The strong coupling. The gluons momentum fraction*

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Abstract

The aim of the work is to estimate some properties of the strong interaction related to the asymptotic freedom. The first one is the running of coupling α_s. We compute the value of the coupling for the next-to-leading order (NLO) and for NNLO using iterative techniques. We find an agreement with existing results from the literature and also comment to some ambiguities related to the procedure. The second property we are focusing on is the dependence of momentum fraction of gluons inside a nucleon on α_s and small-x behaviour of the gluonic density. We solve the DGLAP equation and obtain the dependence for LO, NLO, NNLO.

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1 The strong coupling

The strong coupling α_s is a fundamental parameter of the Standard Model characterising the strength of the strong interaction. To remove infinities coming from loop diagrams and define physical parameters contributing to Lagrangian one should make use of the renormalization procedure. This procedure, apart from everything else, introduces an arbitrary energy scale Q at which the calculation should be performed (the physical coupling becomes a function of this scale). If the renormalized coupling $\alpha_s(Q_0^2)$ can be fixed (i.e. measured) at a given scale Q_0^2, then one can predict the value of α_s at any other energy scale Q^2 via the renormalization group equation [2, 5]

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta\left(\alpha_s(Q^2)\right). \tag{1}$$

In QCD the β function can be expanded as

$$\beta\left(\alpha_s(Q^2)\right) = -\beta_0\,\alpha_s^2(Q^2) - \beta_1\,\alpha_s^3(Q^2) - \beta_2\,\alpha_s^4(Q^2) - \beta_3\,\alpha_s^5 - \dots, \tag{2}$$

where

$$\beta_0 = \frac{33 - 2\,n_f}{12\,\pi}, \quad \beta_1 = \frac{153 - 19\,n_f}{24\,\pi^2}, \quad \beta_2 = \frac{77138 - 15099\,n_f + 325\,n_f^2}{3456\,\pi^3},$$

$$\beta_3 = \frac{\frac{149753}{6} + 3564\,\zeta_3 - \left(\frac{1078361}{162} + \frac{6208}{27}\zeta_3\right) n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) n_f^2 + \frac{1093}{729}\,n_f^3}{(4\,\pi)^4}. \tag{3}$$

and n_f is the number of flavours for quarks with mass less than the energy scale Q and $\zeta_3 \simeq 1.2020569$ is the Riemann zeta function. The three- and four-loop coefficients, β_2 and β_3 respectively, depend on the renormalization scheme. The result given here is for the minimal-substraction (MS) scheme. The β function coefficients are extracted from higher-order (loop) corrections of the bare vertices of the theory. The minus sign in Eq. (2) is the origin of the asymptotic freedom, determining that the renormalized strong coupling becomes weak for large momentum Q.

Table 1: Expansion coefficients of the β-function for $n_f = 1, 2, \dots, 6$.

n_f	β_0	β_1	β_2	β_3
1	0.822	0.566	0.582	0.910
2	0.769	0.485	0.450	0.681
3	0.716	0.405	0.324	0.485
4	0.663	0.325	0.205	0.322
5	0.610	0.244	0.091	0.193
6	0.557	0.165	-0.016	0.099

Figure 1 shows the beta function with three light flavours, when is retained only the first (one-loop) term, or the first and second (two-loop) terms or the first up fourth (four-loop) terms in the perturbative expansion in Eq. (2). The difference between truncated β functions dicreases with addition of higher order α_s terms.

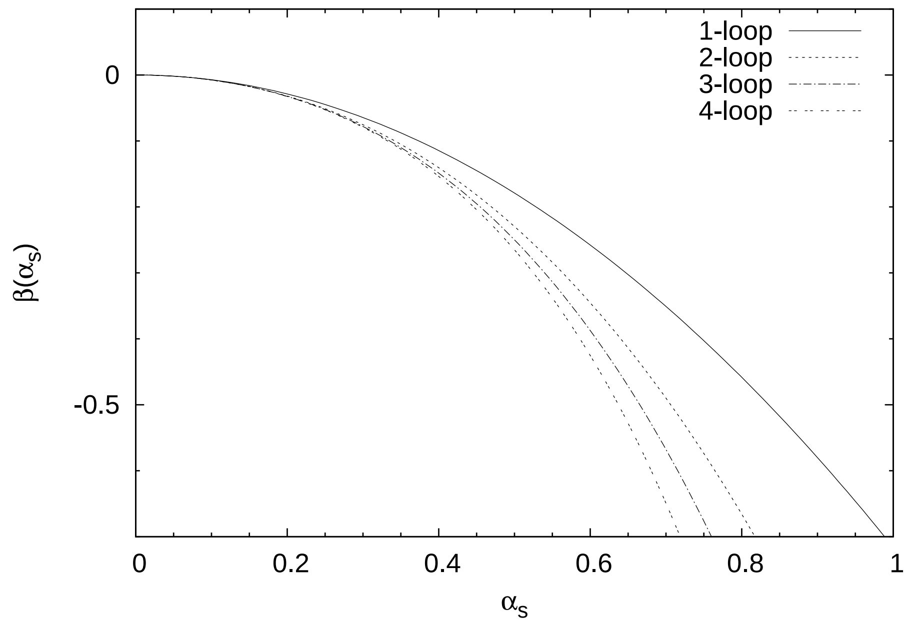

Figure 1: The β function with $n_f = 3$.

The β function coefficients change by discrete amount as flavor thresholds are crossed. In consequence, there is an indirect dependence of α_s on the quark mass. The values of $\alpha_s^{(n_f)}$ and $\alpha_s^{(n_f-1)}$ are related so that a physical quantity calculated in both cases gives the same result [3, 2]. This implies that for the so-called pole mass $Q = Q_0$

$$\alpha_s^{(n_f-1)}(Q_0^2) = \alpha_s^{(n_f)}(Q_0^2) - 0.0295 \left(\alpha_s^{(n_f)}(Q_0^2)\right)^3 + O\left(\left(\alpha_s^{(n_f)}(Q_0^2)\right)^4\right). \tag{4}$$

In leading and in next-to-leading order the matching condition is $\alpha_s^{(n_f)}(Q_0) = \alpha_s^{(n_f)}(Q_0)$.

In solving the differential equation (1) for $\alpha_s(Q^2)$, a constant of integration is introduced. This constant is the fundamental constant of QCD that must be determined experimentally. A possible choice for this constant is the value of α_s at a fixed-reference scale Q_0[1]. The strong coupling at other values of Q can be obtained from

$$\ln \frac{Q^2}{Q_0^2} = \int\limits_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \frac{\mathrm{d}x}{\beta(x)}. \tag{5}$$

[1]It has become standard to choose $Q_0 = M_Z$.

An alternative approach is to introduce a dimensionful parameter by the definition of $\alpha_s(Q^2)$. By convention this parameter is called Λ and is the constant of integration

$$\ln\frac{Q^2}{\Lambda^2} = -\int\limits_{\alpha_s(Q^2)}^{\infty}\frac{\mathrm{d}x}{\beta(x)} = \int\limits_{\alpha_s(Q^2)}^{\infty}\frac{\mathrm{d}x}{\beta_1 x^2(1+b_1x+b_2x^2+\dots)}. \tag{6}$$

where $b_k = \frac{\beta_k}{\beta_0}$ $(k=1,2,\dots)$. Λ represents the scale at which the coupling would diverge. The introduction of Λ allows us to writhe the asymptotic solution for α_s in terms of this parameter. In leading order (LO), i.e. retaining only the β_0 coefficient in the β function, Eq. (6) can be integrated out leading to

$$\ln\frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\alpha_s(Q^2)}. \tag{7}$$

By including the b_1 coefficient in the integral (6) the definition of the Λ is extended to the next-to-leading order (NLO)

$$\ln\frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\,\alpha_s(Q^2)} + \frac{b_1}{\beta_0}\ln\frac{b_1\alpha_s(Q^2)}{1+b_1\alpha_s(Q^2)}, \tag{8}$$

for next-to-next-leading order (NNLO), i.e. b_2 is included, the integration gives:

$$\ln\frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\,\alpha_s(Q^2)} + \frac{b_1}{\beta_0}\ln\left(\alpha_s(Q^2)\right) - \frac{b_1}{2\,\beta_0}\ln\left(1+b_1\alpha_s(Q^2)+b_2\alpha_s^2(Q^2)\right) -$$
$$-\frac{b_1^2-2b_2}{\beta_0\sqrt{4b_2-b_1^2}}\arctan\frac{2b_2\alpha_s(Q^2)+b_1}{\sqrt{4b_2-b_1^2}} + \mathrm{C}_0, \tag{9}$$

where

$$\mathrm{C}_0 = \frac{b_1}{2\,\beta_0}\ln b_2 + \frac{b_1^2-2b_2}{\beta_0\sqrt{4b_2-b_1^2}}\cdot\frac{\pi}{2}.$$

Eq. (8) and (9) allow a numerical determination of $\alpha_s(Q^2)$ for a fixed value of Λ.

Furthermore one can Taylor expand (9) for small α_s and obtain an approximate solution of (8) and (6) in terms of powers of $\alpha_s(Q^2)$

$$\ln\frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\,\alpha_s(Q^2)} + \frac{b_1}{\beta_0}\ln\beta_0\alpha_s(Q^2) + \mathrm{C}_1 + O\left(\alpha_s(Q^2)\right), \quad \mathrm{C}_1 = \frac{b_1}{\beta_0}\ln\frac{b_1}{\beta_0} \tag{10}$$

$$\ln\frac{Q^2}{\Lambda^2} = \frac{1}{\beta_0\,\alpha_s(Q^2)} + \frac{b_1}{\beta_0}\ln\beta_0\alpha_s(Q^2) - \frac{(b_1^2-b_2)}{\beta_0}\alpha_s(Q^2) + \mathrm{C}_0 + \mathrm{C}_1 + O\left(\alpha_s^2(Q^2)\right), \tag{11}$$

where

$$\mathrm{C}_1 = -\frac{b_1^2-2b_2}{\beta_0\sqrt{4b_2-b_1^2}}\arctan\frac{b_1}{\sqrt{4b_2-b_1^2}} - \frac{b_1}{\beta_0}\ln\beta_0.$$

At this point one can redefine Λ parameter in a way that no constant term appreas in Eq. (8) and (9). To obtain an approximate iterative solution for α_s in inverse powers of L one has to substitute α_s form Eq. (7) to logarithmic term of Eq. (10) (without the constant term). Making Taylor expansion[2] by large L one then obtains first two terms of the NLO iterative solution (first two terms in Eq. given below). This NLO iterative solution of α_s can be again substituted in NNLO eq. (11) and after the same chain of steps one obtains

$$\alpha_s(Q^2) = \frac{1}{\beta_0 L} - \frac{b_1 \ln L}{(\beta_0 L)^2} + \frac{1}{(\beta_0 L)^3}\left(b_1^2(\ln^2 L - \ln L - 1) + b_2\right) + O\left(\frac{1}{L^4}\right) \quad (12)$$

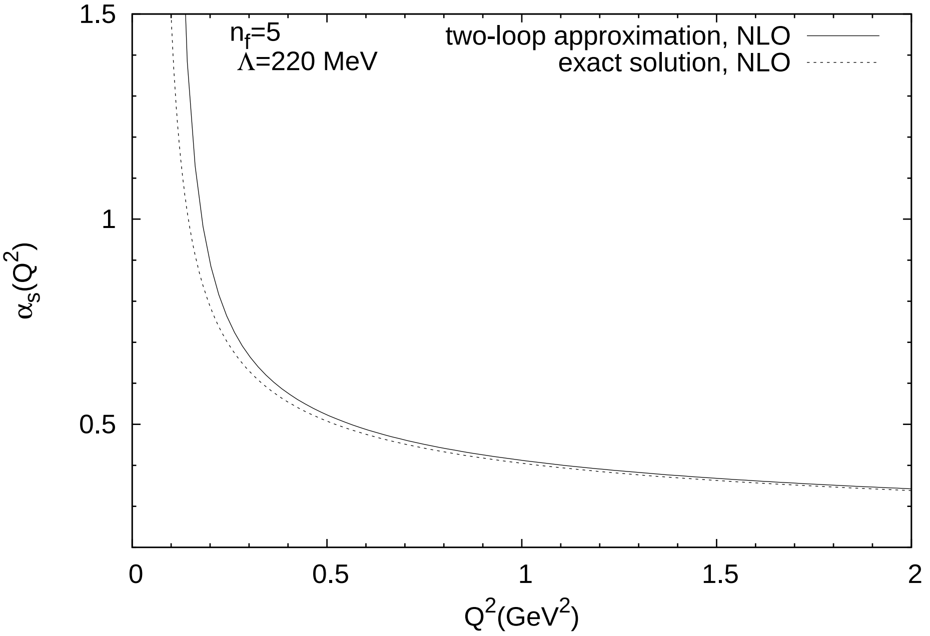

Figure 2: α_s dependence of Q^2. Two-loop approximation courve is calculated using first two term of Eq. (12). Exanct solution represents numerically solved Eq. (8) minus the constant C_1.

2 The momentum fraction of gluons in the nucleon

2.1 Motivation

The structure of a hadron can be probed by the deep inelastic scattering (DIS) where an electron emits a virtual photon which scatters on a quark in a hadron. The momentum and

[2] We assume that Λ in Eq. (7) is the same (or sufficiently close to) the new redefined Λ.

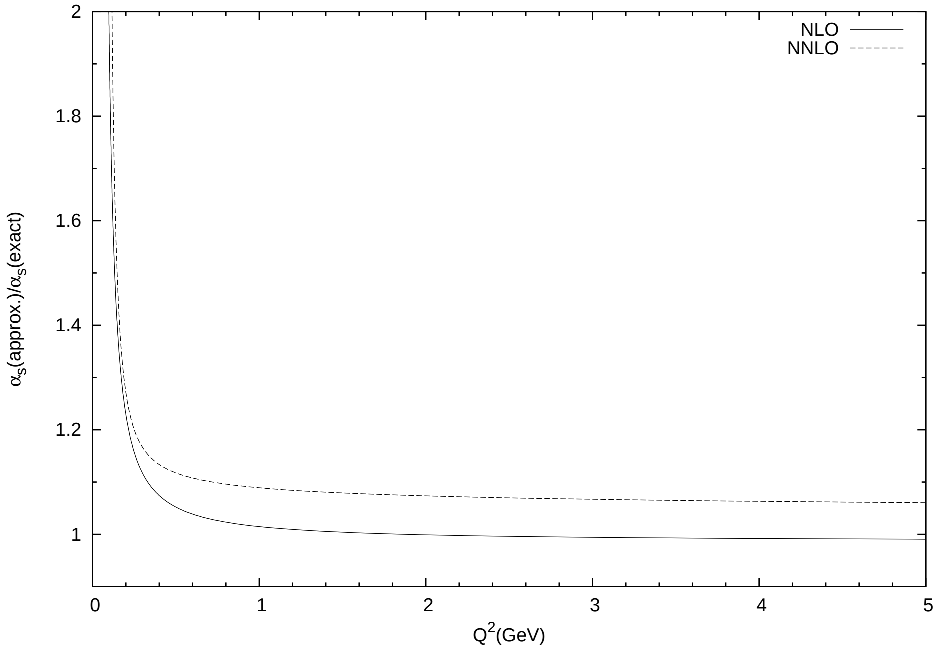

Figure 3: Dependence of the ratio between approximative solution of α_s given by Eq.(12) and exact solution Eq. (8), (9) for NLO, NNLO respectively, minus the constant terms on Q^2.

energy transver of the electron can be then measured and provide an information about the structure functions of the hadron. The hadron itself consists of gluons and (anti-)quarks, which can be characterized by their typical energy fraction and rapidity

$$x \equiv \frac{E_{constituent}}{E_{hadron}}, \tag{13}$$

$$y = y_{hadron} - \ln\left(\frac{1}{x}\right). \tag{14}$$

The larger is the total energy the smaller could be x. The number of small x partons per unit rapidity is given by

$$\frac{dN}{dy} = x\, g(x, Q^2), \tag{15}$$

where we also take into account the energy scale Q^2 because of the dependence of the number of partons on the resolution scale. The density of gluons measured at HERA accelerator is shown in Figure 4.

We clearly see from the plot that the density of gluons increases for small x, which means that for increasing energies the low momentum gluon density grows. As the density of gluons per unit area per unit rapidity increases, the typical transverse separation of gluons decreases, so the matter becomes very dense and the strong coupling becomes relatively small. Because of the Bose-Einstein statistics we can expect that at large phase space density gluons can be described classically. This opens a door for a new phenomenology and motivates us to measure the properties of gluonic flields at small x.

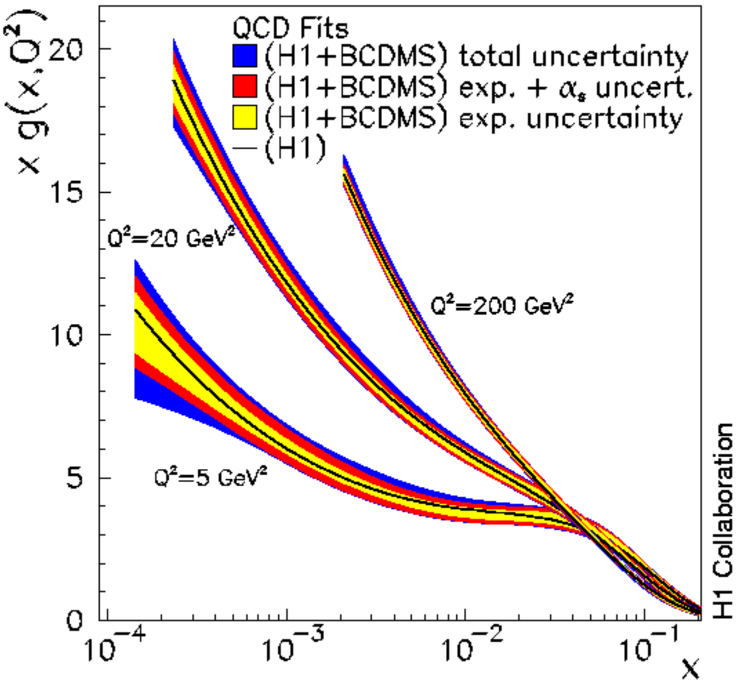

Figure 4: Gluon density distribution as a function of x at fixed values of Q^2 (by H1 collaboration [1])

2.2 Computation

The DGLAP equations, that describes the Q^2 dependence of partons distribution function at a fixed fraction x of the nucleon momentum, can be rewritten in terms of gluon $g(x, Q^2)$, a flavour non-singlet $q^{(ns)}(x, Q^2)$ and a flavour singlet $\Sigma(x, Q^2)$ distribution function as following

$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2} q^{(ns)}(Q^2) = P_{qq}^{(ns)} \otimes q^{(ns)}(Q^2),$$
$$\frac{\mathrm{d}}{\mathrm{d}\ln Q^2} \begin{pmatrix} \Sigma(x, Q^2) \\ g(x, Q^2) \end{pmatrix} = \left(\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} \Sigma \\ g \end{pmatrix} \right)(x, Q^2), \tag{16}$$

with the Mellin convolution defined as

$$(f \otimes g)(x, Q^2) = \int_0^1 \frac{\mathrm{d}x}{y} f\left(\frac{x}{y}\right) g(x, Q^2), \tag{17}$$

and the splitting function P_{ij}. Valence quark distributions are non-singlet distributions and their evolution does not depend on the gluon distribution.

Using the momentum fraction carried by a parton distribution

$$[q](Q^2) \equiv \int_0^1 \mathrm{d}x\, x\, q(x, Q^2), \tag{18}$$

and the renormalization group equation (1), one obtains the evolution equations for momentum fraction carried by gluons $[g]$ and quarks $[\Sigma]$ from (16)

$$\beta(\alpha_s(Q^2)) \frac{\mathrm{d}}{\mathrm{d}\alpha_s(Q^2)} \begin{pmatrix} [\Sigma] \\ [g] \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} \begin{pmatrix} [\Sigma] \\ [g] \end{pmatrix}, \tag{19}$$

where the anomalous dimensions $\gamma_{ij} = \int_0^1 \mathrm{d}u\, u\, P_{ij}(u)$ can be preturbativelly expanded in $\alpha_s(Q^2)$

$$\begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix} = -\alpha_s(Q^2) \begin{pmatrix} a_0 & -b_0\, n_f \\ -a_0 & b_0\, n_f \end{pmatrix} - \alpha_s^2(Q^2) \begin{pmatrix} a_1 - b_1\, n_f & -c_1\, n_f \\ -a_1 + b_1\, n_f & c_1\, n_f \end{pmatrix} -$$

$$-\alpha_s^3(Q^2) \begin{pmatrix} a_2 - b_2\, n_f - c_2\, n_f^2 & -d_2\, n_f + e_2\, n_f^2 \\ -a_2 + b_2\, n_f + c_2\, n_f^2 & d_2\, n_f - e_2\, n_f^2 \end{pmatrix}, \tag{20}$$

with the entries

a_0	b_0	a_1	b_1	c_1	a_2
$\frac{8}{9\pi}$	$\frac{1}{6\pi}$	$\frac{734}{243\pi}$	$\frac{26}{81\pi}$	$\frac{611}{1296\pi}$	$\frac{344638}{6561\pi} + \frac{80}{81\pi}\zeta_3$

b_2	c_2	d_2	e_2
$\frac{33722}{8748\pi} + \frac{160}{27\pi}\zeta_3$	$\frac{71}{243\pi}$	$\frac{670871\pi}{69984\pi} - \frac{650}{108\pi}\zeta_3$	$\frac{8830}{11664\pi}$

The sum rule $[\Sigma] + [g] = 1$ imposes the constraint $\gamma_{qq} - \gamma_{qg} = \gamma_{gg} - \gamma_{gq}$ on the linear system of ODE's (19). A general form for the gluon momentum fraction can be written as

$$[g](Q^2) = \mathrm{e}^{p(\alpha_s(Q^2))} \left(\int_{\alpha_s(Q_0^2)}^{\alpha_s(Q^2)} \mathrm{d}x\, \mathrm{e}^{-p(x)} q(x) + [g](Q_0^2) \right), \tag{21}$$

where

$$p(x) = \int_{\alpha(Q^2)}^{x} \frac{\gamma_{gg} - \gamma_{gq}}{\beta(y)}\, \mathrm{d}y, \quad q(x) = \frac{\gamma_{gq}}{\beta(x)}$$

and the initial conditions $\alpha_s(Q_0^2)$, $[g](Q_0^2)$.

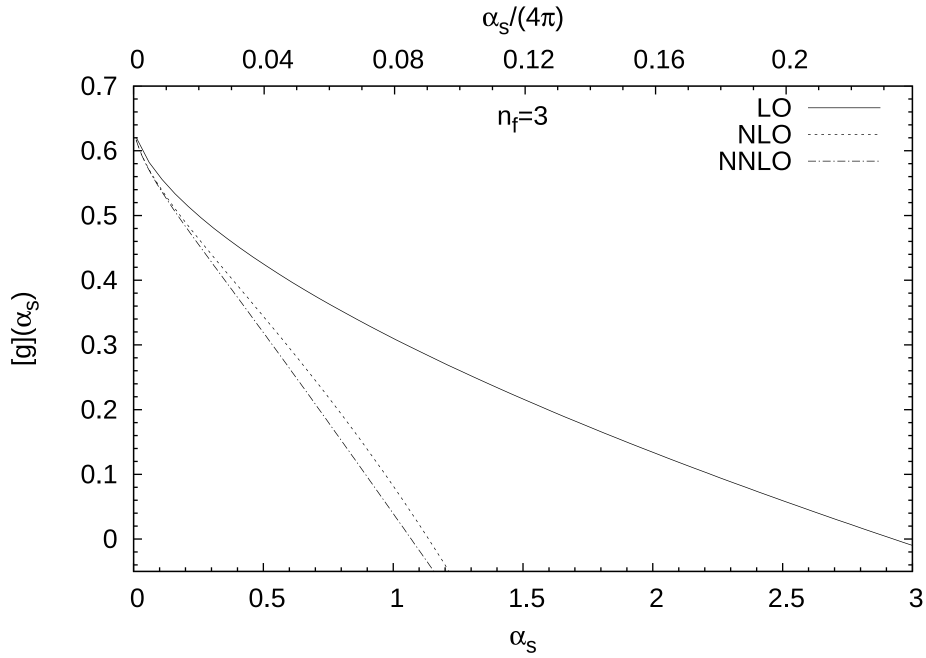

Figure 5: The dependence of gluons momentum fraction on strong coupling α_s for LO, NLO, NNLO approximation of anomalous dimensions.

The results at LO, NLO and NNLO, employing the perturbative expansion (2) for β function are given in Figure 5. The input data for initial condition for $n_f = 3$ are $[g](2\,\mathrm{GeV}^2) = 0.479$ and $\alpha_s(2\,\mathrm{GeV}^2) = 0.313$ [4].The deviation of LO from NLO and NNLO of $[g]$ increases with growing α_s, i. e. in the region where the perturbative calculations are not valid. The behavior of gluons' momentum fraction remains similar going from NLO to NNLO. The difference between curves becomes smaller as α_s decreases and they approach the same value for $\alpha_s \to 0$.

Table 2: The values of α_s, Q^2 and Q (calculated with Eq. (12)) at $[g](\alpha_s) = 0$.

	LO	NLO	NNLO
α_s	2.93	1.14	1.07
$Q^2(\mathrm{GeV}^2)$	0.19	0.63	0.73
$Q(\mathrm{GeV})$	0.44	0.79	0.85

The values for Q^2 in Table 2 are calculated using Eq. (12). A significant difference between Q^2 at LO from NLO and NNLO is observed.

3 Summary

In the present work we desctibe a dependence of the running coupling α_s on the energy scale Q^2 for the LO, NLO and NNLO. We also found the dependence of gluonic momentum fraction on strong coupling α_s for LO, NLO, NNLO approximation of anomalous dimensions using the DGLAP equation. We also commented on a possible application of the small-x results to the estimation of the classical gluonic density inside a nucleon.

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