Pion matrix element in chiral perturbation theory

Laura Laulumaa

University of Jyväskylä,
Finland

September 11, 2014

Supervisor: Markus Diehl

Abstract

In this report I shortly write down calculations I have done during my summer student program at DESY. Main purpose is calculate pion matrix element with two vector and axial vector currents at lowest order using chiral perturbation theory. These matrix elements are needed when one study multiparton interactions. We start with derivation of Feynman rules for vector and axial vector currents and then use them for determining of matrix element.

Contents

1 Introduction

When one studies hadron-hadron collisions, one usually uses parton model to describe situation. Collisions where one parton from both hadrons scatters, single parton scattering, is quite well-known. Methods of determining probability functions to find such partons in hadron, single parton distribution functions (PDFs), are also known.

Considering hardon-hadron collisions in parton model it has found that one needs to examine multiparton interactions, where more than one parton from both hadrons interacts independently [1]. In this case, things are more complicated since one has to take into account probability to find more than one certain partons in hadron at the specific transverse distance. Concentrate now on double parton scatterings and the special case where two interacting partons are quarks and hadron is proton.

Define the double parton distribution function (DPDF) F for which

$$F \sim \langle p| \mathcal{O}_{\alpha_1}(0, z_2)\mathcal{O}_{\alpha_2}(y, z_1) |p\rangle \ ,$$

where operators $\mathcal{O}_\alpha(z, y) = \bar{q}(y - \frac{1}{2}z)\Gamma_\alpha q(y + \frac{1}{2}z)$ and α labels the quark polarization that is $\Gamma_\alpha \in [\frac{1}{2}\gamma^+, \frac{1}{2}\gamma^+\gamma_5, \frac{1}{2}i\sigma^{j+}\gamma_5]$. The DPDF 'tells' the probability of finding two quarks in hadron at relative distance y. The cross section of proton-proton collision is now related to two DPDFs, one for each proton [2]. Problem with these DPDFs is that matrix elements in it can not be exactly calculated and thus nor the cross section. The only way is guess the form of matrix elements or approximate them using chiral perturbation theory as we done in this report. Simplify situation by taking $z_1 = z_2 = 0$. In this case matrix element in DPDF can be calculated using chiral perturbation theory in lattice QCD.

In this report these matrix elements are calculated for pions. We consider two cases where operators $\mathcal{O}$ are vector and axial vector currents. First we take a look to basics of theory in section 2. Then Feynman rules that are needed are derived in chiral perturbation theory in section 3. Using them in section 4 pion matrix elements are calculated.

1.1 Conventions and definitions

For Fourier transformations we use the convention

$$f(x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, e^{-ik\cdot x}\tilde{f}(k)$$
$$\tilde{f}(k) = \int \mathrm{d}^4x\, e^{ik\cdot x} f(x)$$

and for δ-function representation

$$\int \mathrm{d}^4x\, e^{ik\cdot x} = (2\pi)^4\delta^4(k) \ .$$

For derivative with respect to component x^μ we use shorthand notation

$$\frac{\partial}{\partial x^\mu} \doteq \partial_{x,\mu} \; .$$

Denote any scalar field as ϕ. For initial state fields we use LSZ-reduction formula

$$|\phi(p)\rangle = i \int \mathrm{d}^4 y \, e^{-ip\cdot y} (\partial_y^2 + M^2) \phi(y) \, |0\rangle \;\; ,$$

where $\partial_y^2 = \partial_{y,\mu} \partial_y^\mu$.

Define the *Feynman propagator* for scalar field ϕ as

$$\langle 0| \, T(\phi(x)\phi(y)) \, |0\rangle \doteq D(x-y) \; ,$$

where T denotes time ordering. This propagator satisfies equations

$$(\partial_x^2 + m^2) D(x-y) = (\partial_y^2 + m^2) D(x-y) = -i\delta^4(x-y) \; .$$

2 Chiral perturbation theory

Chiral perturbation theory (ChPT) is an effective theory of QCD at low energies. When chiral symmetry of Lagrangian of lightest quarks is spontaneously broken, Goldstone bosons are three pions.

Let us discuss about chiral symmetries in QCD and take the low energy limit as it is done in [3], [4, p.667]. When we ignore all but lightest quarks u and d, the QCD Lagrangian is

$$\mathcal{L} = \bar{u} i \not{D} u - m_u \bar{u} u + \bar{d} i \not{D} d - m_d \bar{d} d - \frac{1}{4} G_{\mu\nu,a} G_a^{\mu\nu} \; .$$

Since u and d are very light we can take so-called chiral limit $m_u, m_d \to 0$ and fermionic part of our Lagrangian is

$$\mathcal{L} = \bar{u} i \not{D} u + \bar{d} i \not{D} d \; .$$

Let us define projection operators

$$P_R = \frac{1}{2}(1+\gamma_5) \quad \text{and} \quad P_L = \frac{1}{2}(1-\gamma_5) \; ,$$

which clearly satisfies properties

$$\begin{cases} P_R = P_R^\dagger \quad , \quad P_L = P_L^\dagger \\ P_R^2 = P_R \quad , \quad P_L^2 = P_L \\ P_R + P_L = 1 \\ P_R P_L = P_L P_R = 0 \ , \end{cases}$$

and are indeed projection operators which project field q to its left and right handed components

$$q_r = P_R q \quad , \quad q_L = P_L q \ .$$

Our Lagrangian can now be written as

$$\mathcal{L} = \bar{u}_R i \not{D} u_R + \bar{d}_R i \not{D} d_R + \bar{u}_L i \not{D} u_L + \bar{d}_L i \not{D} d_L \ .$$

Form doublet $Q \doteq \binom{u}{d}$ of quarks u and d and write Lagrangian $\mathcal{L}$ with it. $\mathcal{L}$ is now symmetric under unitary transformations

$$Q_R \to U_R(x) Q \quad , \quad Q_L \to U_L(x) Q \ ,$$

where $U_R, U_L \in SU(2)$. Since the vacuum expectation value

$$\langle 0 | \bar{Q} Q | 0 \rangle = \langle 0 | \bar{Q}_L Q_R + \bar{Q}_R Q_L | 0 \rangle \neq 0 \ ,$$

the lowest energy states do not obey the same symmetry than Lagrangian. Thus symmetry is spontaneously broken and by Goldstone's theorem there must be three massless bosons corresponding this symmetry breaking. We can do the interpretation that these bosons are three light mesons, pions, which form the isospin triplet

$$\pi = (\pi^1, \pi^2, \pi^3) \ .$$

By change of basis they can be written

$$\begin{cases} \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2) \\ \pi^0 = \pi^3 \ . \end{cases}$$

Since u and d quarks are not massless, this theory describes nature only approximatively. Indeed ChPT is a perturbation theory and perturbations are done in masses of quarks, thus in masses of pions, and at low energies.

2.1 Operators in ChPT

Next we find out how to expand operators like in section 1 in ChPT. Ideas in this section are from [5] and operators and quantities defined in this section are special cases of those defined in [5].

Define the isovector operator

$$\mathcal{O}^a_\mu = \frac{1}{2}\bar{q}\gamma_\mu\tau^a q \ ,$$

where fields are taken in the same space time coordinate. This operator we would like to expand.

Take the Lagrangian of pions in ChPT

$$\mathcal{L}^2_{\pi\pi} = \frac{1}{4}F^2\,\mathrm{Tr}(D_\mu U D^\mu U^\dagger + \chi^\dagger U + \chi U^\dagger) \ ,$$

where $U = \exp(i\tau^a\pi^a/F)$ and F is a pion decay constant, as a starting point. Here covariant derivative D_μ and external field χ can be chosen such that $D_\mu U = \partial_\mu U$ and $\chi = M^2\mathbb{I}$, where M is pion mass and $\mathbb{I}$ is unit matrix. Matrix elements with operators $\mathcal{O}^a_\mu$ could be calculated in ChPT introducing corresponding external sources in Lagrangian, but we do it in different.

For that define left- and right-handed operators

$$\begin{aligned}
\left(\mathcal{O}^L_{ij}\right)_\mu &= S\bar{q}_j\gamma_\mu\frac{1-\gamma_5}{2}q_i \\
\left(\mathcal{O}^R_{ij}\right)_\mu &= S\bar{q}_j\gamma_\mu\frac{1+\gamma_5}{2}q_i \ .
\end{aligned}$$

With these definitions

$$\mathcal{O}^a_\mu = \frac{1}{2}\,\mathrm{Tr}[\tau^a(\mathcal{O}^L_\mu + \mathcal{O}^R_\mu)] \ .$$

Operators $\mathcal{O}^L_\mu$ and $\mathcal{O}^R_\mu$ indeed are left- and right-handed, since they transform as

$$\mathcal{O}^L_\mu \to V_L\mathcal{O}^L_\mu V_L^\dagger \quad \text{and} \quad \mathcal{O}^R_\mu \to V_R\mathcal{O}^R_\mu V_R^\dagger$$

under chiral rotations of pion field in which $U \to V_R U V_L^\dagger$. We now define two quantities L_μ and R_μ such that

$$\begin{aligned}
L_\mu &= U^\dagger\partial_\mu U = i\tau^a\left(\frac{1}{F}\partial_\mu\pi^a + \frac{\epsilon^{abc}}{F^2}\pi^b\partial\pi^c + \frac{2}{3F^3}\pi^b(\pi^a\partial_\mu\pi^b - \pi^b\partial_\mu\pi^a) + O(\pi^3)\partial_\mu\pi\right) \\
&\doteq \tau^a L^a_\mu \\
R_\mu &= U^\dagger\partial_\mu U = i\tau^a\left(\frac{-1}{F}\partial_\mu\pi^a + \frac{\epsilon^{abc}}{F^2}\pi^b\partial\pi^c - \frac{2}{3F^3}\pi^b(\pi^a\partial_\mu\pi^b - \pi^b\partial_\mu\pi^a) + O(\pi^3)\partial_\mu\pi\right) \\
&\doteq \tau^a R^a_\mu \ ,
\end{aligned}$$

where we expanded U as powers of π. An important remark is that these quantities transform similarly as operators $\mathcal{O}^L_\mu$ and $\mathcal{O}^R_\mu$ and thus they might have some connection to those operators. Indeed it turns out after few steps discussed in [5] that at lowest order

$$\mathcal{O}^L_\mu = c_1 L_\mu \quad \text{and} \quad \mathcal{O}^R_\mu = c_1 R_\mu \ ,$$

where c_1 is constant, which need to be determined somehow.

Hence we have expansion for operators $\mathcal{O}^L_\mu$ and $\mathcal{O}^R_\mu$ as powers of pion fields π via quantities L_μ and R_μ and thus for every operator we can form using them similarly than $\mathcal{O}^a_\mu$.

3 Feynman rules

In this section we derive Feynman rules for three tree-level graph of pions using the results derived in section 2.1 for operators in chiral perturbation theory. We still need to know how to deal with derivatives of fields in vacuum expectation values and this is discussed in next section.

3.1 Result for derivative of field

Let us consider the problem concerning derivatives of fields in vacuum expectation values. It turns out that derivatives can be replaced by momentum of the initial (or final) state pion which is contracted with differentiated field.

Consider the expectation value

$$\begin{aligned}\langle 0| \left(\partial_{x,\mu}\pi^a(x)\right) \left|\pi^b(p)\right\rangle =& i\int \mathrm{d}^4 y\, e^{-ip\cdot y}(\partial_y^2 + M^2)\, \langle 0| T[(\partial_{x,\mu}\pi^a(x))\pi^b(y)] \left|0\right\rangle \\ =& i\int \mathrm{d}^4 y\, \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}^4 z\, e^{-ip\cdot y - ik\cdot x + ik\cdot z}(\partial_y^2 + M^2) \\ & \times \langle 0| T[(\partial_{z,\mu}\pi^a(z)\pi^b(y)] \left|0\right\rangle \ .\end{aligned}$$

Here $\pi^a(x)$ denotes the pion field and $\pi^a(p)$ the particle. Since

$$\begin{aligned}\langle 0| T[(\partial_{z,\mu}\pi^a(z))\pi^b(y)] \left|0\right\rangle &= \lim_{h\to 0}\frac{1}{h}\, \langle 0| T[(\pi^a(z+he^{(\mu)}) - \pi^a(z))\pi^b(y)] \left|0\right\rangle \\ &= \lim_{h\to 0}\frac{1}{h}[D(z+he^{(\mu)} - y) - D(z-y)]\delta^{ab} \\ &= \partial_{z,\mu} D(z-y)\delta^{ab} \ ,\end{aligned}$$

where $e^{(\mu)}$ is unit vector in direction of μ, we get

$$\begin{aligned}\langle 0| \left(\partial_{x,\mu}\pi^a(x)\right) \left|\pi^b(p)\right\rangle =& i\int \mathrm{d}^4 y\, \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}^4 z\, e^{-ip\cdot y - ik\cdot x + ik\cdot z}(\partial_y^2 + M^2) \\ & \times \partial_{z,\mu} D(z-y)\delta^{ab} \ .\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}\int \mathrm{d}^4z\, e^{ik\cdot z}\partial_{z,\mu}D(z-y) &= -\int \mathrm{d}^4y\,(\partial_{z,\mu}e^{ik\cdot z})D(z-y)\\ &= -ik_\mu \int \mathrm{d}^4z\, e^{ik\cdot z}D(z-y)\ .\end{aligned}$$

In first step we neglect the boundary term since propagator vanishes at infinities. Using this result our vacuum expectation value is

$$\begin{aligned}\langle 0|\,(\partial_{x,\mu}\pi^a(x))\,\left|\pi^b(p)\right\rangle =& i(-ik_\mu)\int \mathrm{d}^4y\,\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{d}^4z\, e^{-ip\cdot y-ik\cdot x+ik\cdot z}(\partial_y^2+M^2)\\ &\times D(z-y)\delta^{ab}\\ =& k_\mu\int \mathrm{d}^4y\,\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{d}^4z\, e^{-ip\cdot y-ik\cdot x+ik\cdot z}(-i)\delta^4(z-y)\delta^{ab}\\ =& -ik_\mu\int \mathrm{d}^4y\,\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-i(p-k)\cdot y-ik\cdot x}\delta^{ab}\\ =& -ik_\mu\int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{-ik\cdot x}(2\pi)^4\delta^4(p-k)\delta^{ab}\\ =& -ip_\mu e^{-ip\cdot x}\delta^{ab}\ ,\end{aligned}\tag{1}$$

and indeed we have a component of initial state momentum in final result.

3.2 Pion propagator

Our pion field is a three component vector which components are scalar fields. Thus for pions as for all scalar fields the propagator in momentum space is [4, p.31]

$$\frac{i}{p^2-M^2+i\epsilon}\ .\tag{2}$$

3.3 Vector current

Consider the isotriplet vector current

$$V_\mu^a = \frac{1}{2}\bar{q}\gamma_\mu\tau^a q\ .$$

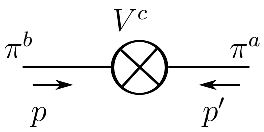
Using the definitions from section 2.1 we notice that

$$V_\mu^a = \mathcal{O}_\mu^a = \frac{1}{2}\,\mathrm{Tr}[\tau^a(\mathcal{O}_\mu^L+\mathcal{O}_\mu^R)]$$

and thus

$$
\begin{aligned}
V_\mu^a &= \frac{c_1}{2}(L_\mu^a + R_\mu^a) \\
&= \frac{ic_1\epsilon^{abc}}{F}\pi^b(\partial\pi^c) + O(\pi^3)\partial_\mu\pi \ .
\end{aligned}
$$

At tree-level the connected graph with vector current is



and the corresponding matrix element is $\langle 0|\,\tilde{V}_\mu^c(k)\,\left|\pi^a(p')\pi^b(p)\right\rangle$. Using the result derived in section 3.1

$$
\begin{aligned}
\langle 0|\,\tilde{V}_\mu^c(k)\,\left|\pi^a(p')\pi^b(p)\right\rangle =& \frac{ic_1\epsilon^{cde}}{F^2}\int \mathrm{d}^4x\, e^{ik\cdot x}\,\langle 0|\,\pi^d(x)(\partial_{x,\mu}\pi^e(x))\,\left|\pi^a(p')\pi^b(p)\right\rangle \\
=& i^2\frac{ic_1\epsilon^{cde}}{F^2}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4y'\, e^{ik\cdot x}e^{-ip'\cdot y'}e^{-ip\cdot y}(\partial_y^2+M^2)(\partial_{y'}^2+M^2) \\
&\times \langle 0|\,T(\pi^d(x)(\partial_{x,\mu}\pi^e(x))\pi^a(y')\pi^b(y))\,|0\rangle \ .
\end{aligned}
$$

Since

$$
\begin{aligned}
\langle 0|\,T[\pi^d(x)(\partial_{x,\mu}\pi^e(x))\pi^a(y')\pi^b(y)]\,|0\rangle =& D(y'-x)(\partial_{x,\mu}D(y-x))\delta^{ad}\delta^{eb} \\
&+ (\partial_{x,\mu}D(y'-x))D(y-x)\delta^{ae}\delta^{db}
\end{aligned}
$$

we get

$$
\begin{aligned}
\langle 0|\,\tilde{V}_\mu^c(k)\,\left|\pi^a(p')\pi^b(p)\right\rangle &= \frac{ic_1\epsilon^{cde}}{F^2}\int \mathrm{d}^4x\, e^{-i(p'+p-k)\cdot x}[-ip_\mu\delta^{ad}\delta^{eb} - ip'_\mu\delta^{ae}\delta^{db}] \\
&= \frac{c_1}{F^2}(2\pi)^4\delta^4(p+p'-k)[p_\mu\epsilon^{cab} + p'_\mu\epsilon^{cba}] \\
&= \frac{c_1}{F^2}(2\pi)^4\epsilon^{abc}(p_\mu - p'_\mu)\delta^4(p+p'-k) \\
&= i(2\pi)^4\epsilon^{abc}(p_\mu - p'_\mu)\delta^4(p+p'-k)\ , \qquad (3)
\end{aligned}
$$

where we used $c_1 = iF^2$ (see next section).

3.4 Axial vector current

For axial vector current

$$
A_\mu^a = \frac{1}{2}\bar{q}\gamma_\mu\gamma_5\tau^a q
$$

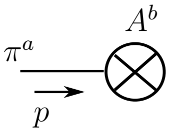
results in section 2.1 give

$$A^a_\mu = \frac{1}{2}\,\mathrm{Tr}[\tau^a(\mathcal{O}^R_\mu - \mathcal{O}^L_\mu)]$$

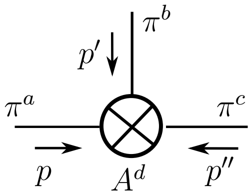
and

$$\begin{aligned} A^a_\mu &= \frac{c_1}{2}(R_\mu - L_\mu) \\ &= -\left[\frac{c_1 i}{F}(\partial_\mu \pi^a) + \frac{2c_1 i}{3F^3}\pi^b(\pi^a(\partial_\mu\pi^b) - \pi^b(\partial_\mu\pi^a)) + O(\pi^4)\partial_\mu\pi\right] \\ &\doteq A^a_{1,\mu} + A^a_{3,\mu} + O(\pi^4)\partial_\mu\pi \ . \end{aligned}$$

The only connected graph at lowest order that can be formed using the first term is



and using the second term is



Matrix element corresponding graph of fist term is

$$\begin{aligned} \langle 0|\,\tilde{A}^b_{1,\mu}(k)\,|\pi^a(p)\rangle &= \int \mathrm{d}^4x\, e^{ik\cdot x}\,\langle 0|\,\tilde{A}^b_\mu(x)\,|\pi^a(p)\rangle \\ &= -\frac{ic_1}{F}\int \mathrm{d}^4x\, e^{ik\cdot x}\,\langle 0|\,\partial_\mu\pi^b(x)\,|\pi^a(p)\rangle \\ &= -\frac{ic_1}{F}\int \mathrm{d}^4x\, e^{i(k-p)\cdot x}(-ip_\mu)\delta^{ab} \ , \end{aligned}$$

where the result from section 3.1 is used. Since [4, p.670]

$$\langle 0|\,A^b_{1,\mu}(x)\,|\pi^a(p)\rangle = -ip_\mu F e^{-ip\cdot x}\delta^{ab}$$

we get

$$-\frac{ic_1}{F} = F \quad \Rightarrow c_1 = iF^2 \ .$$

Plugging this in, Feynman rule for the first term is

$$\langle 0 | \tilde{A}^b_{1,\mu}(k) | \pi^a(p) \rangle = -iF(2\pi)^4 \delta^{ab} p_\mu \delta^4(p-k) \ . \tag{4}$$

For the second term

$$\begin{aligned}
\langle 0 | \tilde{A}^d_{3,\mu}(k) \left| \pi^a(p)\pi^b(p')\pi^c(p'') \right\rangle &= \int \mathrm{d}^4x \, e^{ik\cdot x} \langle 0 | A^d_{2,\mu}(x) \left| \pi^a(p)\pi^b(p')\pi^c(p'') \right\rangle \\
&= \frac{-2ic_1}{3F^3} \int \mathrm{d}^4x \, \mathrm{d}^4y \, \mathrm{d}^4y' \, \mathrm{d}^4y'' \, e^{ik\cdot x - ip\cdot y - ip'\cdot y' - ip''\cdot y''} (\partial_y + M^2)(\partial_{y'} + M^2)(\partial_{y''} + M^2) \\
&\quad \times \langle 0 | \pi^e(x)(\pi^d(x)(\partial_\mu \pi^e(x)) - \pi^e(x)(\partial_\mu \pi^d(x)))\pi^a(y)\pi^b(y')\pi^c(y'') | 0 \rangle \ .
\end{aligned}$$

The vacuum expectation value looks quite complicated in this case, but it is straightforward to do, so we go straight to the result which is

$$\begin{aligned}
&\langle 0 | \tilde{A}^d_{3,\mu}(k) \left| \pi^a(p)\pi^b(p')\pi^c(p'') \right\rangle \\
&= \frac{-2i}{3F^3} iF^2 \int \mathrm{d}^4x \, e^{-i(-k+p+p'+p'')\cdot x} [i\delta^{ad}\delta^{bc}(p'_\mu + p''_\mu) + i\delta^{bd}\delta^{ac}(p_\mu + p''_\mu) \\
&\quad + i\delta^{cd}\delta^{ab}(p_\mu + p'_\mu) - 2i\delta^{ad}\delta^{bc}p_\mu - 2i\delta^{bd}\delta^{ac}p'_\mu - 2i\delta^{cd}\delta^{ab}p''_\mu] \\
&= \frac{2i}{3F} [\delta^{ad}\delta^{bc}(-2p_\mu + p'_\mu + p''_\mu) + \delta^{bd}\delta^{ac}(p_\mu - 2p'_\mu + p''_\mu) + \delta^{cd}\delta^{ab}(p_\mu + p'_\mu - 2p''_\mu)] \\
&\quad \times (2\pi)^4 \delta^4(-k + p + p' + p'') \ .
\end{aligned} \tag{5}$$

Notice that when deriving these Feynman rules we have chosen convention where all external momentums are flowing in.

4 Matrix element in ChPT

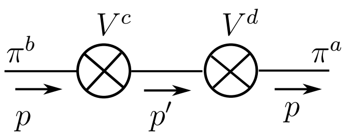
Our purpose is to calculate matrix elements for pions where we act on pion state by two vector current or axial vector current operators. Let us start with vector currents.

4.1 Matrix element with vector currents

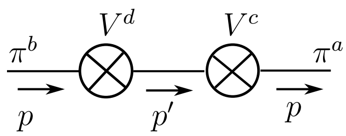
The matrix element we would like to find out is

$$\langle \pi^a(p) | V^c_\mu(0) V^d_\nu(x) \left| \pi^b(p) \right\rangle \ .$$

Here $x_0 = 0$ as we have in lattice. At lowest order the corresponding Feynman graphs are



and



Using Feynman rule (3) for vector current and (2) for propagator we get

$$
\begin{aligned}
\langle \pi^a(p)| V_\mu^c(0)V_\nu^d(x) \left|\pi^b(p)\right\rangle &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4} e^{-iq\cdot x} \langle \pi^a(p)| V_\mu^c(k)V_\nu^d(q) \left|\pi^b(p)\right\rangle \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4} e^{-iq\cdot x}\Big[[i(2\pi)^4\epsilon^{aec}(p_\mu + p'_\mu)\delta^4(p'-p-k)] \\
&\quad \times \frac{i}{(p')^2 - M^2 + i\epsilon}[i(2\pi)^4\epsilon^{ebd}(p_\nu + p'_\nu)\delta^4(p'-p-q)] \\
&\quad + [i(2\pi)^4\epsilon^{aed}(p_\nu + p'_\nu)\delta^4(p'-p-q)] \\
&\quad \times \frac{i}{(p')^2 - M^2}[i(2\pi)^4\epsilon^{ebc}(p_\mu + p'_\mu)\delta^4(p'-p-k)]\Big] \\
&= i(\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc}) \int \mathrm{d}^4q\, e^{-iq\cdot x}\Big[\frac{(2p_\mu - q_\mu)(2p_\nu - q_\nu)}{(p-q)^2 - M^2 + i\epsilon} \\
&\quad + \frac{(2p_\mu + q_\mu)(2p_\nu + q_\nu)}{(p+q)^2 - M^2 + i\epsilon}\Big] .
\end{aligned}
$$

Do the change of variable such that for the first term in integral $p - q = -k$ and for the second term $p + q = k$. Besides use the fact that

$$k_\mu e^{-ik\cdot x} = i\partial_{x,\mu} e^{-ik\cdot x} .$$

After these steps the matrix element is of the form

$$
\begin{aligned}
\langle \pi^a(p)| V_\mu^c(0)V_\nu^d(x) \left|\pi^b(p)\right\rangle = i(\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc})\Big[&e^{-ip\cdot x}(2p_\mu - i\partial_{x,\mu})(2p_\nu - i\partial_{x,\nu}) \\
&+ e^{ip\cdot x}(2p_\mu + i\partial_{x,\mu})(2p_\nu + i\partial_{x,\nu})\Big] \int \mathrm{d}^4k\, \frac{e^{-ik\cdot x}}{k^2 - M^2 + i\epsilon} .
\end{aligned}
$$

There still is one integral to be done. Because the result is needed also in the case of axial vector currents, we do that separately.

4.1.1 Integral

The goal of this section is to do integral

$$\int \mathrm{d}^4k\, \frac{e^{-ik\cdot x}}{k^2 - M^2 + i\epsilon} .$$

The first thing to take into account is that this integral has poles when $k_0 = \pm(M - i\epsilon)$. We use the common trick and turn the k_0-axis imaginary, that is do the Wick rotation $k_0 \to ik_0$, under which $\mathrm{d}^4k \to i\mathrm{d}^4k_E$ and $k^2 \to -k_E^2$. Since we have $x_0 = 0$, after Wick rotation $q \cdot x \to -q_E \cdot x_E$ and for whole integral

$$\int \mathrm{d}^4k \, \frac{e^{-ik\cdot x}}{k^2 - M^2 + i\epsilon} \to -i \int \mathrm{d}^4k_E \, \frac{e^{ik_E\cdot x_E}}{k_E^2 + M^2} \, .$$

Use the four dimensional spherical coordinate system

$$\begin{cases} x_0 &= r\cos(\psi) \\ x_1 &= r\sin(\psi)\cos(\theta) \\ x_2 &= r\sin(\psi)\sin(\theta)\cos(\phi) \\ x_3 &= r\sin(\psi)\sin(\theta)\sin(\phi) \, , \end{cases}$$

where $\psi, \theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The volume element in these coordinates is $dV = r^3 \sin^2(\psi)\sin(\theta)\mathrm{d}r \, \mathrm{d}\psi \, \mathrm{d}\theta \, \mathrm{d}\phi$. Choosing coordinate system such that $x_E \| x_0$ one get

$$\begin{aligned} -i \int \mathrm{d}^4k_E \, \frac{e^{ik_E\cdot x_E}}{k_E^2 + M^2} &= -i \int \mathrm{d}k \, \mathrm{d}\psi \, \mathrm{d}\theta \, \mathrm{d}\phi \, k^3 \sin^2(\psi)\sin(\theta) \frac{e^{ik|x|\cos(\psi)}}{k^2 + M^2} \\ &= -4\pi i \int_0^\infty \mathrm{d}k \, \frac{k^3}{k^2 + M^2} \int_0^\pi \mathrm{d}\psi \, \sin^2(\psi) e^{ik|x|\cos(\psi)} \\ &= -4\pi i \int_0^\infty \mathrm{d}k \, \frac{k^3}{k^2 + M^2} \frac{\pi}{k|x|} J_1(k|x|) \, , \end{aligned}$$

where J is Bessel function of first kind. By change of variables such that $k|x| = q$ integral is

$$\begin{aligned} -i \int \mathrm{d}^4k_E \, \frac{e^{ik_E\cdot x_E}}{k_E^2 + M^2} &= -\frac{4\pi^2 i}{|x|^2} \int_0^\infty \mathrm{d}q \, \frac{q^2}{q^2 + |x|^2 M^2} J_1(q) \\ &= -\frac{4\pi^2 i}{|x|^2} M|x| K_1(M|x|) \, , \end{aligned}$$

where K is modified Bessel function of second kind. Hence

$$\int \mathrm{d}^4k \, \frac{e^{-ik\cdot x}}{k^2 - M^2 + i\epsilon} = -\frac{4\pi^2 i M}{|x|} K_1(M|x|) \, .$$

4.1.2 Back to the pion matrix element

Using the result derived in the previous section the pion matrix element with vector currents is

$$\begin{aligned} \langle \pi^a(p)| V_\mu^c(0) V_\nu^d(x) \left|\pi^b(p)\right\rangle = 4\pi^2 M (\delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc}) \Big[& e^{-ip\cdot x}(2p_\mu - i\partial_{x,\mu})(2p_\nu - i\partial_{x,\nu}) \\ & + e^{ip\cdot x}(2p_\mu + i\partial_{x,\mu})(2p_\nu + i\partial_{x,\nu}) \Big] \frac{K_1(M|x|)}{|x|} \, . \end{aligned} \tag{6}$$

We are left with derivatives with respect to only x_μ or x_ν and with respect to both of them. First derivative gives

$$\partial_{x,\mu}\frac{K_1(M|x|)}{|x|} = -\frac{x_\mu}{|x|^3}K_1(M|x|) - \frac{x_\mu M}{2|x|^2}\big(K_0(M|x|)+K_2(M|x|)\big)$$
$$= -\frac{x_\mu M}{|x|^2}K_2(M|x|) \tag{7}$$

where we used relation

$$K_{n+1}(y) = K_{n-1}(y) + \frac{2n}{y}K_n(y) \tag{8}$$

for Bessel functions. Taking derivative of (7) with respect to x_ν and using relation (8) we have

$$\partial_{x,\nu}\partial_{x,\mu}\frac{K_1(M|x|)}{|x|} = -\frac{-g_{\mu\nu}M}{|x|^2}K_2(M|x|) + \frac{x_\mu x_\nu M^2}{|x|^3}K_3(M|x|)\ . \tag{9}$$

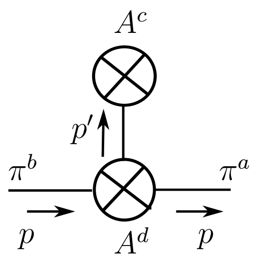
Plugging in these derivatives does not make matrix elements looking simpler, so we leave derivatives on our final results.

4.2 Matrix element with axial vector currents

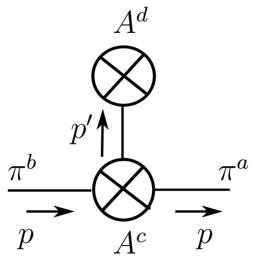
For axial vector current the matrix element we would like to find out is

$$\langle \pi^a(p)|\, A^c_\mu(0)A^d_\nu(x)\, |\pi^b(p)\rangle\ ,$$

again with $x_0 = 0$, which corresponds to graphs



and



at lowest order. Using Feynman rules (4) and (5) for axial vector current derived in section 3.4 and (2) for the propagator the matrix element is

$$\begin{aligned}
\langle \pi^a(p)|A^c_\mu(0)A^d_\nu(x)\left|\pi^b(p)\right\rangle &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iq\cdot x}\left\langle \pi^a(p)\right| A^c_\mu(k)A^d_\nu(q)\left|\pi^b(p)\right\rangle \\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}e^{-iq\cdot x}\Big[\frac{2i}{3F}[\delta^{ae}\delta^{bd}(-2p_\nu - p'_\nu - p_\nu) + \delta^{ab}\delta^{de}(p_\nu + 2p'_\nu - p_\nu) \\
&\quad + \delta^{ad}\delta^{be}(p_\nu - p'_\nu + 2p_\nu)](2\pi)^3\delta^4(p - p' - p - q)\frac{i}{(p')^2 - M^2 + i\epsilon} \\
&\quad \times [-iF\delta^{ce}(2\pi)^4\delta^4(k - p')p'_\mu] \\
&\quad + \frac{2i}{3F}[\delta^{ae}\delta^{bc}(-2p_\mu - p'_\mu - p_\mu) + \delta^{ab}\delta^{ce}(p_\mu + 2p'_\mu - p_\mu) \\
&\quad + \delta^{ac}\delta^{be}(p_\mu - p'_\mu + 2p_\mu)](2\pi)^3\delta^4(p - p' - p - k)\frac{i}{(p')^2 - M^2 + i\epsilon} \\
&\quad \times [-iF\delta^{de}(2\pi)^4\delta^4(q - p')p'_\nu]\Big] \\
&= \frac{2i}{3}\int \mathrm{d}^4q\, e^{-iq\cdot x}\Big\{\big[2q_\nu\delta^{ab}\delta^{cd} + (3p_\nu - q_\nu)\delta^{ac}\delta^{bd} - (3p_\nu + q_\nu)\delta^{ad}\delta^{bc}\big]\frac{q_\mu}{q^2 + M^2 + i\epsilon} \\
&\quad + \big[2q_\mu\delta^{ab}\delta^{cd} + (3p_\mu - q_\mu)\delta^{ac}\delta^{bd} - (3p_\mu + q_\mu)\delta^{ad}\delta^{bc}\big]\frac{q_\nu}{q^2 + M^2 + i\epsilon}]\Big\}
\end{aligned}$$

Using again the fact that

$$k_\mu e^{-ik\cdot x} = i\partial_{x,\mu}e^{-ik\cdot x}\ .$$

and integral done in section 4.1.1 we end up with

$$\begin{aligned}
&\langle \pi^a(p)|A^c_\mu(0)A^d_\nu(x)\left|\pi^b(p)\right\rangle \\
&= \frac{8\pi^2 M}{3}\Big\{\big[2i\partial_{x,\nu}\delta^{ab}\delta^{cd} + (3p_\nu - i\partial_{x,\nu})\delta^{ac}\delta^{bd} - (3p_\nu + i\partial_{x,\nu})\delta^{ad}\delta^{bc}\big]i\partial_{x,\mu} \\
&\quad + \big[2i\partial_{x,\mu}\delta^{ab}\delta^{cd} + (3p_\mu - i\partial_{x,\mu})\delta^{ac}\delta^{bd} - (3p_\mu + i\partial_{x,\mu})\delta^{ad}\delta^{bc}\big]i\partial_{x,\nu}\Big\}\frac{K_1(M|x|)}{|x|}\ .
\end{aligned}$$

5 Summary

Let us sum up this report by writing matrix elements for pions in basis $\{\pi^+, \pi^-, \pi^0\}$ in special case when $c = d = 3$.

For vector currents

$$\begin{aligned}
\left\langle \pi^+(p)\right| V^3_\mu(0)V^3_\nu(x)\left|\pi^+(p)\right\rangle = 4\pi^2 M\Big[&e^{-ip\cdot x}(2p_\mu - i\partial_{x,\mu})(2p_\nu - i\partial_{x,\nu}) \\
&+ e^{ip\cdot x}(2p_\mu + i\partial_{x,\mu})(2p_\nu + i\partial_{x,\nu})\Big]\frac{K_1(M|x|)}{|x|}\ ,
\end{aligned}$$

$$\left\langle \pi^-(p) \right| V_\mu^3(0) V_\nu^3(x) \left| \pi^-(p) \right\rangle = 4\pi^2 M \Big[e^{-ip\cdot x}(2p_\mu - i\partial_{x,\mu})(2p_\nu - i\partial_{x,\nu}) + e^{ip\cdot x}(2p_\mu + i\partial_{x,\mu})(2p_\nu + i\partial_{x,\nu}) \Big] \frac{K_1(M|x|)}{|x|}$$

and

$$\left\langle \pi^0(p) \right| V_\mu^3(0) V_\nu^3(x) \left| \pi^0(0) \right\rangle = 0 \ .$$

For axial vector currents

$$\left\langle \pi^+(p) \right| A_\mu^3(0) A_\nu^3(x) \left| \pi^+(p) \right\rangle = -\frac{32\pi^2 M}{3} \partial_{x,\mu}\partial_{x,\nu} \frac{K_1(M|x|)}{|x|} \ ,$$

$$\left\langle \pi^-(p) \right| A_\mu^3(0) A_\nu^3(x) \left| \pi^-(p) \right\rangle = -\frac{32\pi^2 M}{3} \partial_{x,\mu}\partial_{x,\nu} \frac{K_1(M|x|)}{|x|}$$

and

$$\left\langle \pi^0(p) \right| A_\mu^3(0) A_\nu^3(x) \left| \pi^0(p) \right\rangle = 0 \ .$$

Acknowledgements

For always having time to discuss with me about my project and for all good advices and help given to me I would like to thank my supervisor Markus Diehl. For this great opportunity to study and work with physics I would like to thank DESY summer student program organizers. And many thanks to all summies for making my summer student program unique and memorable.

References

[1] A. Del Fabbro and D. Treleani. A Double parton scattering background to Higgs boson production at the LHC. *Phys.Rev.*, D61:077502, 2000.

[2] Markus Diehl and Andreas Schafer. Theoretical considerations on multiparton interactions in QCD. *Phys.Lett.*, B698:389–402, 2011.

[3] Stefan Scherer and Matthias R. Schindler. A Chiral perturbation theory primer. 2005.

[4] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory*. Advanced book classics. Addison-Wesley Publishing Company, 1995.

[5] M. Diehl, A. Manashov, and A. Schafer. Generalized parton distributions for the pion in chiral perturbation theory. *Phys.Lett.*, B622:69–82, 2005.