined with the renormalized expression one arrives at the RGEs and RRGEs[2],

$$\mu\frac{d}{d\mu} S^R(\vec{p},\mu,\nu) = \gamma^S_\mu(\vec{p},\mu) S^R(\vec{p}_s,\mu,\nu), \tag{8}$$
$$\nu\frac{d}{d\nu} S^R(\vec{p},\mu,\nu) = \gamma^S_\nu(\vec{p},\mu,\nu) \otimes_\perp S^R(\vec{p}_s,\mu,\nu), \tag{9}$$
$$\mu\frac{d}{d\mu} f^{R}_{g/P}{}^{\mu\nu}(z,\vec{p}_i,\mu,\nu) = \gamma^f_\mu(\vec{p},\mu) f^{R}_{g/P}{}^{\mu\nu}(z,\vec{p}_i,\mu,\nu), \tag{10}$$
$$\nu\frac{d}{d\nu} f^{R}_{g/P}{}^{\mu\nu}(z,\vec{p}_i,\mu,\nu) = \gamma^f_\nu(\vec{p},\mu,\nu) \otimes_\perp f^{R}_{g/P}{}^{\mu\nu}(z,\vec{p}_i,\mu,\nu), \tag{11}$$
$$\mu\frac{d}{d\mu} H^R(m_h,\mu) = \gamma^H_\mu(\mu) H^R(m_h,\mu). \tag{12}$$

[2]The $\gamma_\mu^{f/S}$ are really proportional to $\mathbb{I}_{\vec{p}}$ which cancels the convolutions. But I have omitted that in the notation

Where the anomalous dimensions are defined as

$$\gamma_\mu^S(\vec{p},\mu) = -(Z^S)^{-1}\mu\frac{d}{d\mu}Z^S, \tag{13}$$

$$\gamma_\nu^S(\vec{p},\mu,\nu) = -(Z^S)^{-1}\otimes_\perp \nu\frac{d}{d\nu}Z^S, \tag{14}$$

$$\gamma_\mu^f(\vec{p},\mu) = -(Z^f)^{-1}\mu\frac{d}{d\mu}Z^f, \tag{15}$$

$$\gamma_\nu^f(\vec{p},\mu,\nu) = -(Z^f)^{-1}\otimes_\perp \nu\frac{d}{d\nu}Z^f, \tag{16}$$

$$\gamma_\mu^H(\mu) = -(Z^H)^{-1}\mu\frac{d}{d\mu}Z^H. \tag{17}$$

The soft, beam and hard function are all defined at their respective characteristic μ and ν scale. With solutions of the RGE and RRGE they can all be evolved to the same point in (ν,μ) space. What point does not matter, so for convenience one can evolve the soft and hard function to the beam function and leave the beam function unchanged. But a different evolution choice would reach the same result since the cross-section is μ and ν independent.

The anomalous dimensions can thus be derived from the renormalization factors and they have been calculated to 1-loop in [1]. There however they are only including LL terms. This can be matched to the more general form which for the soft function can be written

$$\gamma_\mu^S(\mu) = 4\Gamma_{\text{cusp}}^g[\alpha_s(\mu)]\log\frac{\mu}{\nu} + \gamma^S[\alpha_s(\mu)], \tag{18}$$

$$\gamma_\nu^S(p,\mu\nu) = 4\Gamma_{\text{cusp}}^g[\alpha_s(\mu)]\frac{2\pi}{\mu^2}\left[\frac{\mu^2}{p^2}\right]_+. \tag{19}$$

Definition and details of the plus distribution can be found in appendix B and the formulas for the Γ_{cusp}^g, γ_F and $\alpha_s(\mu)$ functions are collected in appendix A.

Similarly the anomalous dimensions for the beam function are

$$\gamma_\mu^f(\mu) = 2\Gamma_{\text{cusp}}^g[\alpha_s(\mu)]\log\frac{\nu}{m_h} + \gamma^f[\alpha_s(\mu)], \tag{20}$$

$$\gamma_\nu^f(p,\mu\nu) = -2\Gamma_{\text{cusp}}^g[\alpha_s(\mu)]\frac{2\pi}{\mu^2}\left[\frac{\mu^2}{p^2}\right]_+, \tag{21}$$

and for the hard function

$$\gamma_H(\mu) = -4\Gamma_{\text{cusp}}^g\log\frac{\mu}{m_h} + \gamma_H[\alpha_s(\mu)]. \tag{22}$$

2.2. Factorization

Assuming that the only relevant Higgs production occurs through gluon fusion, the cross section can be written as

$$\frac{d\sigma}{d^2p_T dy} = \frac{\pi C_t^2}{2v^2 s^2(N_c^2-1)} \int d^2\vec{p}_1 \int d^2\vec{p}_2 \int d^2\vec{p}_s \delta(p_T^2 - |\vec{p}_1 + \vec{p}_2 + \vec{p}_s|^2)$$
$$\times f_{g/P}^{\mu\nu}\left(\frac{m_h}{\sqrt{s}}e^{-y}, \vec{p}_1\right) f_{g/P\,\mu\nu}\left(\frac{m_h}{\sqrt{s}}e^{y}, \vec{p}_2\right) S(\vec{p}_s) H(m_h). \tag{23}$$

3. Solving the RGE & RRGE

The μ RGEs are homogeneous Ordinary Differential Equations(ODE) which can easily be solved. Care must be taken to include the α_s running as well when integrating over μ. The general solution for all the functions can be expressed as

$$F(\mu) = \exp\left[\int_{\mu_0}^{\mu} d\log\mu' \gamma_F^{\mu}(\mu')\right] F(\mu_0) \equiv U_F(\mu,\mu_0) F(\mu_0). \tag{24}$$

Where the function U_F evolve the function F from μ_0 to μ. The full evolution functions are

$$U_S(\mu,\mu_S,\nu) = \exp\left[4\int_{\alpha_s(\mu_S)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\Gamma_{\text{cusp}}^g[\alpha] \int_{\alpha_s(\mu_S)}^{\alpha} \frac{d\alpha'}{\beta[\alpha']} + 4\log\frac{\mu_S}{\nu}\int_{\alpha_s(\mu_S)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\Gamma_{\text{cusp}}^g[\alpha] \right.$$
$$\left. + \int_{\alpha_s(\mu_S)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\gamma_S[\alpha]\right], \tag{25}$$
$$U_f(\mu,\mu_S,\nu) = \exp\left[-2\log\frac{\nu}{m_h}\int_{\alpha_s(\mu_B)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\Gamma_{\text{cusp}}^g[\alpha] + \int_{\alpha_s(\mu_B)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\gamma_f[\alpha]\right], \tag{26}$$
$$U_H(\mu,\mu_S,\nu) = \exp\left[-4\int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\Gamma_{\text{cusp}}^g[\alpha] \int_{\alpha_s(\mu_H)}^{\alpha} \frac{d\alpha'}{\beta[\alpha']} - 4\log\frac{\mu_H}{m_h}\int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\Gamma_{\text{cusp}}^g[\alpha] \right.$$
$$\left. + \int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta[\alpha]}\gamma_H[\alpha]\right]. \tag{27}$$

These can be computed to whatever order is needed.

In the case of the RRGEs it is a little bit more complicated since they involve a convolution. However the equation can be simplified if one goes to b-parameter space with a fourier transform. In the case of the soft function, this yields the multiplicative ODE

$$\nu\frac{d}{d\nu}\tilde{S} = \tilde{\gamma}_\nu^S \tilde{S}. \tag{28}$$

Where the fourier transformed functions are defined by

$$\tilde{f}(b) = \int \frac{d^2\vec{p}}{(2\pi)^2} e^{i\vec{p}\cdot\vec{b}} f(p). \tag{29}$$

For the soft function, the solution in b-space is given by

$$\tilde{S}(b,\mu,\nu) = \exp\left[\tilde{\gamma}_S^{\nu}(b,\mu)\log\frac{\nu}{\nu_S}\right]\tilde{S}(b,\mu,\nu_S) = \left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{-\omega_s(\nu,\nu_S,\mu)}\tilde{S}(b,\mu,\nu_S). \quad (30)$$

Where

$$\omega_s(\nu,\nu_S,\mu) = 2\Gamma^g_{\text{cusp}}[\alpha_s(\mu)]\log\frac{\nu}{\nu_S}. \quad (31)$$

Transforming back to momentum space gives a convolution again

$$S(\vec{p},\mu,\nu) = V_S(\vec{p},\nu,\nu_S,\mu)\otimes_\perp S(\vec{p},\mu,\nu_S), \quad (32)$$

with

$$\begin{aligned} V_S(\vec{p},\nu,\nu_S,\mu) &\equiv 2\pi\int_0^\infty db b J_0(bp)\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{-\omega_s(\nu,\nu_S,\mu)} \\ &= 4\pi e^{-2\gamma_E\omega_S}\frac{\Gamma(1-\omega_S)}{\Gamma(\omega_S)}\frac{1}{\mu^2}\left(\frac{\mu^2}{p^2}\right)^{1-\omega_S}. \end{aligned} \quad (33)$$

The evolution of the beam function gives a similar expression but with

$$\begin{aligned} V_f(\vec{p},\nu,\nu_B,\mu) &\equiv 2\pi\int_0^\infty db b J_0(bp)\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right)^{\omega_B(\nu,\nu_B,\mu)} \\ &= 4\pi e^{2\gamma_E\omega_B}\frac{\Gamma(1+\omega_B)}{\Gamma(-\omega_B)}\frac{1}{\mu^2}\left(\frac{\mu^2}{p^2}\right)^{1+\omega_B}. \end{aligned} \quad (34)$$

and

$$\omega_B(\nu,\nu_B,\mu) = \Gamma^g_{\text{cusp}}[\alpha_s(\mu)]\log\frac{\nu}{\nu_B}. \quad (35)$$

3.1. Unwanted Singularity

However eq. 33 and eq. 34 poses a problem because they are singular at $\omega_S = 1$ and $\omega_B = -1$. This is relevant since $\omega_{B/S}$ are of $\mathcal{O}(1)$, see fig. 1. Which can be seen to cause trouble if you consider the integrals that govern the inverse transform. For example $\omega_S = 1$ gives a singular contribution from the region of $b \ll 1/p$. The problem can be reduced to only one integral if the soft function is evolved to the beam function or vice verse. The authors in [1] suggests a solution to this problem by suppressing the singular region by setting $\nu_S = 1/b$.

It is somewhat surprising that this singularity shows up here and it is not fully understood why it does. With no better solution as a suggestion, this project deals mainly with numerically investigating this "$1/b$" solution.

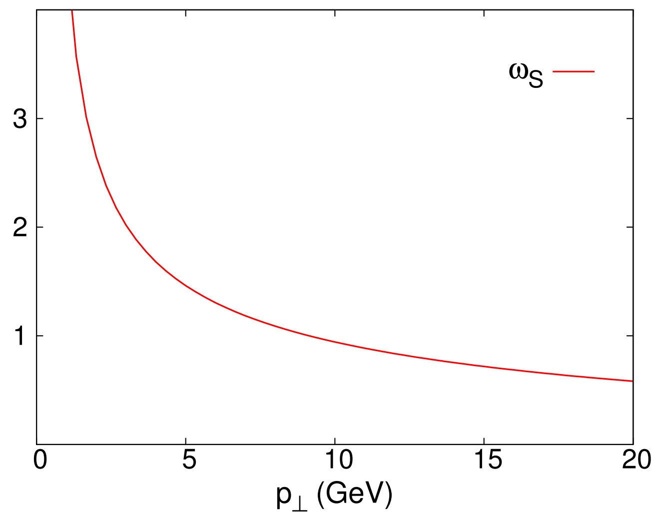

Figure 1: $\omega_S(\nu, \nu_S, \mu_S)$ is of order 1 with $\mu_S = \nu_S = p_\perp$ and $\nu = m_h$.

4. Numerical calculations

To compute the cross-section all the factors have to be resummed. This can be done in several ways and here below we present one convenient way of doing it.

4.1. Evolution of the Cross-Section

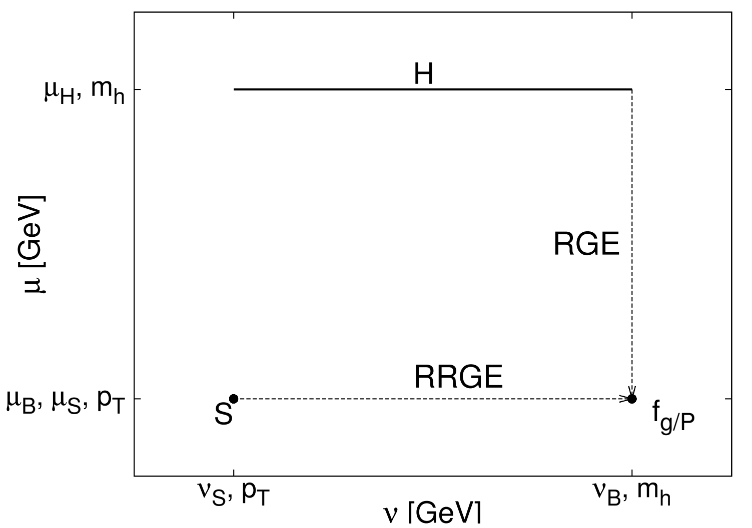

Figure 2: All the functions in the factorized cross-section should be evolved to a common point in the (ν, μ) space. Here is the evolution to the beam scale $\mu_B \sim p_\perp$, $\nu_B \sim m_h$.

The total evolution in (ν, μ) space is path independent and the cross section is independent of the final value of ν and μ. So one is free to choose how to evolve the soft, beam and hard function. Their characteristic scales are set by minimizing the logarithms that appear to higher order in α_s. This is done by setting [1]

$$\nu_S \sim \mu_S \sim \mu_B \sim p_\perp, \qquad \nu_B \sim \mu_H \sim m_h. \tag{36}$$

A convenient choice is then to only evolve the soft and hard function to the scale of the beam function, see fig. 2. Then there is no need to evolve the beam function at all. The total evolution function would then be[3]

$$E_{\text{tot}}(\vec{p}, \nu_B, \nu_S, \mu_B, \mu_S, \mu_H) = U_H(\mu_B, \mu_H) U_S(\mu_B, \mu_S, \nu_B) V_S(\nu_B, \nu_S, \mu_S). \tag{37}$$

To NLL it is sufficient to only include the lowest order of $f_{g/P}$ and S which are [1]

$$S^{(0)}(\vec{p}) = \delta^{(2)}(\vec{p}), \tag{38}$$

$$f_{g/P}^{(0)\ \mu\nu}\left(\frac{m_h}{\sqrt{s}}e^y, \vec{p}\right) = \left(\frac{\sqrt{s}}{m_h}\right) \frac{g_\perp^{\mu\nu}}{2} \delta^{(2)}(\vec{p}) f_{g/P}\left(\frac{m_h}{\sqrt{s}}e^y\right). \tag{39}$$

The gluon fusion hard function can be found in [3] and is to lowest order

$$H(m_h) = 4m_h^4 \left| \alpha_s(m_h) \left[\frac{6m_t^2}{m_h^2} - \frac{6m_t^2}{m_h^2} \left| 1 - \frac{4m_t^2}{m_h^2} \right| \arcsin^2\left(\frac{m_h}{2m_t}\right) \right] \right|^2. \tag{40}$$

With everything inserted into the cross-section eq. 23, it becomes

$$\frac{d\sigma}{d^2p_\perp dy} = \frac{\pi C_t^2}{2v^2 s^2 (N_c^2 - 1)} \frac{s}{8\pi m_h^2} f_{g/P}\left(\frac{m_h}{\sqrt{s}}e^{-y}\right) f_{g/P}\left(\frac{m_h}{\sqrt{s}}e^{y}\right) E_{\text{tot}}(\vec{p}_\perp) H(m_h). \tag{41}$$

4.2. Scale Variations

To compute the cross-section and avoid the mentioned singular contribution in the soft evolution, we will fix $\nu_S = 1/b$. For a realistic investigation of the cross-section for Higgs production you would want a prediction over the entire $p_\perp$ range. However the resummation is only valid for low $p_\perp$. For the entire range one should match the NLL to NLO. This can be done by switching of the resummation[4] at some scale that can be decided by looking at the singular and non-singular pieces of the fixed order calculation. However we will not do this in this project, but instead only look at the variation of the characteristic scales to see how it affects the resummation.

We will consider the canonical choice of scaling each parameter with a factor of 2, see fig. 3. Each renormalization scale is then varied and the total variation is the envelope of them all.

[3] It is really a convolution with the soft function.

[4] Setting all the renormalization scales to the fixed order scale.

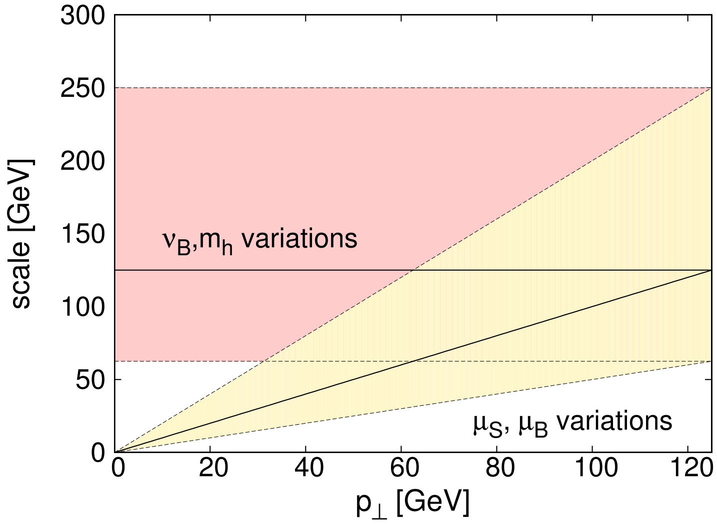

Figure 3: Each of the parameters μ_B, μ_S, μ_H and ν_B is scaled by a factor 2. ν_B and μ_H are shown in red shading while μ_S and μ_B are shown in gold shading.

4.3. Results

The cross-section has been computed for a variation of the renormalization scales according to the scaling profile in fig. 3 to see how it affect the resummation, see fig. 4. At ~ 1 GeV the variation is bigger than at the higher scales because at the very low scale there are non-perturbative effects.

The computation of the cross-section was done in Mathematica. The MSTW package [4] was used to generate the parton distribution functions and the RunDec package [5] was used to generate $\alpha_s(\mu)$.

5. Conclusion

In this report the RGE and RRGE of the soft, beam and hard functions for the cross-section for low $p_\perp$ spectrum of Higgs production were presented. The general solutions were also derived however no solution to get rid of the surprising singularity in [1] was found. Instead some numerical calculations of the resummed cross-section were done up to NLL.

The solution of setting $\nu_S = 1/b$ works for numerical calculations. However it is not clear how one shuts off the resummation with this choice.

Further analysis could be done by comparing the results to LHC data of the $p_\perp$ spectrum for Higgs production. But because of time issues this was omitted in this project.

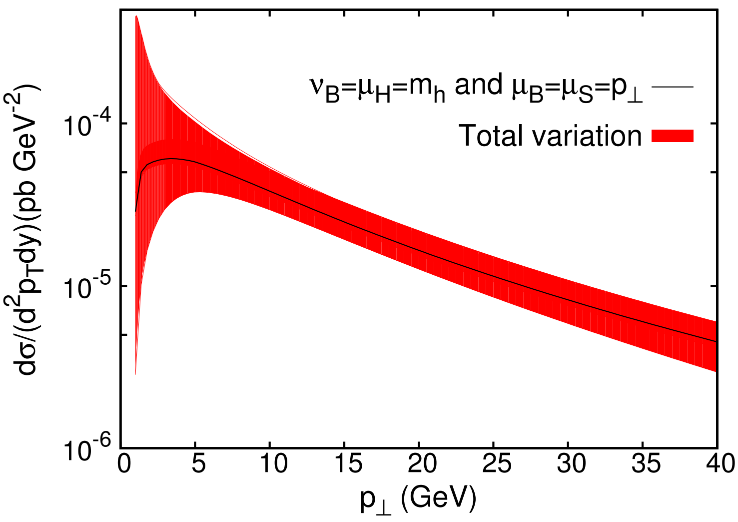

Figure 4: Cross-section for Higgs production. The red shading is the total variation of the renormalization scales according to fig 3. At low $p_\perp$ non-perturbative effects come into play causing a bigger variation. Other parameters: $\sqrt{s} = 8$ TeV and $y = 0$.

Appendices

A. Formulas Γ_{cusp}, γ_F and β functions

The anomalous dimensions are written in the form of the $\Gamma^g_{\text{cusp}}[\alpha_s]$ function. It, as well as the non-cusp piece $\gamma_F[\alpha_s]$, are power series in α_s,

$$\gamma_F[\alpha] = \sum_{n=0}^{\infty} \gamma_n^F \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \qquad \Gamma^g_{\text{cusp}}[\alpha] = \sum_{n=0}^{\infty} \Gamma^g_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \tag{42}$$

For completeness, the lowest order terms that was used in this project are listed below. All terms up to 3-loop order can be found in [3]. The coefficients up to 2-loop for $\Gamma^g_{\text{cusp}}[\alpha_s]$ are

$$\Gamma^g_0 = 4C_A, \tag{43}$$

$$\Gamma^g_1 = 4C_A\left[\left(\frac{67}{9} - \frac{\pi^2}{3}\right)C_A - \frac{10}{9}n_f\right] \tag{44}$$

and to 1-loop for $\gamma_F[\alpha_s]$ are

$$\gamma^S_0 = 0, \tag{45}$$

$$\gamma^B_0 = 2\beta_0, \tag{46}$$

$$\gamma^H_0 = -2\beta_0. \tag{47}$$

Where β_0 is the first coefficient in the beta function for α_s

$$\beta[\alpha_s] = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \tag{48}$$

$$\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f, \tag{49}$$

$$\beta_1 = \frac{34}{3}C_A^2 - \left(\frac{10}{3}C_A + 2C_F\right) n_f. \tag{50}$$

B. Plus Distributions

Plus distributions are used to render the 2 dimensional convolutions finite. They can be defined with dimensional regularization as

$$\int \frac{d^2\vec{p}}{(2\pi)^2} f(\vec{p}) \left[P(\vec{p},\mu)\right]_+ \equiv \lim_{\epsilon\to 0^+} \mu^{-2\epsilon} \left\{ \int \frac{d^{2+2\epsilon}\vec{p}}{(2\pi)^{2+2\epsilon}} f(\vec{p}) P(\vec{p},\mu) - f(0) \int_{\mathcal{D}\mu} \frac{d^{2+2\epsilon}\vec{p}}{(2\pi)^{2+2\epsilon}} P(\vec{p},\mu) \right\}. \tag{51}$$

Where $\mathcal{D}\mu = \{\vec{p} : |\vec{p}| < \mu\}$.

In the solution of the RRGE we are in need of the fourier transform of $\frac{1}{\mu^2}\left[\frac{\mu^2}{\vec{p}^2}\right]_+$. We can derive it by first evaluating for $\alpha < 0$,

$$\begin{aligned}\int \frac{d^2\vec{p}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}} \frac{1}{\mu^2}\left[\left(\frac{\mu^2}{\vec{p}^2}\right)^{1+\alpha}\right]_+ &= \int \frac{d^2\vec{p}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{p}^2}\right)^{1+\alpha} - \int_{\mathcal{D}\mu} \frac{d^2\vec{p}}{(2\pi)^2} \frac{1}{\mu^2}\left(\frac{\mu^2}{\vec{p}^2}\right)^{1+\alpha} \\ &= \frac{1}{4\pi}\frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}\left(\frac{b^2\mu^2}{4}\right)^{\alpha} + \frac{1}{4\pi\alpha}.\end{aligned} \tag{52}$$

Now if we expand both sides in α we get

$$\int \frac{d^2\vec{p}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}} \left\{ \frac{1}{\mu^2}\left[\frac{\mu^2}{\vec{p}^2}\right]_+ + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!\mu^2}\left[\frac{\mu^2}{\vec{p}^2}\log^n \frac{\mu^2}{\vec{p}^2}\right]_+ \right\} = -\frac{1}{4\pi}\log\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right) + \mathcal{O}(\alpha). \tag{53}$$

Matching the powers of α gives us

$$\int \frac{d^2\vec{p}}{(2\pi)^2} e^{i\vec{b}\cdot\vec{p}} \frac{1}{\mu^2}\left[\frac{\mu^2}{\vec{p}^2}\right]_+ = -\frac{1}{4\pi}\log\left(\frac{b^2\mu^2 e^{2\gamma_E}}{4}\right). \tag{54}$$

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