Two point correlation functions on the torus in SU(2) WZW theory

Piotr Kucharski

4th September 2012

Supervisor: Ingo Kirsch

Abstract

We formulated a conjecture about the correlation function on the torus in SU(2) WZW theory. After obtaining the differential equations which must be satisfied, we checked that the conjecture has a proper behaviour up to third order around fixed point.

1 Introduction

The conjecture of existence of the $\mathrm{AdS}_5/\mathrm{CFT}$ duality encourages to study also simpler, lower dimensional versions of this relationship. Finding the correlation functions in two dimensional Conformal Field Theories would help to understand the structure of the duality.

2 SU(2) WZW model

2.1 Two-dimensional conformal field theory

2.1.1 Conformal transformations

Conformal Field Theories (CFT) are theories in which the symmetry transformations are conformal transformations. There are defined by the condition [1] that if $x \mapsto x'$, then:

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x) g_{\mu\nu}(x) \tag{1}$$

where g is the metric tensor and the function $\Lambda(x)$ is called the scale factor. This means that these transformations preserves oriented angles.

In this paper we focus on the two dimensional CFT, in which the identification $\mathbb{C} \simeq \mathbb{R}^2$:

$$z = x^1 + ix^2$$

$$\bar{z} = x^1 - ix^2$$

is very useful. For example, in the case of $\mathbb{C} \simeq \mathbb{R}^2$ the condition (1) is equivalent to demading that the function f which defines the conformal transformation $z \mapsto f(z)$ is holomorphic. Conformal transformations can map some points to infinity, so adding the infinity and identification with the Riemann sphere $\mathbb{C} \cup \{\infty\} \simeq \mathbb{S}^2$ is also sometimes convinient.

2.1.2 Primary fields and conformal dimension

The basic objects in CFT are primary fields. A field is called a primary field of conformal dimension $(h, \bar{h})$ if it transforms under the conformal transformations $z \mapsto f(z)$ according to [1]:

$$\phi(z,\bar{z}) \mapsto \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))$$

2.2 SU(2) WZW model

The two-dimensional $SU(2) \times SU(2)$ Wess-Zumino-Witten model is described by the action [2]:

$$S_{\lambda,k} = \frac{1}{4\lambda^2}\int \mathrm{Tr}(\partial_\mu g^{-1}\partial_\mu g)d^2\xi + k\Gamma[g] \tag{2}$$

where g is the field valued in $SU(2)$ group, $\xi^\mu = (\xi^1, \xi^2)$ are the coordinates, k and λ are dimensionless coupling constants. $\Gamma[g]$ is defined by:

$$\Gamma[g] = \int_A \frac{d^3x}{24\pi}\epsilon^{abc}\mathrm{Tr}(\tilde{g}^{-1}\partial_a\tilde{g}\tilde{g}^{-1}\partial_b\tilde{g}\tilde{g}^{-1}\partial_c\tilde{g})$$

where A is the three dimensional half-space $A = \{(\xi^\mu, t), t \geq 0\}$ and $\tilde{g}(\xi,t) \in SU(2)$ is an arbitrary smooth function which satisfies the conditions: $\lim\limits_{t\to\infty}\tilde{g}(\xi,t) = 1$, $\tilde{g}(\xi,0) = g(\xi)$.

2.3 Correlation functions on the sphere

2.3.1 Genus expansion

The genus expansion is an analog of the quantum field-theoretical loop expansion. When the world-sheet represents the world-line, loops are equivalent to holes in the world-sheet. If we add the infinity point and work on the Riemann sphere $\mathbb{C}\cup\{\infty\}\simeq \mathbb{S}^2$, number of loops (number of holes in the world-sheet) is equal to the genus of the manifold. For instance, the sphere (genus=0) is an analog of the tree level.

Our work will be focused on the case of the torus (genus=1) which represents the one-loop level. The two point correlation functions on the sphere can be found in [2], eq. (2.25).

3 Correlation functions on the torus

3.1 Procedure of obtaining differential equations

Let us consider two-point correlation function on the torus. The correlator is the sum of modulus squared n holomorphic blocks (because holomorphic blocks depends only on the difference $z-w$, from now on we will take $w=0$,):

$$\langle\phi(z,q)\phi(w,q)\rangle_{\mathbb{T}^2} = \sum_{k=1}^{n}|f_k(z-w,q)|^2$$

and each of them satisfies the equation [4, 5]:

$$W(z,q)\partial_z^n f + \sum_{r=0}^{n-1}(-1)^{n-r}W_r(z,q)\partial_z^r f = 0$$

where $W = W_n$ and $W_k$is defined by:

$$W_k = \det\begin{bmatrix} f_1 & \cdots & f_n \\ \partial_z f_1 & \cdots & \partial_z f_n \\ \vdots & & \vdots \\ \partial_z^{k-1} f_1 & \cdots & \partial_z^{k-1} f_n \\ \partial_z^{k+1} f_1 & \cdots & \partial_z^{k+1} f_n \\ \vdots & & \vdots \\ \partial_z^n f_1 & \cdots & \partial_z^n f_n \end{bmatrix}$$

Following the procedure described in [5] (which containes expressing W_k in the basis of derivatives of Weierstrass P-functions and using the asymptotic behaviour of f around $z = 0$ and $z = i\infty$) we can obtain the differential equation for the holomorphic blocks.

3.2 Equations

3.2.1 k=1, j=1/2

The differential equation for holomorphic blocks for $\langle\phi_{j=\frac{1}{2}}(z)\phi_{j=\frac{1}{2}}(0)\rangle$ and $k = 1$ (where k is a level of WZW model; see (2)) is:

$$4\partial_z^2 f(z,q) - 3\wp(z,q)f(z,q) = 0 \tag{3}$$

where $\wp(z,q)$ is the Weierstrass P-function (see Appendix). Equation (3) stays in agreement with the result in [3] obtained by a different method.

3.2.2 k=2, j=1

The differential equation for and $k = 2, j = 1$ is:

$$2\partial_z^3 f(z,q) - 6\wp(z,q)\partial_z f(z,q) - 3f(z,q)\partial_z\wp(z,q) = 0 \tag{4}$$

and it stays in agreement with the result in [3] obtained by a different method.

3.2.3 k=3, j=3/2

We also obtained new equations. The equation for $k = 3, j = 3/2$ reads:

$$\frac{15}{2}\wp(z,q)\partial_z^2 f(z,q) + \frac{15}{2}\left[\partial_z\wp(z,q)\right]\left[\partial_z f(z,q)\right] + \frac{45}{32}\left[\partial_z^2\wp(z,q)\right]f(z,q) +$$

$$-\partial_z^4 f(z,q) - \frac{9\pi^4 E_4(q)}{16}f(z,q) = 0 \tag{5}$$

where $E_4(q)$ is an element of Eisenstein series.

3.2.4 k=4, j=2

The equation for $k=4, j=2$ is:

$$15(z,q)\partial_z^3 f(z,q) + \frac{45}{2}\left[\partial_z \wp(z,q)\right]\left[\partial_z^2 f(z,q)\right] + \frac{15}{2}\left[\partial_z^2 \wp(z,q)\right]\partial_z f(z,q) +$$
$$-\partial_z^5 f(z,q) - 4\pi^4 E_4(q)\partial_z f(z,q) = 0 \tag{6}$$

3.3 Conjecture for j=k/2

In the pinching limit $z \to 0$ the two point function on the torus factorises into the two point function on the sphere and the character (see e.g. [6]):

$$\lim_{z\to 0}\langle \phi_j(z,q)\phi_j(0,q)\rangle_{\mathbb{T}^2} = \sum_l \langle \phi_j(z)\phi_j(0)\rangle_{\mathbb{S}^2} |\chi_l^{(k)}(0,q)|^2 \tag{7}$$

The sum is take over the conformal blocks and the correlator on the sphere is:

$$\langle \phi_j(z)\phi_j(0)\rangle_{\mathbb{S}^2} = \frac{1}{|z^{2\Delta}|^2}$$

the character is:

$$\chi_l^{(k)}(0,q) = \lim_{z\to 0}\chi_l^{(k)}(z,q)$$

$$\chi_l^{(k)}(z,q) = \frac{\Theta_{2l+1,k+2}(z,q) - \Theta_{-2l-1,k+2}(z,q)}{\Theta_{1,2}(z,q) - \Theta_{-1,2}(z,q)}$$

and we use

$$\Theta_{m,n}(z,q) = \sum_{i\in\mathbb{Z}+\frac{m}{2n}} q^{2ni^2}\xi^{ni}$$

where $\xi = e^{i\pi z}$.

Our conjecture about the conformal block labeled by l in the category of $j = k/2, k \in \mathbb{N}$ (then $l \in \left\{0, \frac{1}{2}, 1, ..., \frac{k}{2}\right\}$)for arbitrary z is:

$$f_l(z,q) = \frac{1}{E(z,q)^{2\Delta}}\chi_l^{(k)}(z,q) \tag{8}$$

where $E(z,q)$ is a prime form on the torus given in terms of theta functions with characteristics (see Appendix):

$$E(z,q) = \frac{\theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z,q)}{\partial_z\theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z,q)|_{z=0}}$$

and Δ is a conformal dimension of the field ϕ_j:

$$\Delta = \frac{j(j+1)}{k+2} = \frac{k}{4}$$

The conjecture

$$\langle \phi(z,q)\phi(0,q)\rangle_{\mathbb{T}^2} = \sum_{l=1}^{k/2} \left| \frac{1}{E(z,q)^{2\Delta}} \chi_l^{(k)}(z,q) \right|^2 \tag{9}$$

in natural way satisfies the equation pinching limit, since

$$\lim_{z\to 0} \frac{E(z,q)}{z} = 1$$

.

3.4 Solutions - checking the conjecture

Using the Wolfram Mathematica we checked that the equations (3-6) are satisfied by our conjecture (8) up to the order of z^3 and q^3 around $z = 0 = q$ for all $l \in \left\{0, \frac{1}{2}, 1, ..., \frac{k}{2}\right\}$.

4 Conclusions

The conjecture (9) has a proper behaviour around $z = 0 = q$ up to the order of z^3 and q^3 in the cases $j = k/2, k \in \{1,2,3,4\}$. This is a strong argument that it satisfies differential equations for all z and q, which means that it is a true correlation function on the torus. Probably the conjecture is also proper for the whole class of cases $j = k/2$, but both mentioned assumptions need to be confirmed in further research.

Appendix

Special functions

In our considerations we used some special functions, which are defined below.

Theta functions with characteristics

Theta functions with characteristics are defined by:

$$\theta \left[\begin{array}{c} p \\ r \end{array} \right] (z,q) = \sum_{n\in\mathbb{Z}} q^{(p+n)^2} e^{i2\pi(p+n)(r+z)}$$

(notation with τ, in which $q = e^{i\pi\tau}$ is also common). Functions with the lowest values of p, r have thier own names:

$$\theta \left[\begin{array}{c} 0 \\ 0 \end{array} \right] (z,q) = \theta_{00}(z,q) = \theta_3(z,q)$$

$$\theta \left[\begin{array}{c} 1/2 \\ 0 \end{array} \right] (z,q) = \theta_{10}(z,q) = \theta_2(z,q)$$

$$\theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix}(z,q) = \theta_{01}(z,q) = \theta_4(z,q)$$

$$\theta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}(z,q) = \theta_{11}(z,q) = -\theta_1(z,q)$$

Weierstrass P function

The definition of the Weierstrass P function is [5]:

$$\wp(z,q) = \pi^2 \left[\frac{1}{\sin^2 \pi z} - \frac{1}{3} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} (\cos 2n\pi z - 1) \right]$$

References

[1] R. Blumenhagen, E. Plauschinn, *Introduction to Conformal Field Theory: With Applications to String Theory* Springer, Berlin Heidelberg 2009

[2] A.B. Zamolodchikov, V.A. Fateev, Sov. J. Nucl. Phys. 43, 1031 (1986)

[3] S.D. Mathur, S. Mukhi, A. Sen, Nucl. Phys. B 305, 219 (1988)

[4] S.D. Mathur, S. Mukhi, A. Sen, Nucl. Phys. B 312, 15 (1989)

[5] P. Durganandini, Nucl. Phys. B 344, 446 (1990)

[6] R.P. Malik, N. Behera, R.K. Kaul, Phys. Rev. D 43, 1243 (1991)