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Summer Student Programme 2011 Report

Cohomology of Lie Superalgebras

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September 8, 2011

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In this work we will study the cohomology of Lie algebras and Lie superalgebras, especially of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{gl}(1|1)$. A basic introduction to the theory of Lie (super)-algebras and their representations will be given as well as to the concepts of cohomology.

Contents

Overview

In this work we want to study the cohomology of Lie algebras and Lie superalgebras and apply the results to several examples. Chapters 1 and 2 give an introduction to the theory of Lie algebras and Lie superalgebras as well as to their representations. The reader may skip these chapters if he or she feels sufficiently versed in that topic. In these chapters we will also introduce those examples of Lie (super)algebras whose cohomologies will study later on in the text. For a physical motivation see for example [HT92].

In Chapters 3 and 4 we introduce the concept of cohomology in the cases of Lie algebras and Lie superalgebras and calculate it for the examples of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{gl}(2|2)$.

1 Lie Algebras

In this chapter we will introduce the concept of Lie algebras and look at some elementary properties. Furthermore we will group Lie algebras into different classes. Next, we will look at representations of Lie algebras and will also classify them. The concepts shall be illustrated with the help of the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. An elementary introduction to the theory of Lie algebras can be found in [EW06].

1.1 Basics

Definition 1.1. A *Lie algebra* $(\mathfrak{g}, [\cdot,\cdot])$ is a vector space $\mathfrak{g}$ over a field K together with a binary operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \quad (x, y) \mapsto [x, y]\,,$$

called *Lie bracket* which satisfies:

1. (K**-bilinearity**)
$$[\alpha x + \beta y, z] = \alpha\,[x, z] + \beta\,[y, z] \tag{1.1}$$
and
$$[z, \alpha x + \beta y] = \alpha\,[z, x] + \beta\,[z, y] \tag{1.2}$$
for all scalars $\alpha, \beta \in K$ and all $x, y, z \in \mathfrak{g}$.

2. (**Alternating on** $\mathfrak{g}$)
$$[x, x] = 0 \tag{1.3}$$
for all $x \in \mathfrak{g}$.

3. (**Jacobi identity**)
$$[x, [y, z]] + [y, [x, x]] + [z, [x, y]] = 0 \tag{1.4}$$
for all $x, y, z \in \mathfrak{g}$.

In the following we will simply write $\mathfrak{g}$ for the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$.

In this text we will deal exclusively with (as vector spaces) finite-dimensional Lie algebras, which does dot mean that some of the results are not true in a more general setting.

Remark 1.1. (1) and (2) in the above Definition 1.1 imply the **anti-symmetry** of the Lie bracket, i.e.

$$[x, y] = -[y, x]$$

for all $x, y \in \mathfrak{g}$. Conversely, the implication that (2) follows from the antisymmetry is only true for fields K with $\mathrm{char}(K) \neq 2$. (Set $x = y$.) We will later in the super case (cf. Definition 2.3) use a generalisation of the antisymmetry for the definition of a Lie superalgebra.

Remark 1.2. Let $\mathfrak{g}$ be a Lie algebra with finite vector space dimension. Choose a basis $\{l_1, l_2, \dots, l_n\}$ of $\mathfrak{g}$ and we can evaluate the Lie bracket for pairs of basis vectors and write the result again in terms of the basis vectors. This gives

$$[l_i, l_j] = \sum_{k=1}^{n} f_{ij}^k l_k =: f_{ij}^k l_k \tag{1.5}$$

for $1 \leq i, j \leq n$, which are the defining relations for the f_{ij}^k $(1 \leq i, j, k \leq n)$, called *structure constants*. (In the second step of the above equation we used the Einstein summation convention, which means we sum over indices appearing twice (one upper and one lower index).)

Conversely, a Lie algebra is uniquely defined by giving a basis and the structure constants.

Remark 1.3. It follows directly from the definition of a Lie algebra that the structure constants satisfy the equations

$$f_{ii}^k = 0 \quad \text{and} \quad f_{ij}^k = -f_{ji}^k \tag{1.6}$$

for all $1 \leq i, j, k \leq n$ and

$$f_{ab}^d f_{cd}^e + f_{bc}^d f_{ad}^e + f_{ca}^d f_{bd}^e = 0 \tag{1.7}$$

for all $1 \leq a, b, c, e \leq n$.

In the following we want to introduce an important class of Lie algebras, which is defined starting from an associative algebra.

Definition 1.2. 1. An *algebra over a field* K is a K-vector space A together with a binary operation

$$* : A \times A \to A, \quad (a, b) \mapsto a * b,$$

which is K**-bilinear**, i.e.

$$(\lambda a + \mu b) * c = \lambda (a * c) + \mu (b * c) \tag{1.8}$$

and

$$c * (\lambda a + \mu b) = \lambda (c * a) + \mu (c * b) \tag{1.9}$$

for all scalars $\lambda, \mu \in K$ and all $a, b, c \in A$. For the sake of simplicity we will simply denote the Algebra as A.

2. If the operation $*$ is in addition **associative**, i.e.

$$a * (b * c) = (a * b) * c \tag{1.10}$$

for all $a, b, c \in A$, then A is called an *associative algebra over the field* K.

3. If an algebra A has an element 1 such that $1 * x = x * 1 = 1$ for all $x \in A$, then the algebra is called *unital.* (The element 1 is called *multiplicative identity* and is unique if it exists.)

4. A map $F : A \to B$ between two algebras A and B over K is called *homomorphism of algebras* if F is K-**linear**, i.e. $F(kx + y) = kF(x) + F(y)$ for all $k \in K$ and $x, y \in A$ and if F is **multiplicative**, i.e. $F(x * y) = F(x) * F(y)$ for all $x, y \in A$.

5. If an algebra homomorphism $F : A \to B$ between two unital algebras A and B maps the identity 1 in A to the one in B, then F is called *unital.*

Remark 1.4. Let A be an associative algebra as in the above Definition 1.2. Then A together with the *commutator* $[\cdot, \cdot]$ as Lie bracket defined by

$$[a, b] = a * b - b * a \tag{1.11}$$

for all $a, b \in A$ forms a Lie algebra. In particular, A together with the commutator forms again an algebra, which however is not associative in general.

Remark 1.5. Let V be a vector space over K. The vector space $\mathrm{End}(V)$ of the endomorphisms of V (i.e. the linear maps from V into itself) forms an associative algebra together with function composition $\circ$ as operation.

Definition 1.3. Let V be a vector space over K. We denote by $\mathfrak{gl}(V)$ the Lie algebra formed by the associative algebra (c.f. Remark 1.5) $\mathrm{End}(V)$ of endomorphisms of V together with the commutator as defined in Remark 1.4.

If in particular $V = K^n$, we write $\mathfrak{gl}(V) = \mathfrak{gl}_n(K) = \mathfrak{gl}(n)$.

Definition 1.4. A Lie algebra homomorphism φ is a K-linear map (i.e. a vector space homomorphism)

$$\varphi : \mathfrak{g} \to \mathfrak{h}$$

of a Lie algebra $\left(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}\right)$ to another Lie algebra $\left(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}\right)$ (both over the same field K) with

$$\varphi\left([x, y]_{\mathfrak{g}}\right) = [\varphi(x), \varphi(y)]_{\mathfrak{h}} \tag{1.12}$$

for all $x, y \in \mathfrak{g}$.

If in the above situation φ is a isomorphism of vector spaces, φ is called *Lie algebra isomorphism* and $\mathfrak{g}$ and $\mathfrak{h}$ are called *isomorphic.*

Definition 1.5. A subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, which is closed under the Lie bracket, i.e.

$$[x, y] \in \mathfrak{h} \tag{1.13}$$

for all $x, y \in \mathfrak{h}$ is called *Lie subalgebra of* $\mathfrak{g}$. If $\mathfrak{h}$ fulfils even

$$[x, y] \in \mathfrak{h} \tag{1.14}$$

for all $x \in \mathfrak{g}, y \in \mathfrak{h}$, then $\mathfrak{h}$ is called an *ideal in* $\mathfrak{g}$.

Definition 1.6. Let $\mathfrak{i}$ be an ideal of the lie algebra $\mathfrak{g}$. Then, as can be readily checked, one can define a Lie algebra on the quotient space $\mathfrak{g}/\mathfrak{i}$ by

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}$$

for $x, y \in \mathfrak{g}$, called *quotient algebra.*

Remark 1.6. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then $\ker(\varphi)$ is an ideal in $\mathfrak{g}$. Conversely to every ideal, we have the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, which has $\mathfrak{i}$ as kernel. Hence ideals are exactly kernels of Lie algebra homomorphisms.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra. A *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ consists of a unital associative algebra with the Lie algebra structure defined by the commutator and a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that the following *universal mapping property* holds: If A is an arbitrary unital associative algebra (which we view as a Lie algebra with the commutator), then the Lie Algebra homomorphisms $\psi : \mathfrak{g} \to A$ are in bijection with the unital algebra homomorphisms $F : \mathcal{U}(\mathfrak{g}) \to A$. This bijection is by means of the homomorphism $\varphi : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$. In other words: To every Lie algebra homomorphism $\psi : \mathfrak{g} \to A$ there is a unique algebra homomorphism $F : \mathcal{U}(\mathfrak{g}) \to A$ with $\psi = F \circ \varphi$.

Definition 1.8. Let V be a vector space over the field K. Then the *tensor algebra* $T(V)$ is defined by

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots, \tag{1.15}$$

where $V \otimes W$ denotes the tensor product space of the vector spaces V and W. With the multiplication defined by

$$(x, y) \mapsto x \otimes y \tag{1.16}$$

for all $x \in V^{\otimes i}$ for an i and all $y \in V^{\otimes j}$ for a j (and by bilinearity defined on all of $T(V)$) $T(V)$ becomes a $\mathbb{Z}$-graded, unitary, associative algebra.

Remark 1.7. We can explicitly construct *the* universal enveloping algebra. Let I be the two-sided ideal $T(\mathfrak{g})$, generated by elements of the form

$$x \otimes y - y \otimes x - [x, y]$$

for $x, y \in \mathfrak{g}$. Then $\mathcal{U}(\mathfrak{g})$ is given by the quotient of the tensor algebra $T(\mathfrak{g})$ by the ideal I. (Since I is not homogeneous, $\mathcal{U}(\mathfrak{g})$ does not carry an induced grading, in contrast to the exterior algebra (cf. Remark 3.2).) One can show that $\mathcal{U}(\mathfrak{g})$ does indeed have the properties from Definition 1.7.

Example 1.1. Let us look at the Lie algebra $\mathfrak{gl}_n(K)$, i.e. the endomorphisms of K^n. If we fix a vector space basis, then these correspond to the K-valued $n \times n$-matrices, where we identify the composition of endomorphisms wit matrix multiplication (and accordingly for the commutator).

The traceless endomorphisms ($n \times n$-matrices) form a subspace of $\mathfrak{gl}_n(K)$ and even a subalgebra. If one restricts the Lie bracket of $\mathfrak{gl}_n(K)$ to this subalgebra, one gets the Lie algebra of the traceless endomorphisms ($n \times n$-matrices), called $\mathfrak{sl}_n(K)$. These even form an ideal in $\mathfrak{gl}_n(K)$.

Example 1.2. Let us now look at the example of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, i.e. the Lie algebra of the traceless complex 2×2-matrices. As basis we choose

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The basis elements satisfy

$$[X, Y] = 2H, \quad [H, X] = X \quad \text{and} \quad [H, Y] = -Y.$$

If we extend the basis by

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and continue to use the commutator as Lie bracket, we get all of $\mathfrak{gl}_2(\mathbb{C})$. C fulfils the additional relations

$$[C, X] = [C, Y] = [C, H] = 0.$$

An element like C which commutates with all other elements is called a *central element.*

1.2 Types of Lie Algebras

We shall have a look at some types of Lie algebras and their properties in a nutshell.

Definition 1.9. A Lie algebra $\mathfrak{g}$ is called *abelian* if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

Definition 1.10. A Lie algebra $\mathfrak{g}$ is called *simple* if it is not abelian and if $\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$.

Definition 1.11. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras. We define a Lie bracket on the direct sum

$$\mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{1.17}$$

of the vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ via

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2]) \tag{1.18}$$

for $x_i, y_i \in \mathfrak{g}_i$ $(i = 1, 2)$. The thus defined Lie algebra $\mathfrak{g} := \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is called *direct sum* of the lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

Definition 1.12. Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic 0. $\mathfrak{g}$ is called *semisimple* if $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie algebras, i.e.

$$\mathfrak{g} \cong \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k \tag{1.19}$$

with $\mathfrak{g}_i$ $(i = 1, \ldots, n)$ simple. A Lie algebra $\mathfrak{g}$ is hence semisimple if and only if there exist simple ideals $\mathfrak{h}_i \subseteq \mathfrak{g}$ $(i = 1, \ldots k)$ with $\mathfrak{g} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$.

Remark 1.8. Let $\mathfrak{g}$ be a Lie algebra over a field K with $\mathrm{char}(K) = 0$. Then the following are equivalent:

1. $\mathfrak{g}$ is semisimple.

2. $\mathfrak{g}$ does not contain any non-trivial abelian ideals.

3. $\mathfrak{g}$ does not contain any non-trivial solvable ideals (cf. Definition 1.15).

We will later for the definition of *semisimple* in the case of Lie superalgebras (cf. Definition 2.11) use a generalisation of property (3). Note that the above equivalence then no longer holds.

Definition 1.13. Let $\mathfrak{g}$ be a Lie algebra. We define the *descending central series* $\mathcal{C}^0\mathfrak{g}, \mathcal{C}^1\mathfrak{g}, \ldots$ by

$$\mathcal{C}^0\mathfrak{g} = \mathfrak{g} \tag{1.20}$$

and

$$\mathcal{C}^{m+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^m\mathfrak{g}] \tag{1.21}$$

for $m = 1, 2, \ldots$.

A Lie algebra $\mathfrak{g}$ is called *nilpotent* if its descending central series becomes zero, i.e. there exists an $N \in \mathbb{N}$ with

$$\mathcal{C}^N\mathfrak{g} = \{0\}$$

and thus

$$\mathcal{C}^n\mathfrak{g} = \{0\}$$

for all $n \geq N$.

Definition 1.14. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent Lie subalgebra which is in addition **self-normalising**, i.e. $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ implies $y \in \mathfrak{h}$.

Definition 1.15. Let $\mathfrak{g}$ be a Lie algebra. We define the *derived series* $\mathcal{D}^0\mathfrak{g}, \mathcal{D}^1\mathfrak{g}, \ldots$ by

$$\mathcal{D}^0\mathfrak{g} = \mathfrak{g} \tag{1.22}$$

and

$$\mathcal{D}^{m+1}\mathfrak{g} = [\mathcal{D}^m\mathfrak{g}, \mathcal{D}^m\mathfrak{g}] \tag{1.23}$$

for $m = 1, 2, \ldots$.

A Lie algebra $\mathfrak{g}$ is called *solvable* if its derived series eventually becomes zero, i.e. there is a $N \in \mathbb{N}$ with

$$\mathcal{D}^N\mathfrak{g} = \{0\}$$

and thus

$$\mathcal{D}^n\mathfrak{g} = \{0\}$$

for all $n \geq N$.

A maximal solvable Lie subalgebra is called *Borel algebra*

1.3 Representations of Lie Algebras

In this section we will study representations of Lie algebras and will also classify them into different types. We need these concepts from representation theory to be able to define the cohomology of Lie algebras (cf. Chapter 3).

Definition 1.16. Let $\mathfrak{g}$ be a Lie algebra over a field K and let V be a vector space over K. A *representation of the Lie algebra $\mathfrak{g}$ on V* is a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

from $\mathfrak{g}$ into the Lie algebra of endomorphisms on V. In the case of $\mathfrak{gl}(V) = \mathfrak{gl}_n(K)$ (i.e. $V = K^n$) we speak of an n-dimensional representation of $\mathfrak{g}$.

Definition 1.17. The above Definition 1.16 reads explicitly as

$$\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x) \in \mathfrak{gl}(V) \tag{1.24}$$

for all $x, y \in \mathfrak{g}$. The vector space V together with the representation ρ is called $\mathfrak{g}$-*module*.

Equivalently one can define a $\mathfrak{g}$-module as a vector space V together with a bilinear map $\cdot : \mathfrak{g} \times V \to V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \tag{1.25}$$

for all $x, y \in \mathfrak{g}$ and all $v \in V$. This is equivalent to the above definition via $x \cdot v = \rho(x)v$. "$\cdot$" is called *action* of $\mathfrak{g}$ on V. (One often simply writes xv for $x \cdot v = \rho(x)v$.)

In the following we want to justify the usage of the word *module* in this context.

Definition 1.18. Let A be an associative algebra over the field K. A *left A-module* is a K-vector space V together with a K-bilinear map

$$A \times V \to V, \quad (a, v) \mapsto av$$

such that

$$a_1 (a_2 v) = (a_1 * a_2) v \tag{1.26}$$

for all $a_1, a_2 \in A$ and $v \in V$. (Analogously a *right A-module* is a vector space V together with a K-bilinear map

$$V \times A \to V, \quad (v, a) \mapsto va$$

such that

$$(v a_2) a_1 = v (a_2 * a_1) \tag{1.27}$$

for all $a_1, a_2 \in A$ and $v \in V$.)

A left/right A-module V is called *unital* if A is a unital associative algebra and the identity element 1 in A satisfies $1v = v$ and $v1 = v$ for all $v \in V$.

The following holds:

Theorem 1.1. *The representations of a Lie algebra $\mathfrak{g}$ are in bijection with the unital left modules of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (with the bijection being $\psi \leftrightarrow F$ from Definition 1.7.)*

Proof. We use the notation from Definition 1.7. Let ψ be a representation of a Lie algebra $\mathfrak{g}$ on the vector space V, i.e. $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism. This homomorphism is in turn assigned a unique unital algebra homomorphism $F : \mathcal{U}(\mathfrak{g}) \to \mathfrak{gl}(V)$. V hence becomes a unital left $\mathcal{U}(\mathfrak{g})$-module via

$$uv = \underbrace{F(u)}_{\in \mathrm{End}(V)} v \tag{1.28}$$

for all $u \in \mathcal{U}(\mathfrak{g})$ and all $v \in V$.

Conversely, let V be a unital left $\mathcal{U}(\mathfrak{g})$-module. We then define

$$\underbrace{\psi(x)}_{\in \mathrm{End}(V)} v = \varphi(x) v \tag{1.29}$$

for $x \in \mathfrak{g}$ and $v \in V$, which means ψ is a representation of $\mathfrak{g}$ on the vector space V.

The two constructions are inverse to each other because of $F \circ \varphi = \psi$. $\square$

We now know that representations of $\mathfrak{g}$ are mapped in a 1-to-1 fashion to unital left modules over $\mathcal{U}(\mathfrak{g})$, which is why the name $\mathfrak{g}$-module in Definition 1.17 is justified.

Example 1.3. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. With $\rho(x)v = 0$ for all $x \in \mathfrak{g}$ and $v \in V$, ρ becomes a representation of $\mathfrak{g}$, the *trivial representation.* If in this situation $V = \{0\}$, ρ is called the *zero representation.*

Definition 1.19. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. If a subspace $U \subseteq V$ is $\rho(\mathfrak{g})$-invariant, i.e. $\rho(x)u \in U$ for all $x \in \mathfrak{g}$ and all $u \in U$, we say U is *stable* under $\mathfrak{g}$ (or under the action of $\mathfrak{g}$ on V). In this case U together with the restriction of $\rho(x) \in \mathrm{End}(V)$ to $\mathrm{End}(U)$ is called *subrepresentation.* If $U \subsetneq V$, we speak of a *proper subrepresentation.*

Definition 1.20. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. Then ρ is called *irreducible* if for all subspaces $U \subseteq V$, $\rho(\mathfrak{g})U \subseteq U$ implies $U = V$ or $U = \{0\}$. (With other words: V and $\{0\}$ are the only $\rho(\mathfrak{g})$-invariant subspaces of V. With yet other words: If ρ has a proper subrepresentation, then this must already be the zero representation.) If ρ is not irreducible, ρ is called *reducible.*

Definition 1.21. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V and $\pi : \mathfrak{g} \to \mathfrak{gl}(W)$ a representation of $\mathfrak{g}$ on W. A linear map $\varphi : V \to W$ is called *homomorphism of representations* if

$$\varphi(\rho(x)v) = \pi(x)\varphi(v) \tag{1.30}$$

for all $x \in \mathfrak{g}, v \in V$.

If in the above situation φ is even an isomorphism of vector spaces, then the two representations are called *isomorphic.* If $V = W$, then an isomorphism from V to W corresponds to a change of basis in V. We therefore also speak of *equivalence* of representations.

Definition 1.22. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. If ρ is a direct sum of irreducible representations of $\mathfrak{g}$, ρ is called *completely reducible* (also called semisimple sometimes). In particular every irreducible representation is fully reducible.

A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is completely reducible if and only if there exists a decomposition $V = U_1 \oplus \ldots \oplus U_k$ of V into $\mathfrak{g}$-stable subspaces $U_1, \ldots, U_k$ such that the subrepresentations on them are irreducible.

Definition 1.23. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V. If ρ can be decomposed into a non-trivial direct sum of (not necessarily irreducible) representations, ρ is called *decomposable.* Else ρ is called *indecomposable.*

Definition 1.24. Let $\mathfrak{g}$ be a Lie algebra and $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V and $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W)$ a representation of $\mathfrak{g}$ on W.

The *direct sum representation* $\rho = \rho_V \oplus \rho_W : \mathfrak{g} \to \mathfrak{gl}(V \oplus W)$ is defined by

$$\rho(x)(v + w) = \rho_V(x)v + \rho_W(x)w \tag{1.31}$$

for all $x \in \mathfrak{g}, v \in V, w \in W$.

The *tensor product representation* $\pi = \rho_V \otimes \rho_W : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$ is defined by

$$\pi(x)(v \otimes w) = \rho_V(x)v \otimes w + v \otimes \rho_W(x)w \tag{1.32}$$

for all $x \in \mathfrak{g}, v \in V, w \in W$.

It is easy to see that these are indeed representations of $\mathfrak{g}$.

Definition 1.25. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by

$$\mathrm{ad}(x)y = [x, y] \tag{1.33}$$

for all $x, y \in \mathfrak{g}$ is a representation of $\mathfrak{g}$ on $\mathfrak{g}$, the *adjoint representation of* $\mathfrak{g}$. (In the expression $\mathfrak{gl}(\mathfrak{g})$, $\mathfrak{g}$ is viewed as a vector space.) Moreover $\mathrm{ad}(x) \in \mathcal{D}^1\mathfrak{g}$ for all $x \in \mathfrak{g}$.

If $\mathfrak{g}$ is n-dimensional and if we choose a basis $\{l_1, \ldots, l_n\}$ of $\mathfrak{g}$, then the representation matrices of the basis elements can be given by the structure constants:

$$\mathrm{ad}(l_i) = \left(f_{ij}^k\right)_{k,j=1,\ldots n} \tag{1.34}$$

1.4 Examples

Example 1.4. Let us look at a classical example of a Lie algebra and its representations. $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra of the traceless complex 2×2-matrices with the commutator as Lie bracket, as we have already studied in Example 1.2. We can also choose another often used basis[1] namely

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad a_3 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{1.35}$$

The following elementary commutation relations hold

$$[a_1, a_2] = -a_3, \quad [a_2, a_3] = -a_1 \quad \text{and} \quad [a_3, a_1] = -a_2 \tag{1.36}$$

or in short

$$[a_i, a_j] = -\varepsilon_{ij}{}^k a_k, \tag{1.37}$$

where $\varepsilon_{ij}{}^k = \varepsilon_{ijk}$ is the Levi-Civita symbol with three indices.[2] The structure constants are hence given by

$$f_{ij}^k = -\varepsilon_{ij}{}^k. \tag{1.38}$$

The following holds:

Theorem 1.2. *To every non-negative integer or half-integer number j, i.e. $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ there is a (unique up to isomorphism) irreducible $(2j+1)$-dimensional representation ρ_j of $\mathfrak{sl}_2(\mathbb{C})$, i.e. a representation over a $\mathbb{C}$-vector space V_j of dimension $2j+1$ (here: $V_j = \mathbb{C}^{2j+1}$).*

We can chose an orthonormal basis $\left\{ \psi_m^j \,\middle|\, m = -j, -j+1, \ldots, j-1, j \right\}$ of the representation space V_j such that:

[1]The matrices a_1, a_2, a_3 are related to the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ via $a_i = -\frac{\mathrm{i}}{2}\sigma_i$ $(i = 1, 2, 3)$. The Pauli matrices fulfil the commutation relations $[\sigma_i, \sigma_j] = 2\mathrm{i}\varepsilon_{ij}{}^k\sigma_k$.

[2]In the whole text we will not make a distinction between upper and lower indices but for aesthetic reasons we will write the indices always in a way that a sum according to the Einstein summation convention is over an upper and a lower index.

1. *If we define*
$$A_i = \rho_j(a_i) \in \mathrm{End}(V_j) \tag{1.39}$$
for $i = 1, 2, 3$ *and*
$$A^2 = A_1^2 + A_2^2 + A_3^2 \in \mathrm{End}(V_j), \tag{1.40}$$
the basis vectors satisfy the eigenvalue equations
$$A^2 \psi_m^j = j(j+1)\psi_m^j \tag{1.41}$$
and
$$A_3 \psi_m^j = m\psi_m^j. \tag{1.42}$$

2. *W.r.t. to the basis* $\left\{\psi_{-j}^j, \psi_{-j+1}^j, \ldots, \psi_{j-1}^j, \psi_j^j\right\}$ *of* V_j *the representations* $\rho_j(a_i)$ *have the following matrix structure:*
$$\begin{aligned}
\rho_j(a_1)_{m'm} &= \frac{1}{2}\mathrm{i}\left(\delta_{m',m+1}\sqrt{(j-m)\,(j+m+1)} + \delta_{m',m-1}\sqrt{(j+m)\,(j-m+1)}\right), \\
\rho_j(a_2)_{m'm} &= \frac{1}{2}\left(\delta_{m',m+1}\sqrt{(j-m)\,(j+m+1)} - \delta_{m',m-1}\sqrt{(j+m)\,(j-m+1)}\right), \\
\rho_j(a_3)_{m'm} &= \mathrm{i}m\delta_{m'm}.
\end{aligned} \tag{1.43}$$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is of importance in physics since it is isomorphic to the complexification $\mathfrak{su}(2)_{\mathbb{C}}$ of the real Lie algebra $\mathfrak{su}(2)$. $\mathfrak{su}(2)$ is the Lie algebra of complex, traceless, anti-hermitian[3] 2×2-matrices over $\mathbb{R}$ with the commutator as Lie bracket.

Example 1.5. We can also look at $\mathfrak{sl}_2(\mathbb{C})$ with the basis $\{H, X, Y\}$ from Example 1.2. With this basis we can again study irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $2j+1$ for $j \in \frac{1}{2}\mathbb{C}_{\geq 0}$ (these are of course isomorphic to the representations in the above Example 1.4). The following holds:

Theorem 1.3. *For all* $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ *there is a (unique up to isomorphism) irreducible representation* π_j *of* $\mathfrak{sl}_2(\mathbb{C})$ *on a vector space* V_j *of dimension* $2j+1$ *(this vector space shall have the basis* $\{v_{-j}, v_{-j+1}, \ldots, v_j\}$*) and the following holds:*

1. $\pi_j(H)v_m = mv_m$ *for all* $m = -j, -j+1, \ldots, j$,
2. $\pi_j(X)v_m = v_{m+1}$ *for all* $m = -j, \ldots, j-1$ *and* $\pi(X)v_j = 0$,
3. $\pi_j(Y)v_m = (m+j)(j-m+1)v_{m-1}$ *for all* $m = -j+1, \ldots, j$ *and* $\pi(Y)v_{-j} = 0$.

Example 1.6. Let us also look at tensor product representations of $\mathfrak{sl}_2(\mathbb{C})$. Let two representations π_{j_1} and π_{j_2} on V_{j_1} and V_{j_2} respectively be given as in the above Example 1.5.

[3]In particle physics one often adds a factor i in front of all the basis elements of the Lie algebra such that the matrices are now hermitian instead of anti-hermitian.

We can then look at the $(2j_1+1)(2j_2+1)$-dimensional tensor product representation $\pi_{(j_1,j_2)} = \pi_{j_1} \otimes \pi_{j_2}$ on $V_{(j_1,j_2)} := V_{j_1} \otimes V_{j_2}$, which is defined by

$$\pi_{(j_1,j_2)}(v \otimes v') = \pi_{j_1}(x)v \otimes v' + v \otimes \pi_{j_2}(x)v' \tag{1.44}$$

for all $x \in \mathfrak{g}, v \in V_{j_1}, v' \in V_{j_2}$. One can show:

Theorem 1.4. *The representation $\pi_{(j_1,j_2)} = \pi_{j_1} \otimes \pi_{j_2}$ on $V_{(j_1,j_2)} := V_{j_1} \otimes V_{j_2}$ of $\mathfrak{sl}_2(\mathbb{C})$ has the following decomposition:*

$$\pi_{(j_1,j_2)} \cong \pi_{|j_1-j_2|} \oplus \pi_{|j_1-j_2|+1} \oplus \ldots \oplus \pi_{j_1+j_2}. \tag{1.45}$$

We thus see that $\pi_{(j_1,j_2)}$ is in general not irreducible but can be decomposed into a direct sum of irreducible components. In this decomposition all the irreducible representations π_j with $j = |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2$ appear. The change of basis matrices, which are needed to get from the tensor product basis of $V_{(j_1,j_2)}$ to the basis in which the representation matrices have a block diagonal structure where each block corresponds to a term in the above decomposition, are given in terms of the Clebsch–Gordan coefficients.

2 Lie Superalgebras

Analogously to Lie algebras we can introduce the more general notion of Lie superalgebras. Here we introduce a distinction between even and odd elements in the vector space which is then made a Lie superalgebra. A detailed introduction to the theory of Lie superalgebras is found in [Kac77]. An extensive summary of Lie superalgebra related notions is given in [FSS96].

2.1 Basics

Definition 2.1. A super vector space V is a $\mathbb{Z}_2$-graded vector space, i.e. a vector space V with a direct sum decomposition

$$V = V_0 \oplus V_1, \tag{2.1}$$

where $0, 1 \in \mathbb{Z}_2$.

An element in $x \in V$ is called *homogeneous*, if $x \in V_0$ or $x \in V_1$. We write $|x|$ for the *degree* of a homogeneous element x, i.e. $|x| = 0$ for $x \in V_0$ and $|x| = 1$ for $x \in V_1$. Elements in V_0 are called *even* (or *bosonic*), those in V_1 *odd* (or *fermionic*).

We define the *superdimension of* V as $(\dim(V_0), \dim(V_1))$.

Definition 2.2. A *superalgebra* A *over a field* K is a K-super vector space $A = A_0 \oplus A_1$ together with a **bilinear** binary operation

$$* : A \times A \to A,$$

such that

$$A_i * A_j \subseteq A_{i+j} \tag{2.2}$$

with $i, j \in \mathbb{Z}_2$.

If the binary operation is in addition **associative**, then A is called an *associative superalgebra* A *over the field* K.

Definition 2.3. A *Lie superalgebra* $\mathfrak{g}$ *over a field* K is a superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with a binary operation $[\cdot, \cdot]$, called *Lie superbracket*, which in satisfies the following conditions:

1. (**Super anti-symmetry**)

$$[x, y] = -(-1)^{|x||y|} [y, x] \tag{2.3}$$

   for all homogeneous $x, y \in \mathfrak{g}$.

2. (**Super Jacobi identity**)

$$(-1)^{|z||x|}\,[x,[y,z]] + (-1)^{|y||z|}\,[z,[x,y]] + (-1)^{|x||y|}\,[y,[z,x]] = 0 \qquad (2.4)$$

for all homogeneous $x, y, z \in \mathfrak{g}$.

Remark 2.1. The **super anti-symmetry** (1) in the above definition implies for a field K with $\mathrm{char}(K) \neq 2$ that

$$[x,x] = 0 \qquad (2.5)$$

for all homogeneous $x \in \mathfrak{g}$ with $|x| = 0$.

The **super Jacobi identity** (2) implies for a field K with $\mathrm{char}(K) \neq 3$ that

$$[x,[x,x]] = 0 \qquad (2.6)$$

for all homogeneous $x \in \mathfrak{g}$.

Remark 2.2. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra with finite vector space dimension. If we choose a basis $\{l_1, l_2, \ldots, l_m\}$ of $\mathfrak{g}_0$ and $\{l_{m+1}, l_{m+2}, \ldots, l_{m+n}\}$ of $\mathfrak{g}_1$, we can evaluate the Lie superbracket for pairs of basis vectors and write them again in terms of basis vectors (as for Lie algebras). This gives

$$[l_i, l_j] = \sum_{k=1}^{m+n} f_{ij}^k l_k = f_{ij}^k l_k \qquad (2.7)$$

for $1 \leq i, j \leq m+n$, which are the defining relations for the *structure constants* f_{ij}^k $(1 \leq i, j, k \leq m+n)$.

Also a Lie superalgebra is uniquely determined by choosing a basis and giving the structure constants.

Remark 2.3. The following relations for the structure constants follow directly from the defining relations of a Lie superalgebra:

$$f_{ij}^k = -(-1)^{|i||j|} f_{ji}^k \qquad (2.8)$$

for all $1 \leq i, j, k \leq m+n$,

$$(-1)^{|b||c|} f_{ab}^d f_{cd}^e + (-1)^{|c||a|} f_{bc}^d f_{ad}^e + (-1)^{|a||b|} f_{ca}^d f_{bd}^e = 0 \qquad (2.9)$$

for all $1 \leq a, b, c, e \leq m+n$ and

$$f_{ij}^k \neq 0 \implies |i| + |j| = |k| \in \mathbb{Z}_2 \qquad (2.10)$$

for all $1 \leq i, j, k \leq m+n$. Here we wrote short

$$|i| := |l_i| = \begin{cases} 0 & \text{for } 1 \leq i \leq m \\ 1 & \text{for } m+1 \leq i \leq m+n \end{cases} .$$

Remark 2.4. Given a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, we can look at the restriction of the Lie superbracket on the even part $\mathfrak{g}_0$, which is sensible because of relation 2.2. This makes $\mathfrak{g}_0$ a Lie algebra, since the Lie superbracket simplifies to a Lie bracket. Hence a Lie superalgebra is a generalisation of a Lie algebra where a Lie algebra is a Lie superalgebra with $\dim(\mathfrak{g}_1) = 0$.

Remark 2.5. Analogously to Remark 1.4, starting from an associative superalgebra $A = A_0 \oplus A_1$ with binary operation $*$, we can define the *supercommutator* by

$$[x, y] = x * y - (-1)^{|x||y|} y * x \tag{2.11}$$

for all homogeneous elements $x, y \in A$ (defines $[\cdot, \cdot]$ by linearity for all elements in A). One readily checks that with this construction A becomes indeed a Lie superalgebra with the supercommutator as Lie superbracket.

Remark 2.6. If one looks at the definition of the supercommutator in the above Remark 2.5, one realises that this is just the commutator for two even elements or one even and one odd element. For two odd elements one has the anticommutator $\{\cdot, \cdot\}$, defined for an associative algebra analogously to the commutator as $\{a, b\} = a * b + b * a$.

Remark 2.7. Let $V = V_0 \oplus V_1$ be a K-super vector space. Then there is a natural way of introducing a $\mathbb{Z}_2$-grading on the K-vector space $\mathrm{End}(V)$. If we look at an arbitrary endomorphism T, this can uniquely be written as

$$T = T_{00} + T_{01} + T_{10} + T_{11}$$

where for $x_i \in V_i$, $i \in \mathbb{Z}_2$:

$$T_{ij}(x_0 + x_1) = T_{ij} x_i \in V_j,$$

i.e. T_{ij} maps elements in V_i to elements in V_j and elements in V_{i+1} to 0. T_{00} and T_{11} conserve the degree of a homogeneous element in V, whereas T_{01} and T_{10} reverse the degree. We set $T_0 = T_{00} + T_{11}$ and $T_1 = T_{01} + T_{10}$. We can then define the subspaces $\mathrm{End}_0(V)$ and $\mathrm{End}_1(V)$ as

$$\mathrm{End}(V)_i = \{ T_i(T) \in V | \, T \in V \} .$$

With the above considerations the following holds:

$$\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1. \tag{2.12}$$

Remark 2.8. Given a super vector space $V = V_0 \oplus V_1$ and the corresponding super vector space $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ as in Remark 2.7. $\mathrm{End}(V)$ together with the composition $\circ$ as binary operation forms a superalgebra. In particular, as can be easily checked, condition (2.2) is satisfied.

Definition 2.4. We define analogously to Definition 1.3 starting from an associative superalgebra $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ the Lie superalgebra $\mathfrak{gl}(V_0, V_1)$ by using the supercommutator defined in Remark 2.5.

If in particular $V_0 = K^m$ and $V_1 = K^n$ (i.e. V has superdimension (m, n)), we write $\mathfrak{gl}(V_0, V_1) = \mathfrak{gl}(m|n)$.

Example 2.1. A very simple Lie superalgebra which is not a Lie algebra is $\mathfrak{gl}(1|1)$, in the following viewed over the field $\mathbb{C}$. The underlying vector space $V = V_0 \oplus V_1$ is 2-dimensional with $V_0 = \mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$ and $V_1 = \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$. An arbitrary complex 2×2-matrix can be decomposed as follows:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} + \begin{pmatrix}0 & b\\ c & 0\end{pmatrix},$$

$a, b, c, d \in \mathbb{C}$. The first matrix forms the even part, hence lies in $\mathfrak{gl}(1|1)_0$ and the second matrix defines the odd part and lies in $\mathfrak{gl}(1|1)_1$. We can choose

$$C = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad H = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

as basis for the even subspace $\mathfrak{gl}(1|1)_0$ and

$$X = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

as basis for the odd subspace $\mathfrak{gl}(1|1)_1$.

If we write the supercommutator explicitly as commutator $[\cdot,\cdot]$ and anti-commutator $\{\cdot,\cdot\}$, we have the following elementary (anti)-commutation relations

$$\begin{aligned}
&[H,H] = [H,C] = [C,C] = 0,\\
&[H,X] = X, \quad [H,Y] = -Y, \quad [C,X] = [C,Y] = 0,\\
&\{X,Y\} = C \quad \text{and} \quad \{X,X\} = \{Y,Y\} = 0.
\end{aligned}$$

We cab further decompose $\mathfrak{g}$ by setting $\mathfrak{g}_{(-1)} := \mathbb{C}Y$, $\mathfrak{g}_{(0)} := \mathfrak{g}_0 = \mathbb{C}C + \mathbb{C}H$ and $\mathfrak{g}_{(1)} := \mathbb{C}X$. Then

$$\left\{\mathfrak{g}_{(-1)}, \mathfrak{g}_{(1)}\right\} \subseteq \mathfrak{g}_{(0)} \quad \text{and} \quad \left\{\mathfrak{g}_{(-1)}, \mathfrak{g}_{(-1)}\right\} = \left\{\mathfrak{g}_{(1)}, \mathfrak{g}_{(1)}\right\} = 0 \tag{2.13}$$

hold, which makes $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ a $\mathbb{Z}$-graded algebra. One says $\mathfrak{g}$ is of type I. (In general a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$, which is a $\mathbb{Z}$-graded algebra with that decomposition is called Lie superalgebra of type I.)

Definition 2.5. A Lie superalgebra homomorphism φ is a K-linear map

$$\varphi : \mathfrak{g} \to \mathfrak{h}$$

from a Lie superalgebra $\left(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, [\cdot,\cdot]_{\mathfrak{g}}\right)$ to a Lie superalgebra $\left(\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1, [\cdot,\cdot]_{\mathfrak{h}}\right)$ (both over the same field K) with

$$\varphi\left([x,y]_{\mathfrak{g}}\right) = [\varphi(x), \varphi(y)]_{\mathfrak{h}} \tag{2.14}$$

for all $x, y \in \mathfrak{g}$ and

$$\varphi(\mathfrak{g}_i) \subseteq \mathfrak{h}_i \tag{2.15}$$

for $i \in \mathbb{Z}_2$.

Definition 2.6. A $\mathbb{Z}_2$-graded subspace $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{h}_0 \subseteq \mathfrak{g}_0$ and $\mathfrak{h}_1 \subseteq \mathfrak{g}_1$, which is closed under the Lie superbracket, i.e.

$$[x, y] \in \mathfrak{h} \tag{2.16}$$

for all $x, y \in \mathfrak{h}$ is called *subalgebra of* $\mathfrak{g}$. If $\mathfrak{h}$ satisfies even

$$[x, y] \in \mathfrak{h} \tag{2.17}$$

for all $x \in \mathfrak{g}, y \in \mathfrak{h}$, then $\mathfrak{h}$ is called an *ideal in* $\mathfrak{g}$.

Definition 2.7. Let $\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{i}_1$ be an ideal in the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then analogously to the case of Lie algebras we can define a Lie superalgebra on the quotient space $\mathfrak{g}/\mathfrak{i} = \underbrace{\mathfrak{g}_0/\mathfrak{i}_0}_{=(\mathfrak{g}/\mathfrak{i})_0} \oplus \underbrace{\mathfrak{g}_1/\mathfrak{i}_1}_{=(\mathfrak{g}/\mathfrak{i})_1}$ by

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i},$$

the *quotient algebra.*

Remark 2.9. Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie superalgebra homomorphism. Then $\ker(\varphi)$ is an ideal in $\mathfrak{g}$. Conversely, to every ideal there is the canonical projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$, which has $\mathfrak{i}$ as kernel. Hence ideals are exactly the kernels of Lie superalgebra homomorphisms.

Definition 2.8. We define and construct the universal enveloping algebra for Lie superalgebras analogously to Definition 1.7 and Remark 1.7 for Lie algebras by replacing the commutator by the supercommutator and look at representations and homomorphisms of Lie superalgebras accordingly.

2.2 Types of Lie Superalgebras

Definition 2.9. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called *simple* if $\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$.

Definition 2.10. The terms *direct sum*, *nilpotent*, *Cartan subalgebra* and *solvable* for Lie superalgebras are defined formally identically to the case of ordinary Lie algebras (cf. Definitions 1.11, 1.13, 1.14 and 1.15).

Definition 2.11. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called *semisimple* if it has no non-trivial solvable ideals $\mathfrak{g}$.

Remark 2.10. We should remark that in contrast to Lie algebras (cf. Remark 1.8) semisimplicity of a Lie superalgebra does not imply that the Lie superalgebra can be written as a direct sum of simple Lie superalgebras.

2.3 Representations of Lie Superalgebras

Definition 2.12. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra over the field K. Let $V = V_0 \oplus V_1$ be a K-super vector space and $\mathfrak{gl}(V_0, V_1)$ the canonical endomorphism Lie superalgebra. A Lie superalgebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V_0, V_1)$$

is called *representation of the Lie superalgebra* $\mathfrak{g}$. V together with the representation ρ on V is again called $\mathfrak{g}$-*module* and the *action* of $\mathfrak{g}$ on V is given by $\cdot : \mathfrak{g} \times V \to V$, $x \cdot v = \rho(x)v$.

If in particular $\mathfrak{gl}(V_0, V_1) = \mathfrak{gl}(m|n)$ (i.e. $V_0 = K^m$ and $V_1 = K^n$), we speak of a (m, n)-dimensional representation of $\mathfrak{g}$.

Remark 2.11. We have seen in Remark 2.4 that the even part $\mathfrak{g}_0$ of a Lie superalgebra becomes an ordinary Lie algebra with the restriction of the Lie superbracket to it. Furthermore, because of $[\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$ we can view the odd part $\mathfrak{g}_1$ as $\mathfrak{g}_0$-module with the representation $\rho : \mathfrak{g}_0 \to \mathrm{End}\,(\mathfrak{g}_1)$ defined by

$$\rho(x)y = [x, y] \tag{2.18}$$

for all $x \in \mathfrak{g}_0, y \in \mathfrak{g}_1$.

Definition 2.13. The terms *subrepresentation*, *irreducible*, *homomorphism of representations*, *completely reducible*, *decomposable*, *direct sum representation* and *adjoint representation* for representations of Lie superalgebras are defined formally identically to the case of ordinary Lie algebras (cf. Definitions 1.19, 1.20, 1.21, 1.22, 1.23, 1.24 and 1.25).

Definition 2.14. Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be super vector spaces. Then the tensor product space $V \otimes W$ becomes a super vector space with the $\mathbb{Z}_2$-grading

$$\begin{aligned} (V \otimes W)_0 &:= (V_0 \otimes W_0) \oplus (V_1 \otimes W_1)\,, \\ (V \otimes W)_1 &:= (V_0 \otimes W_1) \oplus (V_1 \otimes W_0) \end{aligned} \tag{2.19}$$

and is called the *tensor product super vector space.*

Definition 2.15. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V_0, V_1)$ a representation $\mathfrak{g}$ of $V = V_0 \oplus V_1$ and $\rho_W : \mathfrak{g} \to \mathfrak{gl}(W_0, W_1)$ a representation of $\mathfrak{g}$ on $W = W_0 \oplus W_1$.

The *tensor product representation* $\pi = \rho_V \otimes \rho_W : \mathfrak{g} \to \mathfrak{gl}((V \otimes W)_0\,, (V \otimes W)_1)$ on $V \otimes W$ is defined by

$$\pi(x)(v \otimes w) = \rho_V(x)v \otimes w + (-1)^{|v|} v \otimes \rho_W(x)w \tag{2.20}$$

for all $x \in \mathfrak{g}$ and homogeneous $v \in V, w \in W$. This indeed defines a representation of $\mathfrak{g}$ on the super vector space $V \otimes W$.

2.4 Examples

Example 2.2. In the following we want to study the (irreducible) representations of $\mathfrak{g} = \mathfrak{gl}(1|1)$ (cf. Example 2.1).

First, there is a 1-dimensional *atypical* representation ρ_λ of $\mathfrak{g}$ on $V = V_0 = \mathbb{C}^1 = \mathbb{C}$ ($V_1 = \{0\}$). It is given by

$$\rho_\lambda(H) = \lambda \quad \text{and} \quad \rho_\lambda(C) = \rho_\lambda(X) = \rho_\lambda(Y) = 0 \tag{2.21}$$

with $\lambda \in \mathbb{C} = \mathrm{End}(\mathbb{C}^1)$.

We now construct a 2-dimensional *typical* representation ρ_Λ, $\Lambda = (\lambda, c)$, of $\mathfrak{g}$ on the vector space $V = \mathbb{C}v_0 + \mathbb{C}v_1$, where $v_0, v_1 \in V$ are linearly independent. Let the action of $\mathfrak{g}_0$ be given by

$$H \cdot v_0 = \lambda v_0 \tag{2.22}$$

and

$$C \cdot v_0 = c v_0 \tag{2.23}$$

with $c, \lambda \in \mathbb{C}$, $c \neq 0$. X shall act trivially on v_0, i.e.

$$X \cdot v_0 = 0. \tag{2.24}$$

Further let v_1 be defined by

$$v_1 := Y \cdot v_0. \tag{2.25}$$

The representation is now already uniquely determined. The representation matrices w.r.t. the basis $\{v_0, v_1\}$ of V are:

$$\rho_\Lambda(H) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda - 1 \end{pmatrix}, \quad \rho_\Lambda(C) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \rho_\Lambda(X) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho_\Lambda(Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.26}$$

It is clear that $V_0 = \mathbb{C}v_0$ and $V_1 = \mathbb{C}v_1$.

Proposition 2.1. *ρ_Λ is irreducible if and only if $c \neq 0$.*

Proof. Let $c = 0$. Then all the representation matrices $\rho_\Lambda(x)$ for $x \in \mathfrak{g}$ have lower triangular form since this is the case for the representation matrices of all basis elements. Then $\mathbb{C}v_1$ is a $\rho_\Lambda(\mathfrak{g})$-invariant subspace.

Let conversely $c \neq 0$. Because of $\rho_\Lambda(Y)v_0 = v_1$ and $\rho_\Lambda(X)v_1 = cv_0$ it is easy to show that there is no non-trivial , i.e. 1-dimensional, $\rho_\Lambda(\mathfrak{g})$-invariant subspace of V. □

For $c = 0$ the representation $\rho_\Lambda = \rho_{(\lambda,0)}$ is hence reducible. It is however not decomposable since for this to be true also a complement of $\mathbb{C}v_1$ would have to be stable under $\mathfrak{g}$. (There would have to be a basis of V w.r.t. which all the representation matrices of elements in $\mathfrak{g}$ are diagonal.) But because of $\rho_\Lambda(Y)v_0 = v_1$ this is not possible. But we can say $\rho_\Lambda = \rho_{(\lambda,0)}$ consists of two atypical representations ρ_λ and $\rho_{\lambda-1}$, which are connected via the action of Y. This can be illustrated in the following diagram:

$$\rho_{(\lambda,0)}: \qquad \rho_\lambda \xrightarrow{Y} \rho_{\lambda-1}$$

Further representations ρ'_Λ which are isomorphic to ρ_Λ can be found by looking at an arbitrary automorphism $\varphi : V \to V$ and choosing ρ'_Λ such that

$$\varphi(\rho_\Lambda(x)v) = \rho'_\Lambda(x)\varphi(v) \tag{2.27}$$

for all $x \in \mathfrak{g}, v \in V$, i.e. one chooses

$$\rho'_\Lambda(x)v = \varphi\left(\rho_\Lambda\left(x\right)\right)\varphi^{-1}(v). \tag{2.28}$$

W.r.t. a basis this corresponds to a similarity transformation of the representation matrix, i.e.

$$\rho'_\Lambda\left(x\right) = S\rho_\Lambda S^{-1} \tag{2.29}$$

with $S \in GL_2(\mathbb{C})$. (The same similarity transformation is obtained by leaving the representation ρ_Λ unchanged and conducting an appropriate change of basis on V.)

With $S = \begin{pmatrix} 0 & 1 \\ \frac{1}{c} & 0 \end{pmatrix}$ we obtain

$$\rho'_\Lambda(H) = \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \rho'_\Lambda(C) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \ \rho'_\Lambda(X) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \rho'_\Lambda(Y) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \tag{2.30}$$

and hence X and Y have changed their roles or with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ one obtains

$$\rho''_\Lambda(H) = \begin{pmatrix} \lambda - 1 & 0 \\ 0 & \lambda \end{pmatrix}, \ \rho''_\Lambda(C) = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \ \rho''_\Lambda(X) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \text{ and } \rho''_\Lambda(Y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.31}$$

Let us compare the representations ρ_Λ and ρ'_Λ for $c = 0$. It is clear that the representations are not isomorphic any more for $c = 0$. For ρ'_Λ we get analogously as for ρ_Λ the following diagram:

$$\rho'_{(\lambda,0)} : \qquad \rho_\lambda \xleftarrow{\ X\ } \rho_{\lambda-1}$$

Let us finally look at the tensor product representation $\rho^{\otimes}_{\Lambda_1,\Lambda_2} := \rho_{\Lambda_1} \otimes \rho_{\Lambda_2}$ of $V \otimes V$ where $\Lambda_1 = (\lambda_1, c_1)$, $\Lambda_2 = (\lambda_2, c_2)$. The representation is calculated according to Definition 2.15. If we choose the standard basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ for $V \otimes V$, the representation matrices have the form

$$\begin{aligned}
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(H) &= \begin{pmatrix} \lambda_1+\lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1+\lambda_2-1 & 0 & 0 \\ 0 & 0 & \lambda_1+\lambda_2-1 & 0 \\ 0 & 0 & 0 & \lambda_1+\lambda_2-2 \end{pmatrix}, \\
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(C) &= \begin{pmatrix} c_1+c_2 & 0 & 0 & 0 \\ 0 & c_1+c_2 & 0 & 0 \\ 0 & 0 & c_1+c_2 & 0 \\ 0 & 0 & 0 & c_1+c_2 \end{pmatrix}
\end{aligned} \tag{2.32}$$

as well as

$$
\begin{aligned}
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(X) &= \begin{pmatrix} 0 & c_2 & c_1 & 0 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(Y) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.33}
$$

Proposition 2.2. *For $c_1 + c_2 \neq 0$ $\rho^{\otimes}_{\Lambda_1,\Lambda_2} = \rho_{(\lambda_1,c_1)} \otimes \rho_{(\lambda_2,c_2)}$ is decomposable and*

$$
\rho_{(\lambda_1,c_1)} \otimes \rho_{(\lambda_2,c_2)} \cong \rho_{(\lambda_1+\lambda_2,c_1+c_2)} \oplus \rho_{(\lambda_1+\lambda_2-1,c_1+c_2)}. \tag{2.34}
$$

Proof. To show the assertion we have to find an appropriate basis of $V \otimes V$ such that the representation matrices are in 2×2-block diagonal form. To find such a basis, we look at the following chains starting from the vectors which are annihilated by X and Y:

$$
\begin{tikzcd}
0 & v_0 \otimes v_0 \arrow[l, "X"'] \arrow[d, "Y"] & & 0 & \\
& v_0 \otimes v_1 + v_1 \otimes v_0 \arrow[d, "Y"] & & c_1 v_0 \otimes v_1 - c_2 v_1 \otimes v_0 \arrow[u, "X"] & \\
& 0 & & v_1 \otimes v_1 \arrow[u, "X"] \arrow[r, "Y"] & 0
\end{tikzcd}
$$

We choose the vectors appearing above (after normalisation) as new basis, i.e.

$$
\left\{ v_0 \otimes v_0, v_0 \otimes v_1 + v_1 \otimes v_0, \frac{c_1}{c_1 + c_2} v_0 \otimes v_1 - \frac{c_2}{c_1 + c_2} v_1 \otimes v_0, v_1 \otimes v_1 \right\}.
$$

This gives the representation matrices

$$
\begin{aligned}
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(X) &= \begin{pmatrix} 0 & c_1 + c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 + c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\rho^{\otimes}_{\Lambda_1,\Lambda_2}(Y) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
\tag{2.35}
$$

$\rho^{\otimes}_{\Lambda_1,\Lambda_2}(H)$ and $\rho^{\otimes}_{\Lambda_1,\Lambda_2}(C)$ stay obviously unchanged. The assertion can now be read off. $\square$

For $c_1 + c_2 = 0$ $\rho^{\otimes}_{\Lambda_1,\Lambda_2}$ is indecomposable. We can however again view it as consisting of the atypical representations $\rho_{\lambda-2}$, 2 times $\rho_{\lambda-1}$ and ρ_λ where we set $\lambda = \lambda_1 + \lambda_2$. The structure is shown in the following diagram:

$$\begin{tikzcd}
& \rho_\lambda \arrow[dr, "Y"] & \\
\rho_{\lambda-1} \arrow[ur, "X"] \arrow[dr, "Y"] & & \rho_{\lambda-1} \\
& \rho_{\lambda-2} \arrow[ur, "X"] &
\end{tikzcd}$$

3 Lie Algebra Cohomology

In this chapter we want to deal with the cohomology of Lie algebras. For this we define starting from a Lie algebra an operator Q on a suitable vector space with $Q^2 = 0$.

An introduction to the theory of Lie algebras and the corresponding cohomology is given in [Kna88].

In this chapter and in the following one we will only study the case of the field $K = \mathbb{C}$, which in particular means it has characteristic zero.

3.1 Basics

Given a finite-dimensional Lie algebra $\mathfrak{g}$ with basis $\{l_1, \ldots, l_n\}$ and structure constants $\left\{ f_{ij}^k \middle| 1 \leq i, j, k \leq n \right\}$. We will study a vector space, which we shall call $\Lambda(\mathfrak{g})$ for the moment and which is spanned by the *vacuum state* $|0\rangle$ and the vectors which arise when applying the operators $b_i \in \mathrm{End}(\Lambda(\mathfrak{g}))$ and $c^j \in \mathrm{End}(\Lambda(\mathfrak{g}))$ $(1 \leq i, j \leq n)$ and arbitrary products of those on $|0\rangle$. The b_i's and c^j's have to fulfil the conditions

$$\left\{b_i, c^j\right\} = \delta_i^j := \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{3.1}$$

and

$$\{b_i, b_j\} = \left\{c^i, c^j\right\} = 0 \tag{3.2}$$

for all $1 \leq i, j \leq n$. In addition

$$b_i |0\rangle = 0 \tag{3.3}$$

shall hold for all $1 \leq i \leq n$, i.e. the b_i's *annihilate the vacuum.* The elements in $\Lambda(\mathfrak{g})$ are called *(fermionic) ghost fields.* The b_i's are called *annihilation operators*, the c^j's are called *creation operators.*

In particular, $c^j c^j = 0$ holds, which is why it is easy to see that $\dim(\Lambda(\mathfrak{g})) = 2^n$. A basis of $\Lambda(\mathfrak{g})$ is given by

$$\{\underbrace{|0\rangle}_{\binom{n}{0}}, \underbrace{c^1|0\rangle, \ldots, c^n|0\rangle}_{\binom{n}{1}}, \underbrace{c^1c^n|0\rangle, c^1c^3|0\rangle, \ldots, c^{n-1}c^n|0\rangle}_{\binom{n}{2}}, \ldots, \underbrace{c^1c^2\ldots c^n|0\rangle}_{\binom{n}{n}}\}.$$

(We explicitly assume all these vectors to be linearly independent.)

We also see now that there is a natural way of making the vector space $\Lambda(\mathfrak{g})$ a graded (i.e. $\mathbb{Z}$-graded) vector space by setting

$$\begin{aligned} \Lambda^0(\mathfrak{g}) &= \operatorname{span}\{|0\rangle\}\,, \\ \Lambda^1(\mathfrak{g}) &= \operatorname{span}\left\{c^1\,|0\rangle\,,\dots,c^n\,|0\rangle\right\}\,, \\ &\vdots \\ \Lambda^n(\mathfrak{g}) &= \operatorname{span}\left\{c^1c^2\dots c^n\,|0\rangle\right\}\,. \end{aligned} \tag{3.4}$$

(For $i \in \mathbb{Z}\setminus\{0,1,\dots,n\}$ is $\Lambda^i(\mathfrak{g}) = \{0\}$.) Then $\dim(G_i) = \binom{n}{i}$ with the convention $\binom{n}{i} = 0$ for $i \in \mathbb{Z}\setminus\{0,1,\dots,n\}$. In addition

$$\Lambda(\mathfrak{g}) = \bigoplus_{i\in\mathbb{Z}} \Lambda^i(\mathfrak{g}) = \bigoplus_{i=0}^{n} \Lambda^i(\mathfrak{g}) = \Lambda^0(\mathfrak{g}) \oplus \Lambda^1(\mathfrak{g}) \oplus \dots \oplus \Lambda^n(\mathfrak{g})$$

and

$$\dim(\Lambda(\mathfrak{g})) = \sum_{i\in\mathbb{Z}} \dim(\Lambda^i(\mathfrak{g})) = \sum_{i=0}^{n} \dim(\Lambda^i(\mathfrak{g})) = \sum_{i=0}^{n} \binom{n}{i} = 2^n$$

hold.

The index i of $\Lambda^i(\mathfrak{g})$ is called *ghost number*. We define the *ghost number operator U* by

$$U = \sum_{i=1}^{n} c^i b_i = c^i b_i. \tag{3.5}$$

Then

$$Ux = ix \tag{3.6}$$

holds for all $x \in \Lambda^i(\mathfrak{g})$ as is easily checked.

We can also define the above vector space $\Lambda(\mathfrak{g})$ rigorously. It also explains the origin of the name $\Lambda(\mathfrak{g})$.

Definition 3.1. Let V be a vector space over the field K. Further, let for $k \in \mathbb{N}$

$$T^k(V) = \bigotimes_{i=1}^{k} V = \underbrace{V\otimes\dots\otimes V}_{k-\text{fold}} \tag{3.7}$$

be the k-fold tensor product of V with itself (with the convention $T^0(V) = K$ and $T^1(V) = V$).

Let the subspace $J^k(V) \subseteq T^k(V)$ be given by

$$J^k(V) := \operatorname{span}\left\{v_1\otimes\dots\otimes v_k\,|v_1,\dots,v_k \in V, \exists i,j \in \{1,\dots,k\} : v_i = v_j\right\}, \tag{3.8}$$

i.e. $J^k(V)$ is spanned by elementary tensors having two equal factors. To each $k \in \mathbb{N}$ the *exterior power* is defined as the quotient space

$$\Lambda^k(V) = T^k(V)/J^k(V). \tag{3.9}$$

Definition 3.2. The direct sum

$$J(V) = \bigoplus_{k=0}^{\infty} J^k(V) \tag{3.10}$$

is a two-sided, homogeneous ideal in the tensor algebra

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V). \tag{3.11}$$

(Alternatively one can define $J(V)$ as the ideal in $T(V)$, generated by all the elements of the form $v \otimes v$ with $v \in V$.) The *exterior algebra* (or *Grassmann algebra*) of the vector space V is the quotient algebra

$$\Lambda(V) := T(V)/J(V). \tag{3.12}$$

Viewed as a vector space it is isomorphic to

$$\bigoplus_{k=0}^{\infty} \Lambda^k(V) = \bigoplus_{k=0}^{\infty} T^k(V)/J^k(V). \tag{3.13}$$

Definition 3.3. The product of two elements $a, b \in \Lambda(V)$ in the exterior algebra is written as $a \wedge b$ and

$$a \wedge b = a \otimes b + J(V). \tag{3.14}$$

Remark 3.1. By definition of $J(V)$ the exterior product is **alternating** on elements in $V = T^1(V) = \Lambda^1(V)$, i.e. we have

$$x \wedge x = 0 \tag{3.15}$$

for all $x \in V$. This implies the **anticommutativity**, i.e.

$$x \wedge y = -y \wedge x \tag{3.16}$$

for all $x, y \in V$. (For $\mathrm{char}(K) \neq 2$ from the anticommutativity of a K-bilinear map it follows that is is alternating. Therefore for $\mathrm{char}(K) \neq 2$ we can define the ideal $J(V)$ as the ideal in $T(V)$ generated by the elements of the form $x \otimes y + y \otimes x$ with $x, y \in V$. In the super case we will use a generalisation of this definition of $J(V)$.)

More generally the exterior product is **anticommutative graded**, i.e.

$$a \wedge b = (-1)^{kl} b \wedge a \tag{3.17}$$

for all $a \in \Lambda^k(V), b \in \Lambda^l(V)$.

Remark 3.2. As we have already seen, the exterior algebra can be decomposed into a direct sum of components of different degrees. $\Lambda^k(V)$ is the subspace of degree k and is generated by all exterior products $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ consisting of k factors $v_i \in V$. (These products are called k-*multivectors*. It follows that every element in $\Lambda(V)$ can be written as a sum of multivectors.) If $\dim(V) = n$ and $\{e_1, \ldots, e_n\}$ is a basis of V, then

$$\{ e_{i_1} \wedge \ldots \wedge e_{i_k} | \, 1 \leq i_1 < \ldots < i_k \leq n \}$$

is a basis of $\Lambda^k(V)$ (and $\Lambda^k(V) = \{0\}$ for $k > n$). Then

$$\dim\left(\Lambda^k\left(V\right)\right) = \binom{n}{k} \tag{3.18}$$

holds and

$$\dim\left(\Lambda\left(V\right)\right) = 2^n. \tag{3.19}$$

The exterior algebra together with the exterior product has the structure of a $\mathbb{Z}$-graded algebra. In particular

$$\Lambda^k(V) \wedge \Lambda^p(V) \subseteq \Lambda^{k+p}(V) \tag{3.20}$$

holds. (The exterior algebra has a grading which is inherited from the tensor algebra because it is formed as the quotient by a homogeneous ideal. For the universal enveloping algebra (cf. Remark 1.7) this was not the case.)

Let us return to our original description. We identify the vector space we had simply called $\Lambda(\mathfrak{g})$ with the vector space $\Lambda(\mathfrak{g})$ by the above construction via

$$c^{i_1} \ldots c^{i_k} \left|0\right\rangle = l_{i_1} \wedge \ldots \wedge l_{i_k} \tag{3.21}$$

for $i_1, \ldots, i_k \in \{1, \ldots, n\}$. (Here $|0\rangle = 1 \in K$ is the empty exterior product.)

This corresponds to the following definitions for the creation and annihilation operators:

Creation Operators: Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ with $\alpha_j \in \{1, \ldots, n\}$ for $k = 1, \ldots, k$ and assume that none of the α_j's appear twice since otherwise the state vanishes. Define for $i \in \{1, \ldots, n\}$:

$$c^i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \begin{cases} l_i \wedge l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k} & \text{if } \alpha_j \neq i \text{ for all } j \\ 0 & \text{if } \alpha_j = i \text{ for a } j. \end{cases} \tag{3.22}$$

Annihilation operators: Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ with $\alpha_j \in \{1, \ldots, n\}$ for $k = 1, \ldots, k$ and assume that none of the α_j's appears twice since otherwise the state vanishes. Define for $i \in \{1, \ldots, n\}$:

$$\begin{aligned} b_i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \\ = \begin{cases} (-1)^{j-1} l_{\alpha_1} \wedge \ldots l_{\alpha_{j-1}} \wedge l_{\alpha_{j+1}} \wedge \ldots \wedge l_{\alpha_k} & \text{if } \alpha_j = i \text{ for a } j \\ 0 & \text{if } \alpha_j \neq i \text{ for all } j. \end{cases} \end{aligned} \tag{3.23}$$

Herefrom we can conversely derive the relations for the creation and annihilation operators (3.1, 3.2, 3.3) by using the properties of the exterior product:

Proposition 3.1. *Let c^i and b_i $(i = 1, \ldots, n)$ be defined as above. The the creation and annihilation operators are well-defined as linear operators in* $\mathrm{End}(\Lambda(\mathfrak{g}))$ *and*

$$\{b_i, c^j\} = \delta_i^j \tag{3.24}$$

as well as

$$\{b_i, b_j\} = \{c^i, c^j\} = 0 \tag{3.25}$$

hold for all $i, j \in \{1, \ldots, n\}$.

Proof. The proof is left to the reader. $\square$

The next theorem shows that the Lie algebra $\mathfrak{g}$ and the vector space $\Lambda(\mathfrak{g})$ are intimately related namely that $\Lambda(\mathfrak{g})$ can be made a (non-trivial) $\mathfrak{g}$-module.

Theorem 3.1. *The Lie algebra $\mathfrak{g}$ has a representation π on $\Lambda(\mathfrak{g})$ given by*

$$\lambda_\alpha := \pi(l_\alpha) := -f^\gamma_{\alpha\beta} c^\beta b_\gamma = f^\gamma_{\beta\alpha} c^\beta b_\gamma \in \mathrm{End}(\Lambda(\mathfrak{g}))\,. \tag{3.26}$$

In particular

$$[\lambda_\alpha, \lambda_\beta] = f^\gamma_{\alpha\beta} \lambda_\gamma. \tag{3.27}$$

Proof. The map π is defined on the basis vectors of $\mathfrak{g}$ and hence defines a unique linear map $\pi : \mathfrak{g} \to \mathrm{End}(\Lambda(\mathfrak{g})) = \mathfrak{gl}(\Lambda(\mathfrak{g}))$. We have to show that this map respects the Lie bracket (here: the commutator) on $\mathfrak{g}$ and $\mathfrak{gl}(\Lambda(\mathfrak{g}))$. Obviously, this follows from

$$[\lambda_\alpha, \lambda_\beta] = f^\gamma_{\alpha\beta} \lambda_\gamma,$$

which is what we will show in the following.

We have

$$[\lambda_\alpha, \lambda_\beta] = \left[-f^\nu_{\alpha\mu} c^\mu b_\nu, -f^\sigma_{\beta\rho} c^\rho b_\sigma\right] = f^\nu_{\alpha\mu} f^\sigma_{\beta\rho} \left(c^\mu b_\nu c^\rho b_\sigma - c^\rho b_\sigma c^\mu b_\nu\right). \tag{3.28}$$

By application of the anticommutation relations (3.1) and (3.2) one gets

$$\begin{aligned}
[\lambda_\alpha, \lambda_\beta] &= f^\nu_{\alpha\mu} f^\sigma_{\beta\rho} \left(c^\mu \left(\delta^\rho_\nu + c^\rho b_\nu\right) b_\sigma - c^\rho \left(\delta^\mu_\sigma + c^\mu b_\sigma\right) b_\nu\right) \\
&= f^\nu_{\alpha\mu} f^\sigma_{\beta\rho} \left(c^\mu b_\sigma \delta^\rho_\nu + \cancel{c^\mu c^\rho b_\nu b_\sigma} - c^\rho b_\nu \delta^\mu_\rho - \cancel{c^\rho c^\mu b_\sigma b_\nu}\right) \\
&= f^\nu_{\alpha\mu} f^\sigma_{\beta\nu} c^\mu b_\sigma - f^\nu_{\alpha\mu} f^\mu_{\beta\rho} c^\rho b_\nu \\
&= \left(f^\nu_{\alpha\mu} f^\sigma_{\beta\nu} - f^\nu_{\beta\mu} f^\sigma_{\alpha\nu}\right) c^\mu b_\sigma,
\end{aligned} \tag{3.29}$$

where we have renamed some indices which are summed over in the last step. By application of the Jacobi identity (cf. (1.7) in Remark 1.3)

$$f^\nu_{\beta\mu} f^\sigma_{\alpha\nu} + f^\nu_{\mu\alpha} f^\sigma_{\beta\nu} + f^\nu_{\alpha\beta} f^\sigma_{\mu\nu} = 0$$

we finally get

$$[\lambda_\alpha, \lambda_\beta] = f^\nu_{\alpha\beta} f^\sigma_{\mu\nu} c^\mu b_\sigma = -f^\nu_{\alpha\beta} f^\sigma_{\nu\mu} c^\mu b_\sigma = f^\nu_{\alpha\beta} \lambda_\nu. \tag{3.30}$$

$\square$

Let us look at the Lie algebra $\mathfrak{g}$ and an arbitrary representation ρ of $\mathfrak{g}$ on a vector space V (write $L_\alpha := \rho(l_\alpha) \in \mathrm{End}(V)$). Furthermore we have the representation π of $\mathfrak{g}$ on the vector space $\Lambda(\mathfrak{g})$ (with $\lambda_\alpha = \pi(l_\alpha) \in \mathrm{End}(\Lambda(\mathfrak{g}))$).

The vector space C, on which we want to define the cohomology be defined as $C = V \otimes \Lambda(\mathfrak{g})$. Then C becomes a graded vector space by

$$C = V \otimes \Lambda(\mathfrak{g}) = V \otimes \left(\bigoplus_{i=0}^{n} \Lambda^i(\mathfrak{g}) \right) = \bigoplus_{i=0}^{n} \left(V \otimes \Lambda^i(\mathfrak{g}) \right) = \bigoplus_{i=0}^{n} C^i, \tag{3.31}$$

where we put $C^i = V \otimes \Lambda^i(\mathfrak{g})$ $(i \in \mathbb{Z})$. (In particular $C^i = V \otimes \{0\} = \{0\}$ for $i \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}$.) C inherits the grading from $\Lambda(\mathfrak{g})$, thus it is ordered by ghost number.

In the following we will define a (linear) operator Q on C with the property

$$Q^2 = 0 \tag{3.32}$$

and

$$[I_V \otimes U, Q] = Q, \tag{3.33}$$

where I_V is the identity operator on V and U is the ghost number operator (cf. (3.5)) on $\Lambda(\mathfrak{g})$. The second property implies that Q raises the ghost number by one 1, i.e.

$$Q(C^i) \subseteq C^{i+1} \tag{3.34}$$

for $i \in 0, 1, \ldots, n$. We put

$$Q^i = Q|_{C^i} . \tag{3.35}$$

Graphically this is written as a sequence:

$$\{0\} \longrightarrow C^0 \xrightarrow{Q^0} C^1 \xrightarrow{Q^1} \ldots \xrightarrow{Q^{n-1}} C^n \xrightarrow{Q^n} \{0\} . \tag{3.36}$$

The condition $Q^2 = 0$ then implies that

$$\mathrm{im}\left(Q^{i-1}\right) \subseteq \ker\left(Q^i\right) . \tag{3.37}$$

If $\mathrm{im}\left(Q^{i-1}\right) = \ker\left(Q^i\right)$, the above sequence is an *exact sequence* (in the category of vector spaces and linear maps).

In general one defines:

Definition 3.4. Let $\left(C^k\right)_{k\in\mathbb{Z}}$ be a sequence of K-vector spaces and let

$$\mathrm{d}^k : C^k \to C^{k+1} \tag{3.38}$$

be a K-vector space homomorphism for all $k \in \mathbb{Z}$ (called *coboundary operator*) with $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$ for all $k \in \mathbb{Z}$. We then call

$$\mathcal{C} = \left(C^k, \mathrm{d}^k : C^k \to C^{k+1}\right)_{k\in\mathbb{Z}} \tag{3.39}$$

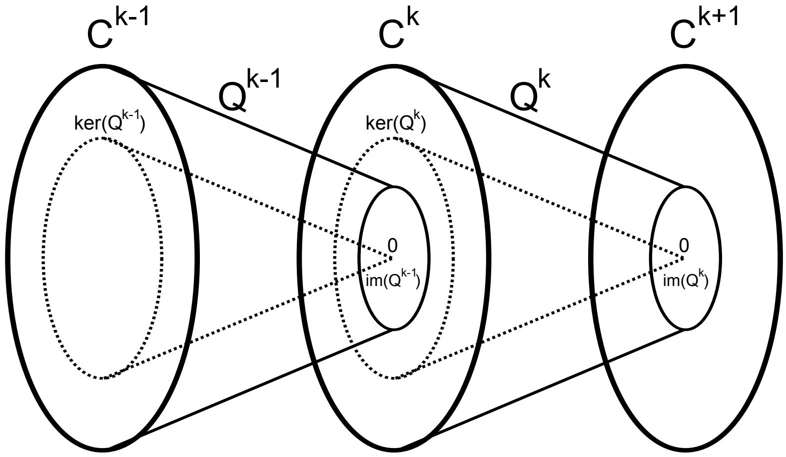

Figure 3.1: The k-th position of the cochain complex $\left(C^k, Q^k\right)_{k\in\mathbb{Z}}$.

a *cochain complex*. The situation is depicted in Figure 3.1. An element in C^k is called *k-cochain*. If for a cochain $\varphi \in C^k$ it holds that $\mathrm{d}^k(\varphi) = 0$, then φ is called a *cocycle*. If there exists a $\psi \in C^{k-1}$ with $\mathrm{d}^{k-1}(\psi) = \varphi$ then φ is called a *coboundary*. Let $Z^k(\mathcal{C}) = \ker(\mathrm{d}^k)$ denote the subspace of all cocycles and $B^k(\mathcal{C}) = \mathrm{im}(\mathrm{d}^{k-1})$ the subspace of all coboundaries.

We define the *k-th cohomological space* of the cochain complex $\mathcal{C}$ as

$$H^k(\mathcal{C}) = Z^k(\mathcal{C})/B^k(\mathcal{C}). \tag{3.40}$$

An element of the cohomological space $H^k(\mathcal{C})$ is called *cohomology class*. A cochain complex is called *exact at position k* if $Z^k(\mathcal{C}) = B^k(\mathcal{C})$ (iff. $H^k(\mathcal{C}) = \{0\}$). A cochain complex is called *exact* if it is exact at every position. In this case we say the cohomology is *trivial*.

We now want to measure how inexact the cochain complex $\left(C^i, Q^i\right)_{i\in\mathbb{Z}}$ is at position i. For this we look at the i-th cohomological space

$$H^i = \ker\left(Q^i\right)/\mathrm{im}\left(Q^{i-1}\right) \tag{3.41}$$

and determine its vector space dimension.

We call $H(\mathfrak{g}, V) := \left(H^i\right)_{i\in\mathbb{Z}}$ the *cohomology of $\mathfrak{g}$ with coefficients in V*.

Remark 3.3. We already know that $\Lambda^i(\mathfrak{g}) = \{0\}$ and thus $C^i = \{0\}$ for $i \in \mathbb{Z} \setminus \{1, 2, \ldots, n\}$. Hence $H^i = \{0\} / \{0\} = \{0\}$ for $i \in \mathbb{Z} \setminus \{1, 2, \ldots, n\}$ and therefore $\left(C^i, Q^i\right)_{i\in\mathbb{Z}}$ is exact at the corresponding positions i. We thus only have to study $H^0, H^1, \ldots, H^n$.

The following theorem shows how one can define an operator Q which fulfils the conditions (3.32) and (3.32).

Theorem 3.2. *The operator Q defined on C as*

$$\begin{aligned} Q &= \rho(l_\alpha) \otimes c^\alpha + \frac{1}{2} I_V \otimes c^\alpha \pi(l_\alpha) \\ &= L_\alpha \otimes c^\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} I_V \otimes c^\alpha c^\beta b_\gamma, \end{aligned} \tag{3.42}$$

fulfils

$$Q^2 = 0$$

and

$$[I_V \otimes U, Q] = Q.$$

Proof. For readability we omit the tensor product sign and I_V and we will ignore the order of the factors of the tensor product. We have

$$[U, Q] = [L_\alpha c^\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta b_\gamma, U] = L_\alpha [c^\alpha, U] - \frac{1}{2} f^\gamma_{\alpha\beta} [c^\alpha c^\beta b_\gamma, U]. \tag{3.43}$$

Furthermore

$$[U, c^\alpha] = [c^\beta b_\beta, c^\alpha] = c^\beta \underbrace{[b_\beta, c^\alpha]}_{=\delta^\alpha_\beta} + \underbrace{[c^\beta, c^\alpha]}_{=0} b_\beta = \delta^\alpha_\beta c c^\beta = c^\alpha \tag{3.44}$$

and analogously one can show

$$[c^\alpha c^\beta b_\gamma, U] = c^\alpha c^\beta b_\gamma. \tag{3.45}$$

This proves the second claim.

The first claim is more difficult to show. We have

$$\begin{aligned} Q^2 &= \left(L_\alpha c^\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta b_\gamma \right) \left(L_\mu c^\mu - \frac{1}{2} f^\rho_{\mu\nu} c^\mu c^\nu b_\rho \right) \\ &= \underbrace{L_\alpha L_\beta c^\alpha c^\beta}_{=:(1)} - \underbrace{\frac{1}{2} f^\gamma_{\alpha\beta} \left(c^\alpha c^\beta b_\gamma c^\mu + c^\mu c^\alpha c^\beta b_\gamma \right) L_\mu}_{=:(2)} + \underbrace{\frac{1}{4} f^\gamma_{\alpha\beta} f^\rho_{\mu\nu} c^\alpha c^\beta b_\gamma c^\mu c^\nu b_\rho}_{=:(3)}. \end{aligned} \tag{3.46}$$

We rewrite (2) and get

$$\begin{aligned} (2) &= \frac{1}{2} f^\gamma_{\alpha\beta} \left(c^\alpha c^\beta b_\gamma c^\mu + c^\mu c^\alpha c^\beta b_\gamma \right) L_\mu = \frac{1}{2} f^\gamma_{\alpha\beta} \left(c^\alpha c^\beta b_\gamma c^\mu + (-1)^2 c^\alpha c^\beta c^\mu b_\gamma \right) L_\mu \\ &= \frac{1}{2} f^\gamma_{\alpha\beta} \left(c^\alpha c^\beta b_\gamma c^\mu + c^\alpha c^\beta \left(\delta^\mu_\gamma - b_\gamma c^\mu \right) \right) L_\mu \\ &= \frac{1}{2} f^\gamma_{\alpha\beta} \left(\cancel{c^\alpha c^\beta b_\gamma c^\mu} - c^\alpha c^\beta \delta^\mu_\gamma - \cancel{c^\alpha c^\beta b_\gamma c^\mu} \right) L_\mu \\ &= \frac{1}{2} f^\mu_{\alpha\beta} c^\alpha c^\beta L_\mu = \frac{1}{2} c^\alpha c^\beta [L_\alpha, L_\beta] = \frac{1}{2} c^\alpha c^\beta L_\alpha L_\beta - \frac{1}{2} c^\alpha c^\beta L_\beta L_\alpha \\ &= \frac{1}{2} c^\alpha c^\beta L_\alpha L_\beta - \frac{1}{2} c^\beta c^\alpha L_\alpha L_\beta = \frac{1}{2} c^\alpha c^\beta L_\alpha L_\beta + \frac{1}{2} c^\alpha c^\beta L_\alpha L_\beta = c^\alpha c^\beta L_\alpha L_\beta \\ &= (1). \end{aligned} \tag{3.47}$$

Hence (1) and (2) cancel.

To show that (3) = 0 we first observe that from the Jacobi identity

$$f^{\delta}_{\beta\gamma} f^{\varepsilon}_{\alpha\delta} + f^{\delta}_{\gamma\alpha} f^{\varepsilon}_{\beta\delta} + f^{\delta}_{\alpha\beta} f^{\varepsilon}_{\gamma\delta} = 0 \tag{3.48}$$

follows that

$$\begin{aligned}
0 &= \left(f^{\delta}_{\beta\gamma} f^{\varepsilon}_{\alpha\delta} + f^{\delta}_{\gamma\alpha} f^{\varepsilon}_{\beta\delta} + f^{\delta}_{\alpha\beta} f^{\varepsilon}_{\gamma\delta} \right) c^{\alpha} c^{\beta} c^{\gamma} b_{\varepsilon} \\
&= f^{\delta}_{\beta\gamma} f^{\varepsilon}_{\alpha\delta} c^{\alpha} c^{\beta} c^{\gamma} b_{\varepsilon} + f^{\delta}_{\gamma\alpha} f^{\varepsilon}_{\beta\delta} c^{\alpha} c^{\beta} c^{\gamma} b_{\varepsilon} + f^{\delta}_{\alpha\beta} f^{\varepsilon}_{\gamma\delta} c^{\alpha} c^{\beta} c^{\gamma} b_{\varepsilon} \\
&= f^{\delta}_{\beta\gamma} c^{\beta} c^{\gamma} f^{\varepsilon}_{\alpha\delta} c^{\alpha} b_{\varepsilon} + f^{\delta}_{\gamma\alpha} c^{\gamma} c^{\alpha} f^{\varepsilon}_{\beta\delta} c^{\beta} b_{\varepsilon} + f^{\delta}_{\alpha\beta} c^{\alpha} c^{\beta} f^{\varepsilon}_{\gamma\delta} c^{\gamma} b_{\varepsilon} \\
&= f^{\delta}_{\beta\gamma} c^{\beta} c^{\gamma} \lambda_{\delta} + f^{\delta}_{\gamma\alpha} c^{\gamma} c^{\alpha} \lambda_{\delta} + f^{\gamma}_{\alpha\beta} c^{\alpha} c^{\beta} \lambda_{\delta} \\
&= 3 c^{\alpha} c^{\beta} [\lambda_{\alpha}, \lambda_{\beta}]
\end{aligned} \tag{3.49}$$

and thus

$$c^{\alpha} c^{\beta} [\lambda_{\alpha}, \lambda_{\beta}] = 0. \tag{3.50}$$

On the other hand

$$\begin{aligned}
c^{\alpha} c^{\beta} \{\lambda_{\alpha}, \lambda_{\beta}\} &= c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} + c^{\alpha} c^{\beta} \lambda_{\beta} \lambda_{\alpha} \\
&= c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} + c^{\beta} c^{\alpha} \lambda_{\alpha} \lambda_{\beta} \\
&= c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} - c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} \\
&= 0,
\end{aligned} \tag{3.51}$$

which also shows

$$c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} = 0. \tag{3.52}$$

A short calculation shows

$$[\lambda_{\alpha}, c^{\mu}] = f^{\mu}_{\alpha\beta} c^{\beta}. \tag{3.53}$$

Together this means

$$\begin{aligned}
4 \cdot (3) &= f^{\gamma}_{\alpha\beta} f^{\rho}_{\mu\nu} c^{\alpha} c^{\beta} b_{\gamma} c^{\mu} c^{\nu} b_{\rho} = c^{\alpha} \lambda_{\alpha} c^{\mu} \lambda_{\mu} = c^{\alpha} \left(f^{\mu}_{\alpha\beta} c^{\beta} + c^{\mu} \lambda_{\alpha} \right) \lambda_{\mu} \\
&= c^{\alpha} c^{\beta} [\lambda_{\alpha}, \lambda_{\beta}] + c^{\alpha} c^{\beta} \lambda_{\alpha} \lambda_{\beta} = 0,
\end{aligned} \tag{3.54}$$

which concludes the proof. □

3.2 Example of $\mathfrak{sl}_2(\mathbb{C})$

In this section we will deal with the cohomology of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We will step by step develop more elaborate concepts (cf. [Tan95b, Tan95a]) to make the calculation of the cohomology easier.

Example 3.1. We want to study the cohomology of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (cf. example 1.4). We choose the basis $\{a_1, a_2, a_3\}$ and look at the irreducible $(2j+1)$-dimensional representations ρ_j on V_j.

We first look at the simplest case $j = 0$. This is in fact the trivial representation on $V_0 = \mathbb{C}^1 = \mathbb{C}$, i.e.

$$\rho_0(a_i) = 0 \in \mathbb{C}^{1\times 1} \tag{3.55}$$

for $i = 1, 2, 3$. The Grassmann algebra $\Lambda(\mathfrak{g})$ has independently of the representation dimension $2^n = 2^3 = 8$ where $n = 3$ is the dimension of the Lie algebra. The vector space C is hence

$$C = V_0 \otimes \Lambda(\mathfrak{g}) = \mathbb{C} \otimes \Lambda(\mathfrak{g}) = \Lambda(\mathfrak{g}) \tag{3.56}$$

and decomposes into its graded components

$$C = C^0 \oplus C^1 \oplus C^2 \oplus C^3 \tag{3.57}$$

with

$$\begin{aligned} C^0 &= \operatorname{span}\{|0\rangle\} \\ C^1 &= \operatorname{span}\left\{c^1 |0\rangle, c^2 |0\rangle, c^3 |0\rangle\right\} \\ C^2 &= \operatorname{span}\left\{c^1c^2 |0\rangle, c^1c^3 |0\rangle, c^2c^3 |0\rangle\right\} \\ C^0 &= \operatorname{span}\left\{c^1c^2c^3 |0\rangle\right\}. \end{aligned} \tag{3.58}$$

The operator Q is of the form

$$Q = L_\alpha \otimes c^\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} I_V \otimes c^\alpha c^\beta b_\gamma = -\frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta b_\gamma \tag{3.59}$$

and with $f^\gamma_{\alpha\beta} = -\varepsilon_{\alpha\beta}{}^\gamma$ one calculates

$$Q = c^1c^2b_3 + c^2c^3b_1 + c^3c^1b_2. \tag{3.60}$$

Theorem 3.3. *The cohomology of $\mathfrak{sl}_2(\mathbb{C})$ w.r.t. the representation ρ_0 is trivial except for the beginning at $i = 0$ and the end at $i = 3$ with $\dim(H^0) = \dim(H^3) = 1$. The situation is as depicted in Table 3.1.*

i	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$\dim(C^i)$	$\cdots$	0	0	1	3	3	1	0	0	$\cdots$
$\dim\left(\ker\left(Q^i\right)\right)$	$\cdots$	0	0	1	0	3	1	0	0	$\cdots$
$\dim\left(\operatorname{im}\left(Q^{i-1}\right)\right)$	$\cdots$	0	0	0	0	3	0	0	0	$\cdots$
$\dim(H^i)$	$\cdots$	0	0	1	0	0	1	0	0	$\cdots$

Table 3.1: Cohomology of $\mathfrak{sl}_2(\mathbb{C})$ w.r.t. ρ_0.

Proof. With the above preparations the calculation of the cocycles and cochains is now simple. We have $\dim\left(\operatorname{im}\left(Q^{-1}\right)\right) = 0$. Further

$$Q\left|0\right\rangle = 0, \tag{3.61}$$

which implies $\dim\left(\operatorname{im}\left(Q^{0}\right)\right) = 0$ and $\dim\left(\ker\left(Q^{0}\right)\right) = \dim(C^0) - \dim\left(\operatorname{im}\left(Q^{0}\right)\right) = 1 - 0 = 1$. From

$$Qc^1\left|0\right\rangle = c^2c^3\left|0\right\rangle, \quad Qc^2\left|0\right\rangle = c^3c^1\left|0\right\rangle, \quad Qc^3\left|0\right\rangle = c^1c^2\left|0\right\rangle \tag{3.62}$$

follows $\dim\left(\operatorname{im}\left(Q^{1}\right)\right) = 3$ and $\dim\left(\ker\left(Q^{1}\right)\right) = 3 - 3 = 0$ and from

$$Qc^1c^2\left|0\right\rangle = Qc^2c^3\left|0\right\rangle = Qc^3c^1\left|0\right\rangle = 0 \tag{3.63}$$

follows that $\dim\left(\operatorname{im}\left(Q^{2}\right)\right) = 0$ and $\dim\left(\ker\left(Q^{2}\right)\right) = 3-0 = 3$. Clearly $\dim\left(\ker\left(Q^{3}\right)\right) = 1 - 0 = 1$. □

In the same way we can calculate the cohomology for representations with higher j, e.g. $j = \frac{1}{2}$. It can be shown however (cf. Example 3.3), that for all other representations with $j > 0$ the cohomology is trivial. For $j = \frac{1}{2}$ we have the situation depicted in Table 3.2.

i	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$\dim(C^i)$	$\cdots$	0	0	2	6	6	2	0	0	$\cdots$
$\dim\left(\ker\left(Q^i\right)\right)$	$\cdots$	0	0	0	2	4	2	0	0	$\cdots$
$\dim\left(\operatorname{im}\left(Q^{i-1}\right)\right)$	$\cdots$	0	0	0	2	4	2	0	0	$\cdots$
$\dim(H^i)$	$\cdots$	0	0	0	0	0	0	0	0	$\cdots$

Table 3.2: Cohomology of $\mathfrak{sl}_2(\mathbb{C})$ w.r.t. $\rho_{\frac{1}{2}}$.

Let us now look at the example of $\mathfrak{sl}_2(\mathbb{C})$ with the finite-dimensional irreducible representation ρ_j we just dealt with from another perspective and see how the cohomology can be obtained without much calculation. First some preliminary considerations:

Let $\mathfrak{g}$ be an arbitrary n-dimensional Lie algebra and ρ a d-dimensional representation on V, i.e. $\dim(V) = d$. Let $\{l_1, \ldots, l_n\}$ be a basis of $\mathfrak{g}$ and f_{ij}^k the corresponding structure constants. We have seen (cf. Theorem 3.1) that by $\pi(l_\alpha) = f_{\beta\alpha}^{\gamma}c^\beta b_\gamma = -f_{\alpha\beta}^{\gamma}c^\beta b_\gamma$ we can define a representation of $\mathfrak{g}$ on $\Lambda(\mathfrak{g})$. Correspondingly, by $(\tilde{\rho}(x))(v \otimes a) = (\rho(x)v) \otimes a$ and $(\tilde{\pi}(x))(v \otimes a) = v \otimes (\pi(x)a)$ for $x \in \mathfrak{g}$, $v \in V$ and $a \in \Lambda(\mathfrak{g})$ representations $\tilde{\rho}$ and $\tilde{\pi}$ on $C = V \otimes \Lambda(\mathfrak{g})$ are defined.

Definition 3.5. We define the *natural representation* ρ_{nat} of $\mathfrak{g}$ on C as

$$\rho_{\text{nat}}(x)(v \otimes a) = (\tilde{\rho}(x) + \tilde{\pi}(x))(v \otimes a) = (\rho(x)v) \otimes a + v \otimes (\pi(x)a). \tag{3.64}$$

Lemma 3.1. *It holds*

$$\{Q, I_V \otimes b_\mu\} = \rho_{nat}(l_\mu). \tag{3.65}$$

Proof. For better readability we omit again the tensor product sign and I_V and set $L_\alpha := \rho(l_\alpha)$ as well as $\lambda_\alpha := \pi(l_\alpha)$. It holds

$$\begin{aligned}\{Q, b_\mu\} &= \{L_\alpha c^\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta b_\gamma, b_\mu\} = L_\alpha \{c^\alpha, b_\mu\} - \frac{1}{2} f^\gamma_{\alpha\beta} \{c^\alpha c^\beta b_\gamma, b_\mu\} \\ &\overset{(!)}{=} L_\alpha \delta^\alpha_\mu - \frac{1}{2} f^\gamma_{\alpha\beta} \left(-\delta^\beta_\mu c^\alpha b_\gamma + \delta^\alpha_\mu c^\beta b_\gamma \right) = L_\mu - f^\gamma_{\mu\beta} c^\beta b_\gamma \\ &= L_\mu + \lambda_\mu = \rho_{\text{nat}}(l_\mu).\end{aligned} \tag{3.66}$$

For (!) we used

$$\{c^\alpha, b_\mu\} = \delta^\alpha_\mu \tag{3.67}$$

on the one hand and

$$\begin{aligned}\left\{ c^\alpha c^\beta b_\gamma, b_\mu \right\} &= c^\alpha c^\beta b_\gamma b_\mu + b_\mu c^\alpha c^\beta b_\gamma = -c^\alpha c^\beta b_\mu b_\gamma + b_\mu c^\alpha c^\beta b_\gamma \\ &= -\delta^\beta_\mu c^\alpha b_\gamma + c^\alpha b_\mu c^\beta b_\gamma + b_\mu c^\alpha c^\beta b_\gamma \\ &= -\delta^\beta_\mu c^\alpha b_\gamma + \delta^\alpha_\mu c^\beta b_\gamma - \cancel{b_\mu c^\alpha c^\beta b_\gamma} + \cancel{b_\mu c^\alpha c^\beta b_\gamma} \\ &= -\delta^\beta_\mu c^\alpha b_\gamma + \delta^\alpha_\mu c^\beta b_\gamma\end{aligned} \tag{3.68}$$

on the other hand. □

Remark 3.4. Obviously $\rho_{\text{nat}}(x)$ leaves the ghost number invariant for all $x \in \mathfrak{g}$. It is therefore sensible to restrict ρ_{nat} to C^k. We denote by ρ^k_{nat} the subrepresentation ρ_{nat} on C^k.

Lemma 3.2. *It holds*

$$[\rho_{nat}(x), Q] = 0 \tag{3.69}$$

for all $x \in \mathfrak{g}$ or equivalently

$$\rho^{k+1}_{\text{nat}}(x) Q^k = Q^k \rho^k_{\text{nat}}(x) \in \text{Hom}\left(C^k, C^{k+1} \right) \tag{3.70}$$

for all $k \in \mathbb{Z}$.

Proof. We prove the second assertion for $x = l_\mu$. The assertion of the lemma then follows by linearity of ρ_{nat}. (We again use the simplified notation as is the proof of Lemma 3.1.) By Lemma 3.1 we have

$$\rho^k_{\text{nat}}(l_\mu) = Q^{k-1} b_\mu + b_\mu Q^k. \tag{3.71}$$

This implies

$$Q^k \rho^k_{\text{nat}}(l_\mu) = \underbrace{Q^k Q^{k-1}}_{=0} b_\mu + Q^k b_\mu Q^k = Q^k b_\mu Q^k. \tag{3.72}$$

Analogously

$$\rho^{k+1}_{\text{nat}}(l_\mu) Q^k = Q^k b_\mu Q^k + b_\mu \underbrace{Q^{k+1} Q^k}_{=0} = Q^k b_\mu Q^k. \tag{3.73}$$

This proves the assertion. □

Proposition 3.2. *Let W be a subspace of C^k, which is stable under the natural representation ρ^k_{nat} of $\mathfrak{g}$, i.e. it forms a $\mathfrak{g}$-module w.r.t. the restriction of ρ^k_{nat} on W. Then $U := Q(W) \subseteq C^{k+1}$ is also stable under $\rho^{k+1}_{\mathrm{nat}}$.*

Proof. Let W be in C^k with $\rho^k_{\mathrm{nat}}(\mathfrak{g})W \subseteq W$. Then the following holds:

$$\rho^{k+1}_{\mathrm{nat}}(\mathfrak{g})U = \rho^{k+1}_{\mathrm{nat}}(\mathfrak{g})Q^k W = Q^k \rho^k_{\mathrm{nat}}(\mathfrak{g})W \subseteq Q^k W = U. \tag{3.74}$$

□

Proposition 3.3. *Let the restriction of ρ^k_{nat} on a subspace W of C^k be an irreducible subrepresentation of dimension d of $\mathfrak{g}$. Then $U := Q(W)$ carries again a d-dimensional irreducible representation of $\mathfrak{g}$, namely the restriction of ρ^{k+1}_{nat} on U or $U = \{0\}$.*

Proof. Let $\{w_1, \ldots, w_d\}$ be a basis of W. Then $\{u_1, \ldots, u_d\}$ with $u_i := Qw_i$ for $i = 1, \ldots, d$ is a spanning set of $Q(W)$. W.l.o.g. we can choose the basis of W such that $\{u_1, \ldots, u_l\}$ $(0 \leq l \leq d)$ is a basis of U and $u_{l+1} = \ldots = u_d = 0$. Then $\rho_{\mathrm{nat}}(l_\alpha)w_i = c^j_{i,\alpha} w_j$ for uniquely determined $c^j_{i,\alpha}$ $(1 \leq i, j \leq d, 1 \leq \alpha \leq n)$. Then

$$\rho^{k+1}_{\mathrm{nat}}(l_\alpha)u_i = \rho^{k+1}_{\mathrm{nat}}(l_\alpha)Qw_i = Q\rho^k_{\mathrm{nat}}(l_\alpha)w_i = Qc^j_{i,\alpha}w_j = c^j_{i,\alpha}u_j. \tag{3.75}$$

$\rho^{k+1}_{\mathrm{nat}}$ hence fulfils on U the same defining relations as ρ^k_{nat} on W. We have $u_{l+1} = \ldots = u_d = 0$ but on the other hand the u_i's with $i \geq l+1$ obey the relation $0 = \rho^{k+1}_{\mathrm{nat}}(l_\alpha)u_i = c^j_{i,\alpha}u_j = \sum_{j=1}^{l} c^j_{i,\alpha}u_j$, which means that $c^j_{i,\alpha} = 0$ for $l+1 \leq i \leq d$ and $1 \leq j \leq l$. This means nothing else but that span $\{u_{l+1}, \ldots, u_d\}$ forms a ρ^k_{nat}-invariant subspace of W. Since the restriction of ρ^k_{nat} on W is irreducible, $l = 0$ or $l = d$ follows. □

We now want to apply the concepts we just learned to the calculation of the cohomology of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.

Example 3.2. We first look at the representation $\pi(l_\alpha) := -f^\gamma_{\alpha\beta}c^\beta b_\gamma = f^\gamma_{\beta\alpha}c^\beta b_\gamma$ on $\Lambda(\mathfrak{g})$. Clearly, this representation is not irreducible since we have just seen that the spaces $\Lambda^k(\mathfrak{g})$ for $k = 0, 1, 2, 3$ are stable under it (π does not change the ghost number). However, it can be easily seen that the representations on the components $\Lambda^k(\mathfrak{g})$ are irreducible. In addition we already know all the irreducible representations of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (cf. Theorem 1.3). Hence, because of $\dim\left(\Lambda^0(\mathfrak{g})\right) = \dim\left(\Lambda^3(\mathfrak{g})\right) = 1$ and $\dim\left(\Lambda^1(\mathfrak{g})\right) = \dim\left(\Lambda^2(\mathfrak{g})\right) = 3$ we have

$$\pi \cong \pi_0 \oplus \pi_1 \oplus \pi_1 \oplus \pi_0. \tag{3.76}$$

Here, π_j denotes the $(2j+1)$-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ from Example 1.5. (One could also take the representations ρ_j from Example 1.4, which are of course equivalent.)

Further, we are given the representation π_j of $\mathfrak{sl}_2(\mathbb{C})$ on $V = V_j$, for which we want to calculate the cohomology. We can then look at the natural representation ρ^k_{nat} on $C^k = V \otimes \Lambda^k(\mathfrak{g})$, which is equivalent to the representation $\pi_j \otimes \pi_l$ with $l = 0, 1$. This

C^k	C^0	C^1	C^2	C^3
$\dim\left(C^k\right)$	$2j+1$	$3(2j+1)$	$3(2j+1)$	$2j+1$
$j=0$	(0)	(1)	(1)	(0)
$j={}^1\!/_2$	$({}^1\!/_2)$	$({}^1\!/_2)\oplus({}^3\!/_2)$	$({}^1\!/_2)\oplus({}^3\!/_2)$	$({}^1\!/_2)$
$j\geq 1$	(j)	$(j-1)\oplus(j)\oplus(j+1)$	$(j-1)\oplus(j)\oplus(j+1)$	(j)

Table 3.3: Irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ on $C^0,\dots,C^3$ starting from a $(2j+1)$-dimensional irreducible representation on V_j where $C^k = V_j \otimes \Lambda^k\left(\mathfrak{sl}_2\left(\mathbb{C}\right)\right)$.

representation decomposes according to Theorem 1.4 again into a sum of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$. The situation is depicted in Table 3.3. We have written (j) for π_j and the corresponding representation space.

We have seen in Proposition 3.2 that Q^k maps a subrepresentation of ρ^k_{nat} on a corresponding representation space $W \subseteq C^k$ again to a subspace $U \subseteq C^{k+1}$, which itself carries a subrepresentation of $\rho^{k+1}_{\mathrm{nat}}$. Because of the uniqueness of the irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ (cf. Theorem 1.3) and Proposition 3.3, when applying the operator Q on C^k and looking at a subspace which carries an irreducible representation (j) of $\mathfrak{sl}_2(\mathbb{C})$, this subspace can only be mapped to a subspace which itself carries such a representation (j) or to $\{0\}$. This enables us to determine the cohomology using the decomposition of C^k in several irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules.

Let us first have a look at the case $j=0$. Here, C^0 carries a representation (0). Since the corresponding subspace cannot be mapped to (0) in C^1, Q has to map it to $\{0\}$. This gives the sequence

$$\{0\} \xrightarrow{Q^{-1}} (0) \xrightarrow{Q^0} \{0\},$$

which however is not exact. If we in contrast look at C^1 and C^2, the following exact sequence is possible:

$$\{0\} \xrightarrow{Q^0} (1) \xrightarrow{Q^1} (1) \xrightarrow{Q^2} \{0\}.$$

Here we assumed that in the second step (1) is in fact mapped to (1), although this does not have to be the case since (1) could have also been mapped to $\{0\}$, which would not give an exact sequence any more. (The following arguments will be based on the assumption that the cohomology is as exact as possible.)

At C^3 we have the same problem as at the beginning at C^0 and we get the following inexact sequence:

$$\{0\} \xrightarrow{Q^2} (0) \xrightarrow{Q^3} \{0\}.$$

As a conclusion we have (where we omitted $\{0\}$ at the beginning and the end of each sequence))

$$
\begin{tikzcd}
C^0 & C^1 & C^2 & C^3 \\
(0) & (1) \arrow[r] & (1) & (0),
\end{tikzcd}
$$

which is exactly the cohomology calculated above.

In the case of $j = \frac{1}{2}$ we get

$$
\begin{tikzcd}
C^0 & C^1 & C^2 & C^3 \\
(1/2) \arrow[r] & (1/2) & (1/2) \arrow[r] & (1/2) \\
 & (3/2) \arrow[r] & (3/2), &
\end{tikzcd}
$$

which is exact at every position.

In the generic case for $j \geq 1$ we get

$$
\begin{tikzcd}
C^0 & C^1 & C^2 & C^3 \\
 & (j-1) \arrow[r] & (j-1) & \\
(j) \arrow[r] & (j) & (j) \arrow[r] & (j) \\
 & (j+1) \arrow[r] & (j+1), &
\end{tikzcd}
$$

which is also exact.

Another way of simplifying the calculation of the cohomology arises when instead of looking at the whole Lie algebra we only need to deal with a subalgebra. The following lemma holds:

Lemma 3.3. *Let $\mathfrak{g}$ be a n-dimensional Lie algebra with a d-dimensional representation ρ on V and let $(C^k, Q^k)_{k\in\mathbb{Z}}$ be the corresponding cochain complex. We look at the natural representation on C^k. The following holds:*

1. $\ker\left(Q^k\right)$ *is stable under* $\mathfrak{g}$.

2. $\mathrm{im}\left(Q^{k-1}\right)$ *is stable under* $\mathfrak{g}$.

3. $\rho^k_{nat}(\mathfrak{g})\left(\ker\left(Q^k\right)\right) \subseteq \mathrm{im}\left(^{k-1}\right)$.

Proof. (1): Let $w \in \ker\left(Q^k\right)$. Then by Lemma 3.2

$$Q^k \rho^k_{\mathrm{nat}}(x) w = \rho^{k+1}_{\mathrm{nat}}(x) Q^k w = 0. \tag{3.77}$$

(2): Let $w \in \mathrm{im}\left(Q^{k-1}\right)$, i.e. $w = Q^{k-1}v$ for a $v \in C^{k-1}$. Then again by Lemma 3.2

$$\rho^k_{\mathrm{nat}}(x) w = \rho^k_{\mathrm{nat}}(x) Q^{k-1} v = Q^{k-1} \rho^{k-1}_{\mathrm{nat}} v \in \mathrm{im}\left(Q^{k-1}\right). \tag{3.78}$$

(3): Let $w \in \ker\left(Q^k\right)$ and $x = \alpha^i l_i \in \mathfrak{g}$ ($\alpha^i \in \mathbb{C}$ for $i = 1, \ldots, n$). Then by Lemma 3.1

$$\rho^k_{\mathrm{nat}}(x) w = \left(Q^{k-1}\alpha^i b_i + \alpha^i b_i Q^k\right) w = Q^{k-1}\alpha^i b_i w \in \mathrm{im}\left(Q^{k-1}\right). \tag{3.79}$$

□

We then have:

Lemma 3.4. *Let* $\mathfrak{g}$ *be an* n*-dimensional Lie algebra with a* d*-dimensional representation* ρ *on* V *and* $(C^k, Q^k)_{k\in\mathbb{Z}}$ *the corresponding cochain complex. Let* $\mathfrak{q}$ *be a subalgebra of* $\mathfrak{g}$ *such that the natural representations* ρ^k_{nat} *of* $\mathfrak{q}$ *on* C^k *are all completely reducible. Then the cohomology of* $H(\mathfrak{g}, V)$ *is identical to the cohomology one obtains from the subcomplex* $\left(C^k_{\mathfrak{q}},\, Q^k|_{C^k_{\mathfrak{q}}}\right)_{k\in\mathbb{Z}}$ *where* $C^k_{\mathfrak{q}}$ *denotes the subspace of the elements of* C^k *which are annihilated by* $\rho^k_{nat}(x)$ *for all* $x \in \mathfrak{q}$.

Proof. The subspace $C^k_{\mathfrak{q}}$ is trivially stable under $\mathfrak{q}$. Since the representation ρ^k_{nat} of $\mathfrak{q}$ on C^k is completely reducible, we can find a $\mathfrak{q}$-stable subspace $\widehat{C}^k$ of C^k such that $C^k = C^k_{\mathfrak{q}} \oplus \widehat{C}^k$. Then $Q^k\left(C^k_{\mathfrak{q}}\right) \subseteq C^{k+1}_{\mathfrak{q}}$ and $Q^k\left(\widehat{C}^k\right) \subseteq \widehat{C}^{k+1}$ since $\rho^{k+1}_{\mathrm{nat}}(x) Q^k w = Q^k \rho^k_{\mathrm{nat}}(x) w = 0$ holds for $w \in C^k_{\mathfrak{q}}$. The cochain complex $(C^k, Q^k)_{k\in\mathbb{Z}}$ is hence a direct sum of the cochain complexes $\left(C^k_{\mathfrak{q}},\, Q^k|_{C^k_{\mathfrak{q}}}\right)_{k\in\mathbb{Z}}$ and $\left(\widehat{C}^k,\, Q^k|_{\widehat{C}^k}\right)_{k\in\mathbb{Z}}$.

By the above Lemma 3.3 (1 and 2) $\ker\left(Q^k\right)$ and $\mathrm{im}\left(Q^{k-1}\right)$ are stable under $\mathfrak{g}$, hence also under $\mathfrak{q}$. We can again choose a $\mathfrak{q}$-stable complement T^k of $\mathrm{im}\left(Q^{k-1}\right)$ in $\ker\left(Q^k\right)$. Then by Lemma 3.3 (3) $\rho^k_{\mathrm{nat}}(x) T^k = T^k \cap \mathrm{im}\left(Q^{k-1}\right) = \{0\}$ for all $x \in \mathfrak{q}$. Hence $T^k \subseteq B^k_{\mathfrak{q}}$. This way every element in the cohomological space $H^k(\mathcal{C}) = \ker\left(Q^k\right)/\mathrm{im}\left(Q^{k-1}\right)$ is represented by an element in $T^k \subseteq B^k_{\mathfrak{q}}$. □

Example 3.3. Let us look again at the cohomology of $\mathfrak{sl}_2(\mathbb{C})$ w.r.t. the $(2j+1)$-dimensional representation π_j on V_j with basis $\{v_{-j}, v_{-j+1}, \ldots, v_j\}$. The element H (from the basis in Example 1.2) in $\mathfrak{g}$ generates a subalgebra $\mathfrak{q} := \mathbb{C}H$. We set $l_1 = H$, $l_2 = X$ and $l_3 = Y$. The structure constants of $\mathfrak{sl}_2(\mathbb{C})$are then given by

$$f^1_{23} = 2 = -f^1_{32}, \quad f^2_{12} = 1 = -f^2_{21} \quad \text{and} \quad f^3_{13} = -1 = -f^3_{31}. \tag{3.80}$$

This gives

$$\lambda_1 = -f^{\gamma}_{1\beta} = -c^2 b_2 + c^3 b_3. \tag{3.81}$$

The natural representation ρ^k_{nat} of $\mathfrak{q}$ on $C^k = V_j \otimes \Lambda^k(\mathfrak{g})$ is hence given by

$$\begin{aligned} \rho^k_{\text{nat}}(H)(v_m \otimes a) &= \pi_j(H) v_m \otimes a + v_m \otimes \lambda_1 a \\ &= m v_m \otimes a + v_m \otimes \left(-c^2 b_2 + c^3 b_3\right) a \end{aligned} \tag{3.82}$$

for $m \in \{-j, -j+1, \ldots, j\}$ and $a \in \Lambda^k(\mathfrak{g})$. If a is a k-multivector, i.e. of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ with $v_i \in V$, then the operator in front of the a simply counts the number of Y's minus the number of X's in the expression a. Since every element in $\Lambda^k(\mathfrak{g})$ can be written as a sum of k-multivectors, we can in particular find a basis in $\Lambda^k(\mathfrak{g})$ of eigenvectors of $\left(-c^2 b_2 + c^3 b_3\right)$. The basis of V_j is already chosen such that $\pi_j(H)$ is diagonal on it and hence $\rho_{\text{nat}}(H)$ is overall diagonalisable on $C^k = V_j \otimes \Lambda^k(\mathfrak{g})$. Hence the restriction of ρ^k_{nat} on $\mathfrak{q} = \mathbb{C}H$ is in particular completely reducible and Lemma 3.4 can be applied. We are now looking for the elements in C^k which are annihilated by $\rho_{\text{nat}}(H)$. These define the subspace $C^k_{\mathfrak{q}}$.

Let first $j = 0$. Then $\pi_j(H) v_0 = 0$ ($V_0 = \text{span}\{v_0\}$) and one finds:

$$\begin{aligned} C^0_{\mathfrak{q}} &= \text{span}\{v_0 \otimes |0\rangle\}, \\ C^1_{\mathfrak{q}} &= \text{span}\{v_0 \otimes H\}, \\ C^2_{\mathfrak{q}} &= \text{span}\{v_0 \otimes X \wedge Y\}, \\ C^3_{\mathfrak{q}} &= \text{span}\{v_0 \otimes H \wedge X \wedge Y\}. \end{aligned} \tag{3.83}$$

The operator Q has the form

$$Q = \pi_j(H) \otimes c^1 + \pi_j(X) \otimes c^2 + \pi_j(Y) \otimes c^3 + I_{V_j} \otimes \left(-2c^2 c^3 b_1 - c^1 c^2 b_2 + c^1 c^3 b_3\right) \tag{3.84}$$

w.r.t. the basis $l_1 = H$, $l_2 = X$ and $l_3 = Y$ of $\mathfrak{sl}_2(\mathbb{C})$ and for $j = 0$

$$Q = I_{V_j} \otimes \left(2c^2 c^3 b_1 + c^1 c^2 b_2 - c^1 c^3 b_3\right) \tag{3.85}$$

holds. One then readily calculates

$$\begin{aligned} QC^0_{\mathfrak{q}} &= \{0\}, \\ QC^1_{\mathfrak{q}} &= C^2_{\mathfrak{q}}, \\ QC^2_{\mathfrak{q}} &= \{0\}, \\ QC^3_{\mathfrak{q}} &= \{0\}, \end{aligned} \tag{3.86}$$

which again gives the cohomology $\dim\left(H^k\right) = 1$ for $k = 0, 3$ and $\dim\left(H^k\right) = 0$ for $k \in \mathbb{Z} \setminus \{0, 3\}$.

For $j \in \frac{1}{2}\mathbb{Z}_{>0}$ obviously all $C^k_{\mathfrak{q}} = \{0\}$ and hence the cohomology must be trivial.

For $j \in \mathbb{Z}_{>0}$ one easily finds:

$$\begin{aligned} C^0_{\mathfrak{q}} &= \operatorname{span}\{v_0 \otimes |0\rangle\}, \\ C^1_{\mathfrak{q}} &= \operatorname{span}\{v_{-1} \otimes Y, v_0 \otimes H, v_1 \otimes X\}, \\ C^2_{\mathfrak{q}} &= \operatorname{span}\{v_{-1} \otimes H \wedge Y, v_0 \otimes X \wedge Y, v_1 \otimes H \wedge X\}, \\ C^3_{\mathfrak{q}} &= \operatorname{span}\{v_0 \otimes H \wedge X \wedge Y\}. \end{aligned} \tag{3.87}$$

One then calculates

$$Q(v_0 \otimes |0\rangle) = v_1 \otimes X + j(j+1)v_{-1} \otimes Y \tag{3.88}$$

and

$$\begin{aligned} Q(v_{-1} \otimes Y) &= v_0 \otimes X \wedge Y, \\ Q(v_0 \otimes H) &= -v_1 \otimes H \wedge X - j(j+1)v_{-1} \otimes H \wedge Y - 2v_0 \otimes X \wedge Y, \\ Q(v_1 \otimes X) &= -j(j+1)v_0 \otimes X \wedge Y \end{aligned} \tag{3.89}$$

and

$$\begin{aligned} Q(v_{-1} \otimes H \wedge Y) &= -v_0 \otimes H \wedge X \wedge Y, \\ Q(v_0 \otimes X \wedge Y) &= 0, \\ Q(v_1 \otimes X) &= j(j+1)v_0 \otimes H \wedge X \wedge Y \end{aligned} \tag{3.90}$$

and

$$Q(v_0 \otimes H \wedge X \wedge Y) = 0. \tag{3.91}$$

With this one sees:

$$\begin{aligned} \dim\left(QC^0_{\mathfrak{q}}\right) &= 1, \\ \dim\left(QC^1_{\mathfrak{q}}\right) &= 2, \\ \dim\left(QC^0_{\mathfrak{q}}\right) &= 1, \\ \dim\left(QC^1_{\mathfrak{q}}\right) &= 0, \end{aligned} \tag{3.92}$$

One calculates that this corresponds to the trivial cohomology as can be seen in Table 3.4. We therefore have proven by the use of Lemma 3.4 what we made plausible in Example 3.2, namely that the cohomology of $\mathfrak{sl}_2(\mathbb{C})$ for irreducible representations with $j \neq 0$ is trivial.

i	$\cdots$	-1	0	1	2	3	4	$\cdots$
$\dim(C_{\mathfrak{q}}^i)$	$\cdots$	0	1	3	3	1	0	$\cdots$
$\dim\left(\ker\left(Q^i\big\|_{C_{\mathfrak{q}}^i}\right)\right)$	$\cdots$	0	0	1	2	1	0	$\cdots$
$\dim\left(\operatorname{im}\left(Q^{i-1}\big\|_{C_{\mathfrak{q}}^i}\right)\right)$	$\cdots$	0	0	1	2	1	0	$\cdots$
$\dim(H^i)$	$\cdots$	0	0	0	0	0	0	$\cdots$

Table 3.4: Cohomology of $\mathfrak{q} = \operatorname{span}\{H\}$ on the subcomplex $\left(C_{\mathfrak{q}}^k,\, Q^k\big|_{C_{\mathfrak{q}}^k}\right)_{k\in\mathbb{Z}}$ for $j \in \mathbb{Z}_{>0}$.

4 Lie Superalgebra Cohomology

We will now generalise the concept of cohomology to be able to apply it to Lie superalgebras. Some care is needed when dealing with the $\mathbb{Z}_2$-grading. There are several conventions as to the definition of the generalised exterior algebra. We will use the definition as in [HT92] rather as the one in [Tan95b, Tan95a].

4.1 Basics

Analogously to the case of ordinary Lie algebras we want to define a cohomology for a finite-dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Let $\{l_1, l_2, \dots, l_m\}$ be a basis of $\mathfrak{g}_0$ and $\{l_{m+1}, l_{m+2}, \dots, l_{m+n}\}$ a basis of $\mathfrak{g}_1$. Further, let $\left\{ f_{ij}^k \middle| 1 \leq i, j, k \leq m+n \right\}$ be the structure constants w.r.t. that basis.

In the following we will again construct a ghost vector space $\Lambda(\mathfrak{g})$. We do so by generalising the notion of exterior algebra.

Definition 4.1. Let $V = V_0 \oplus V_1$ be a super vector space. Let $T(V)$ be again the tensor algebra of V. We look at the two-sided, homogeneous ideal $J(V) \subseteq T(V)$ which is generated by the elements of the form

$$x \otimes y - (-1)^{(|x|+1)(|y|+1)} y \otimes x$$

with $x, y \in V$ homogeneous.

The (generalised) *exterior Algebra* (or *Grassmann algebra*) of the super vector space V is given by

$$\Lambda(V) := T(V)/J(V). \tag{4.1}$$

Alternatively we can again decompose

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V). \tag{4.2}$$

in its graded components $T^k(V) = \bigotimes_{i=1}^k V$ and define the subspace $J^k(V) \subseteq T^k(V)$ on them by

$$\begin{aligned} J^k(V) := \operatorname{span} \Big\{ a \in T^k(V) \Big| \, & \exists i, j \in \{1, \dots, k\} \, \exists v_1, \dots v_k \in V_0 \cup V_1 : a = v_1 \otimes \dots \\ & \otimes v_i \otimes \dots \otimes v_j \otimes \dots \otimes v_k - (-1)^{(|v_i|+1)(|v_j|+1)} v_1 \otimes \dots \otimes v_j \otimes \dots \otimes v_i \otimes \dots \otimes v_k \Big\}. \end{aligned} \tag{4.3}$$

Then

$$J(V) = \bigoplus_{k=0}^{\infty} J^k(V). \tag{4.4}$$

With the (generalised) *exterior power*, the quotient space

$$\Lambda^k(V) = T^k(V)/J^k(V) \tag{4.5}$$

$\Lambda(V)$ is again as a vector space isomorphic to

$$\bigoplus_{k=0}^{\infty} \Lambda^k(V) = \bigoplus_{k=0}^{\infty} T^k(V)/J^k(V). \tag{4.6}$$

Definition 4.2. The product of two elements $a, b \in \Lambda(V)$ in the (generalised) exterior algebra is again written as $a \wedge b$. We have

$$a \wedge b = a \otimes b + J(V). \tag{4.7}$$

Remark 4.1. By definition of $J(V)$ the exterior product is itself supercommutative on elements in $V = T^1(V) = \Lambda^1(V)$, i.e

$$x \wedge y = (-1)^{(|x|+1)(|y|+1)} y \wedge x \tag{4.8}$$

holds for all $x \in V$. However Elements in V of degree 0 are assigned the degree 1 in the exterior algebra and elements in V of degree 1 accordingly the degree 0.

Definition 4.3. As mentioned above we can again define a $\mathbb{Z}_2$-grading on $\Lambda(V)$, i.e. $\Lambda(V)$ can be made a super vector space. The homogeneous elements in $\Lambda(V)$ are of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ with the $v_i \in V$ homogeneous. The degree of such an element is defined by

$$\|v_1 \wedge v_2 \wedge \ldots \wedge v_k\| := (|v_1| + 1) + \ldots (|v_k| + 1) \in \mathbb{Z}. \tag{4.9}$$

Here, $|\cdot|$ denotes the original $\mathbb{Z}_2$-grading in $V = V_0 \oplus V_1$ and–to avoid confusion– $\|\cdot\|$ denotes the degree of an element in $\Lambda(V)$. Especially $\|v\| = |v|+1$ holds for a homogeneous $v \in \mathbb{V}$ where on the left side we consider $v \in \Lambda^1(V) = V$.

Remark 4.2. With the above definition formula (4.8) reads:

$$x \wedge y = (-1)^{\|x\|\|y\|} x \wedge y. \tag{4.10}$$

Remark 4.3. The exterior algebra can be decomposed again into a direct sum of components of different degrees (w.r.t. a $\mathbb{Z}$-grading). Here, $\Lambda^k(V)$ is the subspace of degree k and is spanned by all exterior products $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ consisting of k factors $v_i \in V$, called *k-multivectors*. The exterior algebra together with the exterior product has the structure of a $\mathbb{Z}$-graded algebra, which implies

$$\Lambda^k(V) \wedge \Lambda^p(V) \subseteq \Lambda^{k+p}(V). \tag{4.11}$$

In contrast to the case of an n-dimensional (ordinary) vector space where the vector space $\Lambda(V)$ was finite-, i.e. 2^n-dimensional, $\Lambda(V)$ is infinite-dimensional for a super vector space with $V_1 \neq \{0\}$. This is due to the fact that (assuming $\mathrm{char}(K) \neq 2$) terms of the form $x \wedge x$ vanish only for $x \in V_0$ but not for $x \in V_1$.

The following statement holds:

Proposition 4.1. *The dimension $d(k)$ of the vector space $\Lambda^k(V)$ where $V = V_0 \oplus V_1$ is a super vector space with* $\dim(V_0) = m$ *and* $\dim(V_1) = n$ *obeys:*

$$d(k; m, n) = \sum_{f=0}^{k} \binom{m}{f} \binom{n+k-f-1}{k-f} = \sum_{b=0}^{k} \binom{m}{k-b} \binom{n+b-1}{b}. \tag{4.12}$$

Outline of proof. To prove the above assertion we have to determine which k-multivectors one can write down and use some combinatorics formulae. □

Let us now look at $\Lambda(\mathfrak{g})$ for the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ mentioned at the beginning of the section. As a reminder: $\{l_1, l_2, \ldots, l_m\}$ be a basis of $\mathfrak{g}_0$ and a basis of $\mathfrak{g}_1$ be given by $\{l_{m+1}, l_{m+2}, \ldots, l_{m+n}\}$. Let $\left\{ f_{ij}^k \middle| 1 \leq i, j, k \leq m+n \right\}$ be the structure constants w.r.t. that basis.

We express $\Lambda(\mathfrak{g})$ again by a vacuum state $|0\rangle$ with span $\{|0\rangle\} = \Lambda^0(\mathfrak{g})$ ($|0\rangle$ corresponds to the empty exterior product, which we set $1 \in K$) and an appropriate choice of creation and annihilation operators..

For this, let the *creation operators* c^i $(i = 1, \ldots, m+n)$ be defined by:

Fermionic Case ($|i| = 0$, $\|l_i\| = 1$): Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ be given with $\alpha_j \in \{1, \ldots, m+n\}$ for $j = 1, \ldots, k$ where none of the α_j's with $|\alpha_j| = 0$ (i.e. $\left\|l_{\alpha_j}\right\| = 1$) appears twice since otherwise the state vanishes anyway. Then set for i with $|i| = 0$:

$$c^i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \begin{cases} l_i \wedge l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k} & \text{if } \alpha_j \neq i \text{ for all } j \\ 0 & \text{if } \alpha_j = i \text{ for one } j. \end{cases} \tag{4.13}$$

Bosonic Case ($|i| = 1$, $\|l_i\| = 0$): Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ be given with $\alpha_j \in \{1, \ldots, m+n\}$ for $j = 1, \ldots, k$. Then set for i with $|i| = 1$:

$$c^i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \sqrt{n_i + 1}\, l_i \wedge l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}, \tag{4.14}$$

where n_i is the number of $j \in \{1, \ldots, k\}$ with $\alpha_j = i$.

The *annihilation operators* are defined analogously by:

Fermionic Case ($|i| = 0$, $\|l_i\| = 1$): Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ be given with $\alpha_j \in \{1, \ldots, m+n\}$ for $j = 1, \ldots, k$ where none of the α_j's with $|\alpha_j| = 0$ (i.e. $\left\|l_{\alpha_j}\right\| = 1$) appears twice since otherwise the state vanishes anyway. Then set for i with $|i| = 0$:

$$\begin{aligned} &b_i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \\ &= \begin{cases} (-1)^{|\alpha_1| + \ldots + |\alpha_{j-1}| + j - 1} l_{\alpha_1} \wedge \ldots l_{\alpha_{j-1}} \wedge l_{\alpha_{j+1}} \wedge \ldots \wedge l_{\alpha_k} & \text{if } \alpha_j = i \text{ for a } j \\ 0 & \text{if } \alpha_j \neq i \text{ for all } j. \end{cases} \end{aligned} \tag{4.15}$$

Bosonic Case ($|i| = 1$, $\|l_i\| = 0$): Let $l_{\alpha_1} \wedge l_{\alpha_2} \wedge \ldots \wedge l_{\alpha_k} \in \Lambda^k(\mathfrak{g})$ be given with $\alpha_j \in \{1, \ldots, m+n\}$ for $j = 1, \ldots, k$. Then set for i with $|i| = 1$:

$$b_i \left(l_{\alpha_1} \wedge \ldots \wedge l_{\alpha_k}\right) = \sqrt{n_i}\, l_{\alpha_1} \wedge \ldots l_{\alpha_{j-1}} \wedge l_{\alpha_{j+1}} \wedge \ldots \wedge l_{\alpha_k}, \tag{4.16}$$

where n_i is again the number of $l \in \{1, \ldots, k\}$ with $\alpha_l = i$ and j is (e.g.) the first position for which $\alpha_j = i$.

For the creation and annihilation operators follows:

Proposition 4.2. *Let c^i and b_i ($i = 1, \ldots, m+n$) be defined as above. Then the creation and annihilation operators are well-defined as linear operators in* $\mathrm{End}(\Lambda(\mathfrak{g}))$ *and obey*

$$b_i c^j - (-1)^{(|i|+1)(|j|+1)} c^j b_i = \delta_i^j \tag{4.17}$$

as well as

$$b_i b_j - (-1)^{(|i|+1)(|j|+1)} b_j b_i = c^i c^j - (-1)^{(|i|+1)(|j|+1)} c^j c^i = 0 \tag{4.18}$$

for all $i, j \in \{1, \ldots, m+n\}$.

Proof. The proof is left to the reader. □

The ghost creation and annihilation operators for $|i| = 0$ are called *fermionic creation operators* and *annihilation operators* as they *anticommute* with themselves and each other. In contrast, the creation and annihilation operators for $|i| = 1$ *commute*, which is why we call them *bosonic creation operators* and *annihilation operators*. A fermionic and a bosonic creation or annihilation operator commute with each other.

Definition 4.4. We again introduce a $\mathbb{Z}_2$-grading on $\mathrm{End}(\Lambda(\mathfrak{g}))$ by defining

$$\left|c^i\right| = |b_i| = |i| + 1. \tag{4.19}$$

This means bosonic creation and annihilation operators have degree 0 and the fermionic ones have degree 1, as one would expect. Products of creation and annihilation operators have as degree the sum of the degrees of their factors. Hence, we can make $\mathrm{End}(\Lambda(\mathfrak{g}))$ a Lie superalgebra $\mathfrak{gl}(\Lambda(\mathfrak{g}))$ by using the supercommutator

$$[A, B] = AB - (-1)^{|A||B|} BA$$

for $A, B \in \mathrm{End}(\Lambda(\mathfrak{g}))$ as Lie superbracket.

Remark 4.4. With the help of the super commutator the relations in Proposition 4.2 can be cast in the elegant form

$$\left[b_i, c^j\right] = \delta_i^j \tag{4.20}$$

and

$$[b_i, b_j] = \left[c^i, c^j\right] = 0. \tag{4.21}$$

Also in the super case $\Lambda(\mathfrak{g})$ can be made a (non-trivial) $\mathfrak{g}$-module:

Theorem 4.1. *The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ has a representation $\widetilde{\pi}$ on $\Lambda(\mathfrak{g})$ defined by*

$$\widetilde{\lambda}_\alpha := \widetilde{\pi}(l_\alpha) := f^\gamma_{\beta\alpha} c^\beta b_\gamma \in \mathrm{End}(\Lambda(\mathfrak{g}))\,. \tag{4.22}$$

In particular:

$$[\widetilde{\lambda}_\alpha, \widetilde{\lambda}_\beta] = f^\gamma_{\alpha\beta} \widetilde{\lambda}_\gamma. \tag{4.23}$$

Proof. We have seen that $\mathrm{End}(\Lambda(\mathfrak{g}))$ can be made a Lie superalgebra namely $\mathfrak{gl}(\Lambda(\mathfrak{g}))$. For the map $\widetilde{\pi} : \mathfrak{g} \to \mathfrak{gl}(\Lambda(\mathfrak{g}))$ to be a homomorphism of Lie superalgebras we have to show that even elements in $\mathfrak{g}$ are mapped to even elements in $\mathfrak{gl}(\Lambda(\mathfrak{g}))$ and odd ones to odd ones, and that the map respects the Lie superbracket. The first assertion is obvious since $|\beta| + |\gamma| = |\alpha|$ for $f^\gamma_{\beta\alpha} \neq 0$ and thus $\left|c^\beta b_\gamma\right| = \left|c^\beta\right| + |b_\gamma| = |\beta| + 1 + |\gamma| + 1 = |\alpha| + 1 + 1 = |\alpha|$.

Next, we look at

$$\begin{aligned}
&[\widetilde{\lambda}_\alpha, \widetilde{\lambda}_\mu] \\
&= [f^\gamma_{\beta\alpha} c^\beta b_\gamma, f^\rho_{\nu\mu} c^\nu b_\rho] \\
&= f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} \left(c^\beta b_\gamma c^\nu b_\rho - (-1)^{(|\beta|+1+|\gamma|+1)(|\nu|+1+|\rho|+1)} c^\nu b_\rho c^\beta b_\gamma \right) \\
&= f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} \left(c^\beta \left(\delta^\nu_\gamma + (-1)^{(|\gamma|+1)(|\nu|+1)} c^\nu b_\gamma \right) b_\rho - \right. \\
&\quad \left. - (-1)^{(|\beta|+|\gamma|)(|\nu|+|\rho|)} c^\nu \left(\delta^\beta_\rho + (-1)^{(|\beta|+1)(|\rho|+1)} c^\beta b_\rho \right) b_\gamma \right) \\
&= f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} \left(\delta^\nu_\gamma c^\beta b_\rho + (-1)^{(|\gamma|+1)(|\nu|+1)} c^\beta c^\nu b_\gamma b_\rho - \right. \\
&\quad \left. - (-1)^{(|\beta|+|\gamma|)(|\nu|+|\rho|)} \delta^\beta_\rho c^\nu b_\gamma + (-1)^{(|\beta|+|\gamma|)(|\nu|+|\rho|)} (-1)^{(|\beta|+1)(|\rho|+1)} c^\nu c^\beta b_\rho b_\gamma \right) \\
&= f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} \left(\delta^\nu_\gamma c^\beta b_\rho + (-1)^{(|\gamma|+1)(|\nu|+1)} \cancel{c^\beta c^\nu b_\gamma b_\rho} - \right. \\
&\quad - (-1)^{(|\beta|+|\gamma|)(|\nu|+|\rho|)} \delta^\beta_\rho c^\nu b_\gamma - \\
&\quad \left. - \underbrace{(-1)^{(|\beta|+|\gamma|)(|\nu|+|\rho|)} (-1)^{(|\beta|+1)(|\rho|+1)} (-1)^{(|\beta|+1)(|\nu|+1)} (-1)^{(|\gamma|+1)(|\rho|+1)}}_{=(-1)^{(|\gamma|+1)(|\nu|+1)}} \cancel{c^\beta c^\nu b_\gamma b_\rho} \right) \\
&= f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} \left(\delta^\nu_\gamma c^\beta b_\rho - (-1)^{|\alpha||\mu|} \delta^\beta_\rho c^\nu b_\gamma \right).
\end{aligned} \tag{4.24}$$

By renaming indices we get

$$[\widetilde{\lambda}_\alpha, \widetilde{\lambda}_\mu] = \left(f^\nu_{\beta\alpha} f^\rho_{\nu\mu} - (-1)^{|\alpha||\mu|} f^\nu_{\beta\mu} f^\rho_{\nu\alpha} \right) c^\beta b_\rho. \tag{4.25}$$

The Jacobi identity (2.9) can be cast in the form

$$f^\nu_{\beta\alpha} f^\rho_{\nu\mu} - (-1)^{|\alpha||\mu|} f^\nu_{\beta\mu} f^\rho_{\nu\alpha} = f^\nu_{\alpha\mu} f^\rho_{\beta\nu} \tag{4.26}$$

and thus

$$[\widetilde{\lambda}_\alpha, \widetilde{\lambda}_\mu] = f^\nu_{\alpha\mu} f^\rho_{\beta\nu} c^\beta b_\rho = f^\nu_{\alpha\mu} \widetilde{\lambda}_\nu. \tag{4.27}$$

□

It is easy to derive another representation π from the representation $\widetilde{\pi}$ above.

Corollary 4.1. *The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ has a representation π on $\Lambda(\mathfrak{g})$ given by*

$$\lambda_\alpha := \pi(l_\alpha) := (-1)^{|\alpha|} f^{\gamma}_{\beta\alpha} c^\beta b_\gamma \in \mathrm{End}(\Lambda(\mathfrak{g})\,. \tag{4.28}$$

In particular

$$[\lambda_\alpha, \lambda_\beta] = f^{\gamma}_{\alpha\beta}\lambda_\gamma. \tag{4.29}$$

Proof. We show that π respects the Lie superbracket. We have

$$\begin{aligned} [\lambda_\alpha, \lambda_\beta] &= (-1)^{|\alpha|}(-1)^{|\beta|}[\widetilde{\lambda}_\alpha, \widetilde{\lambda}_\beta] = (-1)^{|\alpha|+|\beta|} f^{\gamma}_{\alpha\beta}\widetilde{\lambda}_\gamma \\ &= (-1)^{|\gamma|} f^{\gamma}_{\alpha\beta}\widetilde{\lambda}_\gamma = f^{\gamma}_{\alpha\beta}\lambda_\gamma. \end{aligned} \tag{4.30}$$

□

Now, let ρ be a representation of $\mathfrak{g}$ on the super vector space $V = V_0 \oplus V_1$. We set

$$\widetilde{C} = V \otimes \Lambda(\mathfrak{g}) \oplus \Lambda(\mathfrak{g}) \otimes V.$$

This vector space is graded by

$$\begin{aligned} \widetilde{C} &= V \otimes \Lambda(\mathfrak{g}) \oplus \Lambda(\mathfrak{g}) \otimes V = V \otimes \left(\bigoplus_{i=0}^{n} \Lambda^i(\mathfrak{g})\right) \oplus \left(\bigoplus_{i=0}^{n} \Lambda^i(\mathfrak{g})\right) \otimes V \\ &= \bigoplus_{i=0}^{n} \big(V \otimes \Lambda^i(\mathfrak{g}) \oplus \Lambda^i(\mathfrak{g}) \oplus V\big) =: \bigoplus_{i=0}^{n} \widetilde{C}^i, \end{aligned} \tag{4.31}$$

where $\widetilde{C}^i = V \otimes \Lambda^i(\mathfrak{g}) \oplus \Lambda^i(\mathfrak{g}) \oplus V$ $(i \in \mathbb{Z})$ and $\widetilde{C}^i = V \otimes \{0\} \oplus \{0\} \otimes V = \{0\}$ for $i \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}$. $\widetilde{C}$ is a super vector space by defining a $\mathbb{Z}_2$-grading on $V \otimes \Lambda(\mathfrak{g})$ and $\Lambda(\mathfrak{g}) \otimes V$ according to Definition 2.14, i.e. via $|v \otimes a| = |v| + \|a\|$ and $|a \otimes v| = \|a\| + |v|$ for $v \in V$ and $a \in \Lambda(V)$ both homogeneous and then putting $\widetilde{C}_0 = (V \otimes \Lambda(\mathfrak{g}))_0 \oplus (\Lambda(\mathfrak{g}) \otimes V)_0$ and $\widetilde{C}_1 = (V \otimes \Lambda(\mathfrak{g}))_1 \oplus (\Lambda(\mathfrak{g}) \otimes V)_1$.

We look at the subspace U of $\widetilde{C}$ spanned by the elements of the form

$$v \otimes a - (-1)^{|v|\|a\|} a \otimes v$$

with $v \in V$ and $a \in \Lambda(V)$ both homogeneous and form the vector space C as the quotient space of $\widetilde{C}$ by U, i.e.

$$C = \widetilde{C}/U.$$

The vector space C is graded by $C = \bigoplus_{i=0}^{n} C^i$ where $C^i = \widetilde{C}^i/U^i$ and $U^i = U \cap C^i$. We write an element (i.e. an equivalence class) in C as

$$v \odot a = v \otimes a + U \in C. \tag{4.32}$$

Then by definition

$$v \odot a = (-1)^{|v|\|a\|} a \odot v. \tag{4.33}$$

C and the C^i's are again super vector spaces by setting

$$|v \odot a| = |a \odot v| := |a \otimes v| \tag{4.34}$$

for $v \in V$ and $a \in \Lambda(V)$ both homogeneous, which is well-defined.

Remark 4.5. We have $\dim(\widetilde{C}^i) = 2\dim(V \otimes \Lambda^i(\mathfrak{g})) = 2\dim(V)\dim(C^i)$ and $\dim U_i = \frac{1}{2}\dim(\widetilde{C}^i)$ and thus

$$\dim(C^i) = \dim(V)\dim(\Lambda(\mathfrak{g})), \tag{4.35}$$

as in the case of ordinary Lie algebras. Our construction of C from V and $\Lambda(\mathfrak{g})$ is however more complicated to account for the correct (anti)-commutation relations between operators on V and operators on $\mathfrak{g}$(cf. Proposition 4.3).

We can now look at the vector space $\mathrm{End}(C)$ of linear operators on $C = C_0 \oplus C_1$. This can be made a super vector space $\mathrm{End}(C) = \mathrm{End}(C)_0 \oplus \mathrm{End}(C)_1$ in a canonical way (cf. Remark 2.7) and with the supercommutator it becomes a Lie superalgebra $\mathfrak{gl}(C)$.

We can embed the operators on V and on $\Lambda(\mathfrak{g})$ in a natural way into C by defining

$$T(v \odot a) = (Tv) \odot a \tag{4.36}$$

for $v \in V$, $a \in \Lambda(\mathfrak{g})$ and $T \in \mathrm{End}(V)$ as well as

$$S(a \odot v) = (Sa) \odot v \tag{4.37}$$

for $v \in V$, $a \in \Lambda(\mathfrak{g})$ and $S \in \mathrm{End}(\Lambda(\mathfrak{g}))$. The degree of T and S as elements in $\mathrm{End}(V)$ and $\mathrm{End}(\Lambda(\mathfrak{g}))$ respectively corresponds, as can be readily checked, to their degree in $\mathrm{End}(C)$.

Proposition 4.3. *For arbitrary $T \in \mathrm{End}(V)$ and $S \in \mathrm{End}(\Lambda(\mathfrak{g}))$ we have with the above embedding*

$$[T, S] = 0, \tag{4.38}$$

where $[\cdot,\cdot]$ is the Lie superbracket, i.e. the supercommutator on $\mathrm{End}(C)$.

Proof. Let $T \in \mathrm{End}(V)$ and $S \in \mathrm{End}(\Lambda(\mathfrak{g}))$ be w.l.o.g. homogeneous. C is generated by elements of the form $v \odot a$ with $v \in V$ and $a \in \Lambda(\mathfrak{g})$ both homogeneous. We have

$$\begin{aligned} TS(v \odot a)) &= (-1)^{|v||a|} TS(a \odot v) = (-1)^{|v||a|} T(Sa \odot v) \\ &= (-1)^{|v||Sa|}(-1)^{|v||a|} T(v \odot Sa) = (-1)^{|v||S|} Tv \odot Sa \\ &= (-1)^{|Tv||Sa|}(-1)|v|\,|S| Sa \odot Tv = (-1)^{|T||S|+|T||a|+|v||a|} S(a \odot Tv) \\ &= (-1)^{|Tv||a|}(-1)^{|T||S|+|T||a|+|v||a|} S(Tv \odot a) \\ &= (-1)^{|T||S|} ST(v \odot a), \end{aligned} \tag{4.39}$$

where we used the relation $|Tv| = |T| + |v|$ and analogously for Sa several times. The assertion follows by linearity of the Lie superbracket. $\square$

In particular we have the relations

$$\left[c^i, L_j\right] = c^i L_j - (-1)^{(|i|+1)|j|} L_j c^i = 0 \tag{4.40}$$

and analogously for $[b_i, L_j]$.

We define the *ghost number operator* U on $\Lambda(\mathfrak{g})$ by

$$U = \sum_{i=1}^{m+n} c^i b_i = c^i b_i \tag{4.41}$$

and again have

$$Ua = ia \tag{4.42}$$

for all $a \in \Lambda^i(\mathfrak{g})$. Moreover we can look at the embedding in $\mathrm{End}(C)$. We have

$$U(v \odot a) = i(v \odot a) \tag{4.43}$$

for $a \in \Lambda^i(\mathfrak{g})$ (iff. $v \odot a \in C^i$).

We can again define an operator Q on C, which generates a cohomology similar to the case of Lie algebras. We have:

Theorem 4.2. *The operator Q defined on C as*

$$Q = c^\alpha L_\alpha + \frac{1}{2} c^\alpha \lambda_\alpha = c^\alpha L_\alpha + \frac{1}{2}(-1)^{|\alpha|} c^\alpha \widetilde{\lambda}_\alpha = c^\alpha L_\alpha + \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha} c^\alpha c^\beta b_\gamma, \tag{4.44}$$

with $L_\alpha := \rho(l_\alpha)$ fulfils

$$Q^2 = 0$$

and

$$[U, Q] = Q.$$

The second term in the operator Q has exactly the opposite sign compared to [HT92]. However, there the commutation relation $\left[b_i, c^j\right] = -\delta_i^j$ holds and not $\left[b_i, c^j\right] = \delta_i^j$ as in this text. By mapping $c^i \mapsto -c^i$ and leaving the b_j's unchanged one can switch between the two conventions. This gives exactly the difference in signs.

Proof. The second assertion is shown analogously to the case of Lie algebras (cf. proof of Theorem 3.2). To prove the first assertion we look at:

$$\begin{aligned} Q^2 = \underbrace{c^\alpha L_\alpha c^\mu L^\mu}_{=:(1)} + \underbrace{\frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha} \left(c^\mu L_\mu c^\alpha c^\beta b_\gamma + c^\alpha c^\beta b_\gamma c^\mu L_\mu\right)}_{=:(2)} + \\ + \underbrace{\frac{1}{4}(-1)^{|\alpha|+|\mu|} f^\gamma_{\beta\alpha} f^\rho_{\nu\mu} c^\alpha c^\beta b_\gamma c^\mu c^\nu b_\rho}_{=:(3)}. \end{aligned} \tag{4.45}$$

We have

$$\begin{aligned}
(2) &= \frac{1}{2}(-1)^{|\alpha|} f^{\gamma}_{\beta\alpha}\left(c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma} + c^{\alpha}c^{\beta}b_{\gamma}c^{\mu}L_{\mu}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|} f^{\gamma}_{\beta\alpha}\left(c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma} + c^{\alpha}c^{\beta}\left(\delta^{\mu}_{\gamma} + (-1)^{(|\gamma|+1)(|\mu|+1)}c^{\mu}b_{\gamma}\right)L_{\mu}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|} f^{\gamma}_{\beta\alpha}\left(c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma} + \delta^{\mu}_{\gamma}c^{\alpha}c^{\beta}L_{\mu} + (-1)^{(|\gamma|+1)(|\mu|+1)}c^{\alpha}c^{\beta}c^{\mu}b_{\gamma}L_{\mu}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|} f^{\gamma}_{\beta\alpha}\left(c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma} + \delta^{\mu}_{\gamma}c^{\alpha}c^{\beta}L_{\mu} + (-1)^{|\gamma|+1}c^{\alpha}c^{\beta}c^{\mu}L_{\mu}b_{\gamma}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|} f^{\gamma}_{\beta\alpha}\left(\cancel{c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma}} + \delta^{\mu}_{\gamma}c^{\alpha}c^{\beta}L_{\mu} + \underbrace{(-1)^{|\alpha|+1+|\beta|+1+|\gamma|+1}}_{=-1}\cancel{c^{\mu}L_{\mu}c^{\alpha}c^{\beta}b_{\gamma}}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|} f^{\mu}_{\beta\alpha}c^{\alpha}c^{\beta}L_{\mu} = \frac{1}{2}(-1)^{|\alpha|}c^{\alpha}c^{\beta}\left[L_{\beta}, L_{\alpha}\right]\\
&= \frac{1}{2}(-1)^{|\alpha|}c^{\alpha}c^{\beta}\left(L_{\beta}L_{\alpha} - (-1)^{|\alpha||\beta|}L_{\alpha}L_{\beta}\right)\\
&= \frac{1}{2}(-1)^{|\alpha|}c^{\alpha}c^{\beta}L_{\beta}L_{\alpha} - \frac{1}{2}(-1)^{|\alpha|(|\beta|+1)}c^{\alpha}c^{\beta}L_{\alpha}L_{\beta}\\
&= \frac{1}{2}(-1)^{|\beta|}c^{\beta}c^{\alpha}L_{\alpha}L_{\beta} - \frac{1}{2}c^{\alpha}L_{\alpha}c^{\beta}L_{\beta}\\
&= -\frac{1}{2}c^{\alpha}L_{\alpha}c^{\beta}L_{\beta} - \frac{1}{2}c^{\alpha}L_{\alpha}c^{\beta}L_{\beta}\\
&= -(1),
\end{aligned} \tag{4.46}$$

which means (1) and (2) cancel.

The Jacobi identity holds:

$$(-1)^{|\gamma||\alpha|}f^{\delta}_{\beta\gamma}f^{\varepsilon}_{\alpha\delta} + (-1)^{|\alpha||\beta|}f^{\delta}_{\gamma\alpha}f^{\varepsilon}_{\beta\delta} + (-1)^{|\beta||\gamma|}f^{\delta}_{\alpha\beta}f^{\varepsilon}_{\gamma\delta} = 0 \tag{4.47}$$

and hence

$$\begin{aligned}
0 &= \left((-1)^{|\gamma||\alpha|}f^{\delta}_{\beta\gamma}f^{\varepsilon}_{\alpha\delta} + (-1)^{|\alpha||\beta|}f^{\delta}_{\gamma\alpha}f^{\varepsilon}_{\beta\delta} + (-1)^{|\beta||\gamma|}f^{\delta}_{\alpha\beta}f^{\varepsilon}_{\gamma\delta}\right)(-1)^{|\beta||\varepsilon|}c^{\alpha}c^{\beta}c^{\gamma}b_{\varepsilon}\\
&= (-1)^{|\gamma|(|\beta|+1)}f^{\delta}_{\beta\gamma}f^{\varepsilon}_{\alpha\delta}c^{\beta}c^{\gamma}c^{\alpha}b_{\varepsilon} + (-1)^{|\alpha|(|\gamma|+1)}f^{\delta}_{\gamma\alpha}f^{\varepsilon}_{\beta\delta}c^{\gamma}c^{\alpha}c^{\beta}b_{\varepsilon}\\
&\quad + (-1)^{|\beta|(|\alpha|+1)}f^{\delta}_{\alpha\beta}f^{\varepsilon}_{\gamma\delta}c^{\alpha}c^{\beta}c^{\gamma}b_{\varepsilon}\\
&= (-1)^{|\gamma|(|\beta|+1)}f^{\delta}_{\beta\gamma}c^{\beta}c^{\gamma}\widetilde{\lambda}_{\delta} + (-1)^{|\alpha|(|\gamma|+1)}f^{\delta}_{\gamma\alpha}c^{\gamma}c^{\alpha}\widetilde{\lambda}_{\delta} + (-1)^{|\beta|(|\alpha|+1)}f^{\delta}_{\alpha\beta}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\delta}\\
&= 3(-1)^{(|\alpha|+1)|\beta|}f^{\gamma}_{\alpha\beta}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\gamma}\\
&= 3(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\left[\widetilde{\lambda}_{\alpha}, \widetilde{\lambda}_{\beta}\right].
\end{aligned} \tag{4.48}$$

Thus we have

$$(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\left(\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta} - (-1)^{|\alpha||\beta|}\widetilde{\lambda}_{\beta}\widetilde{\lambda}_{\alpha}\right) = 0. \tag{4.49}$$

On the other hand

$$
\begin{aligned}
&(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\left(\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}+(-1)^{|\alpha||\beta|}\widetilde{\lambda}_{\beta}\widetilde{\lambda}_{\alpha}\right)\\
&=(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}+(-1)^{|\beta|}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\beta}\widetilde{\lambda}_{\alpha}\\
&=(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}+(-1)^{|\alpha|}c^{\beta}c^{\alpha}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}\\
&=(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}-(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\beta}\\
&=0,
\end{aligned}
\tag{4.50}
$$

which implies

$$
(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\lambda_{\alpha}\lambda_{\beta}=0. \tag{4.51}
$$

A short calculation shows

$$
\left[\widetilde{\lambda}_{\alpha},c^{\mu}\right]=f^{\mu}_{\beta\alpha}c^{\beta}. \tag{4.52}
$$

We have

$$
\begin{aligned}
4\cdot(3)&=(-1)^{|\alpha|+|\mu|}f^{\gamma}_{\beta\alpha}f^{\rho}_{\nu\mu}c^{\alpha}c^{\beta}b_{\gamma}c^{\mu}c^{\nu}b_{\rho}=(-1)^{|\alpha|+|\mu|}c^{\alpha}\widetilde{\lambda}_{\alpha}c^{\mu}\widetilde{\lambda}_{\mu}\\
&=(-1)^{|\alpha|+|\mu|}c^{\alpha}\left(f^{\mu}_{\beta\alpha}c^{\beta}+(-1)^{|\alpha|(|\mu|+1)}c^{\mu}\widetilde{\lambda}_{\alpha}\right)\widetilde{\lambda}_{\mu}\\
&=(-1)^{|\beta|}c^{\alpha}c^{\beta}\left[\widetilde{\lambda}_{\beta},\widetilde{\lambda}_{\alpha}\right]+\underbrace{(-1)^{(|\alpha|+1)|\mu|}c^{\alpha}c^{\mu}\widetilde{\lambda}_{\alpha}\widetilde{\lambda}_{\mu}}_{=0}\\
&=-(-1)^{(|\alpha|+1)|\beta|}c^{\alpha}c^{\beta}\left[\widetilde{\lambda}_{\alpha},\widetilde{\lambda}_{\beta}\right]=0,
\end{aligned}
\tag{4.53}
$$

which completes the proof. □

This enables us to define a cohomology for Lie superalgebras. This is done formally identical to the case of ordinary Lie algebras, i.e.

$$
Q^{i}=Q|_{C^{i}} \tag{4.54}
$$

and

$$
H^{i}=\ker\left(Q^{i}\right)/\mathrm{im}\left(Q^{i-1}\right) \tag{4.55}
$$

as well as

$$
H(\mathfrak{g},V):=\left(H^{i}\right)_{i\in\mathbb{Z}}, \tag{4.56}
$$

which we call the *cohomology of the Lie superalgebra* $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ *with coefficients in* V.

In the following we want to define again some useful concepts, which simplify the calculation of the cohomology.

Definition 4.5. We define the *natural representation* ρ_{nat} of $\mathfrak{g}$ on C via

$$
\rho_{\mathrm{nat}}(l_{\mu})=L_{\mu}+\lambda_{\mu}\in\mathrm{End}(C). \tag{4.57}
$$

Lemma 4.1. *The following holds:*

$$
[b_{\mu},Q]=L_{\mu}+\lambda_{\mu}=\rho_{nat}(l_{\mu}). \tag{4.58}
$$

Proof. We look at

$$Qb_\mu = \left(c^\alpha L_\alpha + \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha} c^\alpha c^\beta b_\gamma\right) b_\mu = c^\alpha L_\alpha b_\mu + \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha} c^\alpha c^\beta b_\gamma b_\mu. \tag{4.59}$$

We now have

$$\begin{aligned} c^\alpha L_\alpha b_\mu &= (-1)^{|\alpha|(|\mu|+1)} c^\alpha b_\mu L_\alpha = (-1)^{|\alpha|(|\mu|+1)}(-1)^{(|\alpha|+1)(|\mu|+1)}\left(b_\mu c^\alpha - \delta^\alpha_\mu\right) L_\alpha \\ &= -(-1)^{|\mu|} b_\mu c^\alpha L_\alpha + (-1)^{|\mu|} L_\mu \end{aligned} \tag{4.60}$$

and

$$\begin{aligned} c^\alpha c^\beta b_\gamma b_\mu &= (-1)^{(|\gamma|+1)(|\mu|+1)} c^\alpha c^\beta b_\mu b_\gamma = (-1)^{(|\gamma|+|\beta|)(|\mu|+1)} c^\alpha \left(b_\mu c^\beta - \delta^\beta_\mu\right) b_\gamma \\ &= (-1)^{(|\gamma|+|\beta|+|\alpha|+1)(|\mu|+1)} \left(b_\mu c^\alpha - \delta^\alpha_\mu\right) c^\beta b_\gamma - (-1)^{(|\gamma|+|\beta|)(|\mu|+1)} \delta^\beta_\mu c^\alpha b_\gamma \\ &= (-1)^{(|\gamma|+|\beta|+|\alpha|+1)(|\mu|+1)} b_\mu c^\alpha c^\beta b_\gamma - (-1)^{(|\gamma|+|\beta|+|\alpha|+1)(|\mu|+1)} \delta^\alpha_\mu c^\beta b_\gamma \\ &\quad - (-1)^{(|\gamma|+|\beta|)(|\mu|+1)} \delta^\beta_\mu c^\alpha b_\gamma. \end{aligned} \tag{4.61}$$

This implies

$$\begin{aligned} Qb_\mu &= -(-1)^{|\mu|} b_\mu c^\alpha L_\alpha + (-1)^{|\mu|} L_\mu + \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha}(-1)^{(|\gamma|+|\beta|+|\alpha|+1)(|\mu|+1)} b_\mu c^\alpha c^\beta b_\gamma \\ &\quad - \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha}(-1)^{(|\gamma|+|\beta|+|\alpha|+1)(|\mu|+1)} \delta^\alpha_\mu c^\beta b_\gamma \\ &\quad - \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha}(-1)^{(|\gamma|+|\beta|)(|\mu|+1)} \delta^\beta_\mu c^\alpha b_\gamma \\ &= -(-1)^{|\mu|} b_\mu c^\alpha L_\alpha + (-1)^{|\mu|} L_\mu - \frac{1}{2}(-1)^{|\alpha|+|\mu|} f^\gamma_{\beta\alpha} b_\mu c^\alpha c^\beta b_\gamma \\ &\quad + \frac{1}{2} f^\gamma_{\beta\mu} c^\beta b_\gamma - \frac{1}{2}(-1)^{|\alpha||\mu|} f^\gamma_{\mu\alpha} c^\alpha b_\gamma \\ &= -(-1)^{|\mu|} b_\mu Q + (-1)^{|\mu|} L_\mu + \widetilde{\lambda}_\mu \\ &= -(-1)^{|\mu|} b_\mu Q + (-1)^{|\mu|}\left(L_\mu + \lambda_\mu\right). \end{aligned} \tag{4.62}$$

Then because of $|Q| = 1$

$$[b_\mu, Q] = b_\mu Q - (-1)^{|\mu|+1} Q b_\mu = b_\mu Q + (-1)^{|\mu|} Q b_\mu = L_\mu + \lambda_\mu \tag{4.63}$$

follows. □

Remark 4.6. Since $\rho_{\mathrm{nat}}(x)$ leaves the ghost number invariant for all $x \in \mathfrak{g}$, it is sensible to restrict ρ_{nat} to the C^k's. We denote by ρ^k_{nat} the subrepresentation of ρ_{nat} on C^k.

Lemma 4.2. *For all $x \in \mathfrak{g}$*

$$[Q, \rho_{nat}(x)] = 0 \tag{4.64}$$

holds or equivalently

$$\rho^{k+1}_{\mathrm{nat}}(x) Q^k = (-1)^{|x|} Q^k \rho^k_{\mathrm{nat}}(x) \in \mathrm{Hom}\left(C^k, C^{k+1}\right) \tag{4.65}$$

for all $k \in \mathbb{Z}$ and x homogeneous.

Proof. We proof the second claim for $x = l_\mu$. The assertion of the lemma follows by linearity of ρ_{nat}. By Lemma 4.1

$$\rho^k_{\text{nat}}(l_\mu) = b_\mu Q^k + (-1)^{|\mu|} Q^{k-1} b_\mu \tag{4.66}$$

holds. This implies

$$Q^k \rho^k_{\text{nat}}(l_\mu) = Q^k b_\mu Q^k + (-1)^{|\mu|} \underbrace{Q^k Q^{k-1}}_{=0} b_\mu = Q^k b_\mu Q^k. \tag{4.67}$$

Analogously

$$\rho^{k+1}_{\text{nat}}(l_\mu) Q^k = b_\mu \underbrace{Q^{k+1} Q^k}_{=0} + (-1)^{|\mu|} Q^k b_\mu Q^k = (-1)^{|\mu|} Q^k b_\mu Q^k. \tag{4.68}$$

This implies the assertion. □

The following assertions hold analogously to the Lie algebra case:

Lemma 4.3. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an n-dimensional Lie superalgebra with a d-dimensional representation ρ on V and let $(C^k, Q^k)_{k\in\mathbb{Z}}$ be the corresponding cochain complex. We look at the natural representation on C^k. The following holds:*

1. $\ker\left(Q^k\right)$ *is stable under* $\mathfrak{g}$.
2. $\operatorname{im}\left(Q^{k-1}\right)$ *is stable under* $\mathfrak{g}$.
3. $\rho^k_{nat}(\mathfrak{g})\left(\ker\left(Q^k\right)\right) \subseteq \operatorname{im}\left(Q^{k-1}\right)$.

Proof. The proof is along the lines of the proof of Lemma 3.3. □

Lemma 4.4. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be an n-dimensional Lie superalgebra with a d-dimensional representation ρ on V and let $(C^k, Q^k)_{k\in\mathbb{Z}}$ be the corresponding cochain complex. Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$ such that the natural representations ρ^k_{nat} of $\mathfrak{q}$ on C^k are all completely reducible. Then the cohomology $H(\mathfrak{g}, V)$ is identical to the one obtained from the subcomplex $(C^k_{\mathfrak{q}}, Q^k|_{C^k_{\mathfrak{q}}})_{k\in\mathbb{Z}}$ where $C^k_{\mathfrak{q}}$ is the subspace of elements in C^k which are annihilated by $\rho^k_{nat}(x)$ for all $x \in \mathfrak{q}$.*

Proof. The proof is analogous to the proof of Lemma 3.4. □

4.2 Example of $\mathfrak{gl}(1|1)$

Example 4.1. We want to determine the cohomology of $\mathfrak{g} = \mathfrak{gl}(1|1)$ w.r.t the 2-dimensional representation $\rho_\Lambda : \mathfrak{gl}(1|1) \to V = \mathbb{C}v_0 + \mathbb{C}v_1$ (cf. Examples 2.2 and 2.1). First, let us choose a basis $\{l_1, l_2, l_3, l_4\}$ of $\mathfrak{g}$. We set

$$l_1 = H, \quad l_2 = C, \quad l_3 = X \quad \text{and} \quad l_4 = Y. \tag{4.69}$$

The elementary (anti)-commutation relations for $\mathfrak{g}$ are given in Example 2.1. From those we can determine the structure constants to be

$$f^3_{13} = -f^3_{31} = 1, \quad f^4_{14} = -f^4_{41} = -1, \quad \text{and} \quad f^2_{34} = f^2_{43} = 1. \tag{4.70}$$

All other 58 of the overall $4^3 = 64$ structure constants vanish. The operator Q is thus given by

$$\begin{aligned} Q &= c^\alpha L_\alpha + \frac{1}{2}(-1)^{|\alpha|} f^\gamma_{\beta\alpha} c^\alpha c^\beta b_\gamma \\ &= c^1 L_1 + c^2 L_2 + c^3 L_3 + c^4 L_4 - c^1 c^3 b_3 + c^1 c^4 b_4 - c^3 c^4 b_2. \end{aligned} \tag{4.71}$$

We now choose a basis of the vector spaces C^k. We determine the dimension of $\Lambda^k(\mathfrak{g})$ using Proposition 4.1 (with $m = n = 2$): We have $\dim\left(\Lambda^k(\mathfrak{g})\right) = 0$ for $k < 0$, $\dim\left(\Lambda^0(\mathfrak{g})\right) = 1$ and $\dim\left(\Lambda^1(\mathfrak{g})\right) = 4$. For $k \geq 2$

$$\begin{aligned} \dim\left(\Lambda^k(\mathfrak{g})\right) &= \sum_{f=0}^{k} \binom{1+k-f}{k+f} \binom{2}{f} \overset{k\geq 2}{=} \sum_{f=0}^{2} \binom{1+k-f}{k+f} \binom{2}{f} \\ &= (1+k) + 2k + (k-1) \\ &= 4k \end{aligned} \tag{4.72}$$

holds and thus

$$\dim\left(C^k\right) = 2\dim\left(\Lambda^k(\mathfrak{g})\right) = \begin{cases} 0 & \text{for } k < 0 \\ 2 & \text{for } k = 0 \\ 8k & \text{for } k \geq 1 \end{cases}. \tag{4.73}$$

We define

$$X^{(i)} = \underbrace{X \wedge X \wedge \ldots \wedge X}_{i \text{ times}} \in \Lambda^i(\mathfrak{g}) \tag{4.74}$$

and analogously for Y. Then, a basis for C^k is given by the following vectors:

$$\alpha^{(k)}_{i,j} := v_j \odot X^{(i)} \wedge Y^{(k-i)} \tag{4.75}$$

for $i = 0, \ldots, k$, $j = 0, 1$, $k \geq 0$,

$$\beta^{(k)}_{i,j} := v_j \odot H \wedge X^{(i)} \wedge Y^{(k-i-1)} \tag{4.76}$$

for $i = 0, \ldots, k-1$, $j = 0, 1$, $k \geq 1$,

$$\gamma^{(k)}_{i,j} := v_j \odot C \wedge X^{(i)} \wedge Y^{(k-i-1)} \tag{4.77}$$

for $i = 0, \ldots, k-1$, $j = 0, 1$, $k \geq 1$ and

$$\delta^{(k)}_{i,j} := v_j \odot C \wedge H \wedge X^{(i)} \wedge Y^{(k-i-2)} \tag{4.78}$$

for $i = 0, \ldots, k-2,\ j = 0, 1,\ k \geq 2$. (This verifies in particular the dimension formula (4.12).)

Next, we calculate the images under Q of the basis vectors of C^k. One gets:

$$\begin{aligned} Q\alpha_{i,0}^{(k)} &= (\lambda + k - 2i)\,\beta_{i,0}^{(k+1)} + c\gamma_{i,0}^{(k+1)} + \alpha_{i,1}^{(k+1)}, \\ Q\alpha_{i,1}^{(k)} &= -(\lambda + k - 2i - 1)\,\beta_{i,1}^{(k+1)} - c\gamma_{i,1}^{(k+1)} + c\alpha_{i+1,0}^{(k+1)} \end{aligned} \tag{4.79}$$

for $i = 0, \ldots, k,\ j = 0, 1,\ k \geq 0$,

$$\begin{aligned} Q\beta_{i,0}^{(k)} &= c\delta_{i,0}^{(k+1)} + \beta_{i,1}^{(k+1)}, \\ Q\beta_{i,1}^{(k)} &= -c\delta_{i,1}^{(k+1)} + c\beta_{i+1,0}^{(k+1)} \end{aligned} \tag{4.80}$$

for $i = 0, \ldots, k-1,\ j = 0, 1,\ k \geq 1$,

$$\begin{aligned} Q\gamma_{i,0}^{(k)} &= -(\lambda + k - 2i - 1)\delta_{i,0}^{(k+1)} + \gamma_{i,1}^{(k+1)} - \alpha_{i+1,0}^{(k+1)}, \\ Q\gamma_{i,1}^{(k)} &= (\lambda + k - 2i - 2)\delta_{i,1}^{(k+1)} + c\gamma_{i+1,0}^{(k+1)} + \alpha_{i+1,1}^{(k+1)} \end{aligned} \tag{4.81}$$

for $i = 0, \ldots, k-1,\ j = 0, 1,\ k \geq 1$ and

$$\begin{aligned} Q\delta_{i,0}^{(k)} &= \delta_{i,1}^{(k+1)} - \beta_{i+1,0}^{(k+1)}, \\ Q\delta_{i,1}^{(k)} &= c\delta_{i+1,0}^{(k+1)} + \beta_{i+1,1}^{(k+1)} \end{aligned} \tag{4.82}$$

for $i = 0, \ldots, k-2,\ j = 0, 1,\ k \geq 2$.

In the following we will only look at the generic case for $k \geq 2$ where all of the above basis vectors appear. The cases $k = 0$ and $k = 1$ are to be treated separately. We also introduce

$$\begin{aligned} A_0^{(k+1)} &:= \operatorname{span}\left\{Q\alpha_{0,0}^{(k)}, \ldots, Q\alpha_{k,0}^{(k)}\right\}, \quad A_1^{(k+1)} := \operatorname{span}\left\{Q\alpha_{0,1}^{(k)}, \ldots, Q\alpha_{k,1}^{(k)}\right\}, \\ B_0^{(k+1)} &:= \operatorname{span}\left\{Q\beta_{0,0}^{(k)}, \ldots, Q\beta_{k-1,0}^{(k)}\right\}, \quad B_1^{(k+1)} := \operatorname{span}\left\{Q\beta_{0,1}^{(k)}, \ldots, Q\beta_{k-1,1}^{(k)}\right\}, \\ \Gamma_0^{(k+1)} &:= \operatorname{span}\left\{Q\gamma_{0,0}^{(k)}, \ldots, Q\gamma_{k-1,0}^{(k)}\right\}, \quad \Gamma_1^{(k+1)} := \operatorname{span}\left\{Q\gamma_{0,1}^{(k)}, \ldots, Q\gamma_{k-1,1}^{(k)}\right\}, \\ \Delta_0^{(k+1)} &:= \operatorname{span}\left\{Q\delta_{0,0}^{(k)}, \ldots, Q\delta_{k-2,0}^{(k)}\right\}, \quad \text{and} \quad \Delta_1^{(k+1)} := \operatorname{span}\left\{Q\delta_{0,1}^{(k)}, \ldots, Q\delta_{k-2,1}^{(k)}\right\} \end{aligned} \tag{4.83}$$

and $A^{(k+1)} = A_0^{(k+1)} + A_1^{(k+1)}$, $B^{(k+1)} = B_0^{(k+1)} + B_1^{(k+1)}$, $\Gamma^{(k+1)} = \Gamma_0^{(k+1)} + \Gamma_1^{(k+1)}$ as well as $\Delta^{(k+1)} = \Delta_0^{(k+1)} + \Delta_1^{(k+1)}$.

Let first $c \neq 0$. One easily checks that in this case $\Delta^{(k+1)}$ is a subspace of $B^{(k+1)}$ and $\Gamma^{(k+1)}$ a subspace of $A^{(k+1)} + B^{(k+1)}$. Moreover, $A^{(k+1)}$ and $B^{(k+1)}$ are linearly independent. Hence $\dim\left(\operatorname{im}\left(Q^k\right)\right) = \dim\left(A^{(k+1)}\right) + \dim\left(B^{(k+1)}\right) = 4k + 2$ and

$$\begin{aligned} \dim\left(H^k\right) &= \dim\left(\ker\left(Q^k\right)\right) - \dim\left(\operatorname{im}\left(Q^{k-1}\right)\right) \\ &= \dim(C^k) - \dim\left(\operatorname{im}\left(Q^k\right)\right) - \dim\left(\operatorname{im}\left(Q^{k-1}\right)\right) \\ &= 8k - (4k + 2) - (4(k-1) + 2) \\ &= 0. \end{aligned} \tag{4.84}$$

for $k \geq 3$. Similarly one gets $\dim\left(H^0\right) = \dim\left(H^1\right) = \dim\left(H^2\right) = 0$. The cohomology is trivial.

Let $c = 0$. In this case

$$\begin{aligned} A_0^{(k+1)} &= \operatorname{span}\left\{ (\lambda + k - 2i)\, \beta_{i,0}^{(k+1)} + \alpha_{i,1}^{(k+1)} \middle| \, i = 0, \ldots, k \right\}, \\ A_1^{(k+1)} &= \operatorname{span}\left\{ (\lambda + k - 2i - 1)\, \beta_{i,1}^{(k+1)} \middle| \, i = 0, \ldots, k \right\} \end{aligned} \tag{4.85}$$

and

$$\begin{aligned} B_0^{(k+1)} &= \operatorname{span}\left\{ \beta_{i,1}^{(k+1)} \middle| \, i = 0, \ldots, k-1 \right\}, \\ B_1^{(k+1)} &= \{0\} \end{aligned} \tag{4.86}$$

and

$$\begin{aligned} \Gamma_0^{(k+1)} &= \operatorname{span}\left\{ -(\lambda + k - 2i - 1)\delta_{i,0}^{(k+1)} + \gamma_{i,1}^{(k+1)} - \alpha_{i+1,0}^{(k+1)} \middle| \, i = 0, \ldots, k-1 \right\}, \\ \Gamma_1^{(k+1)} &= \operatorname{span}\left\{ (\lambda + k - 2i - 2)\delta_{i,1}^{(k+1)} + \alpha_{i+1,1}^{(k+1)} \middle| \, i = 0, \ldots, k-1 \right\} \end{aligned} \tag{4.87}$$

and

$$\begin{aligned} \Delta_0^{(k+1)} &= \operatorname{span}\left\{ \delta_{i,1}^{(k+1)} - \beta_{i+1,0}^{(k+1)} \middle| \, i = 0, \ldots, k-2 \right\}, \\ \Delta_1^{(k+1)} &= \operatorname{span}\left\{ \beta_{i,1}^{(k+1)} \middle| \, i = 1, \ldots, k-1 \right\}. \end{aligned} \tag{4.88}$$

Obviously $\Delta_1^{(k+1)} \subset B_0^{(k+1)}$ and

$$B_0^{(k+1)} + A_1^{(k+1)} = B_0^{(k+1)} + \operatorname{span}\left\{ (\lambda - k - 1)\beta_{i,1}^{(k+1)} \right\}.$$

In addition $A_0^{(k+1)}$, $\Gamma_1^{(k+1)}$ and $\Delta_0^{(k+1)}$ are linearly dependent. More precisely, for $i = 0, \ldots, k-2$

$$\begin{aligned} &\underbrace{\left((\lambda + k - 2i - 2)\, \delta_{i,1}^{(k+1)} + \alpha_{i+1,1}^{(k+1)} \right)}_{\in \Gamma_1^{(k+1)} \text{ for } i=0,\ldots,k-1} = \\ &= \underbrace{(\lambda + k - 2\,(i+1))\, \beta_{i+1,0}^{(k+1)} + \alpha_{i+1,1}^{(k+1)}}_{\in A_0^{(k+1)} \text{ for } i=-1,0,\ldots,k-1} + \underbrace{(\lambda + k - 2i - 2) \left(\delta_{i,1}^{(k+1)} - \beta_{i+1,0}^{(k+1)} \right)}_{\in \Delta_0^{(k+1)} \text{ for } i=0,\ldots,k-2}, \end{aligned} \tag{4.89}$$

which implies

$$A_0^{(k+1)} + \Gamma_1^{(k+1)} + \Delta_0^{(k+1)} = A_0^{(k+1)} + \Delta_0^{(k+1)} + \operatorname{span}\left\{ (\lambda - k)\, \delta_{i,1}^{(k+1)} + \alpha_{i+1,1}^{(k+1)} \right\}. \tag{4.90}$$

Thus we have

$$\begin{aligned} \operatorname{im}\left(Q^k\right) = A_0^{(k+1)} + \Delta_0^{(k+1)} &+ \operatorname{span}\left\{ (\lambda - k)\, \delta_{i,1}^{(k+1)} + \alpha_{i+1,1}^{(k+1)} \right\} + \\ &+ \Gamma_0^{(k+1)} + B_0^{(k+1)} + \operatorname{span}\left\{ (\lambda - k - 1)\beta_{i,1}^{(k+1)} \right\}. \end{aligned} \tag{4.91}$$

The first line is linearly independent from the second one (as is readily checked) and we can determine their contribution to the dimension of $\operatorname{im}\left(Q^k\right)$ separately. One gets

$$\begin{aligned}
&\dim\left(A_0^{(k+1)}+\Delta_0^{(k+1)}+\operatorname{span}\left\{(\lambda-k)\,\delta_{i,1}^{(k+1)}+\alpha_{i+1,1}^{(k+1)}\right\}\right)= \\
&=(k+1)+(k-1)+\left\{\begin{array}{ll} 0, & \text{if } \lambda=n \\ 1, & \text{if } \lambda\neq n \end{array}\right.
\end{aligned} \tag{4.92}$$

and

$$\begin{aligned}
&\dim\left(\Gamma_0^{(k+1)}+B_0^{(k+1)}+\operatorname{span}\left\{(\lambda-k-1)\beta_{i,1}^{(k+1)}\right\}\right)= \\
&=k+k+\left\{\begin{array}{ll} 0, & \text{if } \lambda=n+1 \\ 1, & \text{if } \lambda\neq n+1 \end{array}\right.
\end{aligned} \tag{4.93}$$

and thus

$$\dim\left(\operatorname{im}\left(Q^k\right)\right)=\left\{\begin{array}{ll} 4k+1, & \text{if } \lambda=n \text{ or } \lambda=n+1 \\ 4k+2, & \text{if } \lambda\neq n \text{ and } \lambda\neq n+1 \end{array}\right. \tag{4.94}$$

This implies

$$\begin{aligned}
\dim\left(H^k\right)&=\dim\left(\ker\left(Q^k\right)\right)-\dim\left(\operatorname{im}\left(Q^{k-1}\right)\right) \\
&=\dim(C^k)-\dim\left(\operatorname{im}\left(Q^k\right)\right)-\dim\left(\operatorname{im}\left(Q^{k-1}\right)\right) \\
&=\left\{\begin{array}{ll} 2, & \text{if } \lambda=k \\ 1, & \text{if } \lambda=k-1 \text{ or } \lambda=k+1 \\ 0, & \text{else} \end{array}\right.
\end{aligned} \tag{4.95}$$

for $k\geq 3$. Finally one calculates the special cases

$$\dim\left(H^0\right)=\left\{\begin{array}{ll} 1, & \text{if } \lambda=1 \\ 0, & \text{if } \lambda\neq 1 \end{array}\right. \tag{4.96}$$

and

$$\dim\left(H^1\right)=\left\{\begin{array}{ll} 2, & \text{if } \lambda=1 \\ 1, & \text{if } \lambda=2 \\ 0 & \text{else} \end{array}\right. . \tag{4.97}$$

$\dim\left(H^2\right)$ is as in equation (4.95). We can summarise our findings:

Theorem 4.3. *The cohomology of $\mathfrak{gl}(1|1)$ w.r.t. ρ_Λ is trivial except for $c=0$ and $\lambda\in\mathbb{Z}_{\geq 1}$. In that case*

$$\dim\left(H^k\right)=\left\{\begin{array}{ll} 2, & \textit{if } k=\lambda \\ 1, & \textit{if } k=\lambda\pm 1 \\ 0 & \textit{else} \end{array}\right. . \tag{4.98}$$

Example 4.2. We want to calculate the cohomology of $\mathfrak{gl}(1|1)$ w.r.t. the representation ρ_Λ using Lemma 4.4. For this we look at the subalgebra $\mathfrak{q}=\mathfrak{gl}(1|1)_0=\mathbb{C}H+\mathbb{C}C$.

We must first show that the natural representation ρ^k_{nat} of $\mathfrak{q}$ on the C^k is completely reducible. We have

$$\rho_{\mathrm{nat}}(H) = L_1 + \lambda_1 = L_1 - c^3 b_3 + c^4 b_4. \tag{4.99}$$

It is clear that $\rho_{\mathrm{nat}}(H)$ is diagonal in the basis of C^k we used in the previous example since the two latter terms count the number of Y's minus the number of X's. $\rho_{\mathrm{nat}}(C)$ is simply the multiplication operator by c, i.e. also diagonal (w.r.t. every basis) with eigenvalue c. Hence in particular ρ^k_{nat} is completely reducible as representation restricted to $\mathfrak{q}$. We can thus apply Lemma 4.4.

We are looking for the vectors w in the C^k's for which $\rho_{\mathrm{nat}}(\mathfrak{q})w = 0$. Obviously if $c \neq 0$ this is only the zero vector and hence the cohomology is trivial. So, let $c = 0$ in the following. Then $\rho_{\mathrm{nat}}(C) = 0$ and it suffices to look at $\rho_{\mathrm{nat}}(H)$. One finds

$$\begin{aligned}
\rho^k_{\mathrm{nat}}(H)\alpha^{(k)}_{i,0} &= (\lambda + k - 2i)\,\alpha^{(k)}_{i,0} && \text{for } i = 0, \ldots, k,\\
\rho^k_{\mathrm{nat}}(H)\alpha^{(k)}_{i,1} &= (\lambda + k - 2i - 1)\,\alpha^{(k)}_{i,1} && \text{for } i = 0, \ldots, k,\\
\rho^k_{\mathrm{nat}}(H)\beta^{(k)}_{i,0} &= (\lambda + k - 2i - 1)\,\beta^{(k)}_{i,0} && \text{for } i = 0, \ldots, k-1,\\
\rho^k_{\mathrm{nat}}(H)\beta^{(k)}_{i,1} &= (\lambda + k - 2i - 2)\,\beta^{(k)}_{i,1} && \text{for } i = 0, \ldots, k-1,\\
\rho^k_{\mathrm{nat}}(H)\gamma^{(k)}_{i,0} &= (\lambda + k - 2i - 1)\,\gamma^{(k)}_{i,0} && \text{for } i = 0, \ldots, k-1,\\
\rho^k_{\mathrm{nat}}(H)\gamma^{(k)}_{i,1} &= (\lambda + k - 2i - 2)\,\gamma^{(k)}_{i,1} && \text{for } i = 0, \ldots, k-1,\\
\rho^k_{\mathrm{nat}}(H)\delta^{(k)}_{i,0} &= (\lambda + k - 2i - 2)\,\delta^{(k)}_{i,0} && \text{for } i = 0, \ldots, k-2,\\
\rho^k_{\mathrm{nat}}(H)\delta^{(k)}_{i,1} &= (\lambda + k - 2i - 3)\,\delta^{(k)}_{i,1} && \text{for } i = 0, \ldots, k-2.
\end{aligned} \tag{4.100}$$

We also see now that for $\lambda \notin \mathbb{Z}$ the cohomology can only be trivial since there are no vectors which are annihilated by the action of all of $\mathfrak{q}$. Let $\lambda \in \mathbb{Z}$ be arbitrary but fixed. Using the above equations we determine the vectors which lie in $C_{\mathfrak{q}}$. These are

$$\begin{aligned}
&\alpha^{(k)}_{i,0} \quad \text{for } k = |\lambda|, |\lambda| + 2, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k}{2},\\
&\alpha^{(k)}_{i,1} \quad \text{for } k = |\lambda - 1|, |\lambda - 1| + 2, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-1}{2},\\
&\beta^{(k)}_{i,0} \quad \text{for } k = |\lambda| + 1, |\lambda| + 3, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-1}{2},\\
&\beta^{(k)}_{i,1} \quad \text{for } k = |\lambda - 1| + 1, |\lambda - 1| + 3 \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-2}{2},\\
&\gamma^{(k)}_{i,0} \quad \text{for } k = |\lambda| + 1, |\lambda| + 3, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-1}{2},\\
&\gamma^{(k)}_{i,1} \quad \text{for } k = |\lambda - 1| + 1, |\lambda - 1| + 3 \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-2}{2},\\
&\delta^{(k)}_{i,0} \quad \text{for } k = |\lambda| + 2, |\lambda| + 4, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-2}{2},\\
&\delta^{(k)}_{i,1} \quad \text{for } k = |\lambda - 1| + 2, |\lambda - 1| + 4, \ldots \text{ and } i = \frac{\lambda}{2} + \frac{k-3}{2}.
\end{aligned} \tag{4.101}$$

Obviously we have to distinguish between the cases $\lambda \geq 1$ and $\lambda \leq 0$. The dimensions of the $C^k_{\mathfrak{q}}$'s can be easily read off and are listed in Table 4.1. We want to determine the

$\lambda \geq 1$	k	$\cdots$	$\lambda - 3$	$\lambda - 2$	$\lambda - 1$	λ	$\lambda + 1$	$\lambda + 2$	$\cdots$
	$\dim(C^k_{\mathfrak{q}})$	$\cdots$	0	0	1	3	4	4	$\cdots$
$\lambda \leq 0$	k	$\cdots$	$\lvert\lambda\rvert - 3$	$\lvert\lambda\rvert - 2$	$\lvert\lambda\rvert - 1$	$\lvert\lambda\rvert$	$\lvert\lambda\rvert + 1$	$\lvert\lambda\rvert + 2$	$\cdots$
	$\dim(C^k_{\mathfrak{q}})$	$\cdots$	0	0	0	1	4	4	$\cdots$

Table 4.1: Dimension of the $C^k_{\mathfrak{q}}$'s.

cohomology now. For this, we investigate how Q acts on vectors in $C^k_{\mathfrak{q}}$. We first look at the case $\lambda \geq 1$. One gets:

($k = \lambda - 1$):

$$Q\alpha^{(\lambda-1)}_{\lambda-1,1} = 0, \tag{4.102}$$

($k = \lambda$):

$$\begin{aligned} Q\alpha^{(\lambda)}_{\lambda,0} &= \alpha^{(\lambda+1)}_{\lambda,1}, \\ Q\beta^{(\lambda)}_{\lambda-1,1} &= 0, \\ Q\alpha^{(\lambda)}_{\lambda-1,1} &= \alpha^{(\lambda+1)}_{\lambda,1}, \end{aligned} \tag{4.103}$$

($k \geq \lambda + 1$, $k - \lambda =: 2r + 1$ odd):

$$\begin{aligned} Q\alpha^{(\lambda+2r+1)}_{\lambda+r,1} &= 0 \\ Q\beta^{(\lambda+2r+1)}_{\lambda+r,0} &= \beta^{(\lambda+2r+2)}_{\lambda+r,1}, \\ Q\gamma^{(\lambda+2r+1)}_{\lambda+r,0} &= \gamma^{(\lambda+2r+2)}_{\lambda+r,1} - \alpha^{(\lambda+2r+2)}_{\lambda+r+1,0}, \\ Q\delta^{(\lambda+2r+1)}_{\lambda+r-1,1} &= \beta^{(\lambda+2r+2)}_{\lambda+r,1}, \end{aligned} \tag{4.104}$$

($k \geq \lambda + 2$, $k - \lambda =: 2r$ even):

$$\begin{aligned} Q\alpha^{(\lambda+2r)}_{\lambda+r,0} &= \alpha^{(\lambda+2r+1)}_{\lambda+r,1}, \\ Q\beta^{(\lambda+2r)}_{\lambda+r-1,1} &= 0, \\ Q\gamma^{(\lambda+2r)}_{\lambda+r-1,1} &= \alpha^{(\lambda+2r+1)}_{\lambda+r,1}, \\ Q\delta^{(\lambda+2r)}_{\lambda+r-1,0} &= \delta^{(\lambda+2r+1)}_{\lambda+r-1,1} - \beta^{(\lambda+2r+1)}_{\lambda+r,1}. \end{aligned} \tag{4.105}$$

We can then without further ado read off the dimensions of the images and hence of the kernels of Q^k which leads to the cohomology shown in Table 4.2.

Let us finally look at the case $\lambda \leq 0$: We have to repeat the above steps for $C^k_{\mathfrak{q}}$ with $k = |\lambda|, |\lambda| + 1, \ldots$ and one gets the cohomology in Table 4.3.

The results are of course identical to those in Theorem 4.3.

k	$\cdots$	$\lambda-2$	$\lambda-1$	λ	$\lambda+1$	$\lambda+2$	$\lambda+3$	$\cdots$
$\dim(C_{\mathfrak{q}}^k)$	$\cdots$	0	1	3	4	4	4	$\cdots$
$\dim\left(\ker\left(Q^k\|_{C_{\mathfrak{q}}^k}\right)\right)$	$\cdots$	0	1	2	2	2	2	$\cdots$
$\dim\left(\operatorname{im}\left(Q^{k-1}\|_{C_{\mathfrak{q}}^k}\right)\right)$	$\cdots$	0	0	0	1	2	2	$\cdots$
$\dim(H^k)$	$\cdots$	0	1	2	1	0	0	$\cdots$

Table 4.2: Cohomology for $\lambda \geq 1$.

k	$\cdots$	$\|\lambda\|-2$	$\|\lambda\|-1$	$\|\lambda\|$	$\|\lambda\|+1$	$\|\lambda\|+2$	$\|\lambda\|+3$	$\cdots$
$\dim(C_{\mathfrak{q}}^k)$	$\cdots$	0	0	1	3	4	4	$\cdots$
$\dim\left(\ker\left(Q^k\|_{C_{\mathfrak{q}}^k}\right)\right)$	$\cdots$	0	0	0	1	2	2	$\cdots$
$\dim\left(\operatorname{im}\left(Q^{k-1}\|_{C_{\mathfrak{q}}^k}\right)\right)$	$\cdots$	0	0	0	1	2	2	$\cdots$
$\dim(H^k)$	$\cdots$	0	0	0	0	0	0	$\cdots$

Table 4.3: Cohomology for $\lambda \leq 0$.

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