Summer student program report
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Extremal correlators in string theory on $AdS_3 \times S^3 \times T^4$

Abstract

In this note I will report about my summer project in the string group at DESY Hamburg. In this project I worked on a topic within the so called AdS/CFT correspondence, which is a duality between a gravitational- and a quantum field theory. After a short introduction, in section II I will review the basics of the AdS_3/CFT_2-correspondence [1], which is one of the most studied and elaborated version of the conjecture. Section III shows the first calculation of an at the moment unknown four point correlator in this theory. In section IV we show a recursion relation for a general p-point function, which we will prove by using a worldsheet operator product expansion introduced in [19].

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Contents

1 Introduction

The major problem of the AdS/CFT-correspondence is that it is only a conjecture. There are a few non-trivial ideas on how one could prove this conjecture, but no proof at all. However, one can test the duality by comparing correlation functions computed in the bulk of the AdS space with the corresponding ones in the conformal field theory on the boundary of the AdS space. For a long time, it was not known how to relate correlators in the AdS_3/CFT_2 duality. In [2, 6] extremal and non-extremal three-point functions of chiral primary operators in the worldsheet theory for string theory on $AdS_3 \times S^3 \times T^4$ were successfully matched to the corresponding correlators in the dual boundary theory. Recently, also four-point correlators were tested in [3].

In this project, we will extend the test to higher p-point functions. In principle, we could just repeat the procedure for five-, six-, etc. point functions, but this may be cumbersome due to the complexity of the correlators. Instead we will construct an operator product expansion (OPE) for chiral primary operators on the worldsheet. The idea is to use this OPE to find a simple recursion relation for the p-point correlators.

2 Basics of the AdS_3/CFT_2-correspondence

The AdS/CFT correspondence can be understood as an gauge theory/gravity correspondences, because it describes a duality between a $AdS_n \times \mathcal{M}$ string theory ($\mathcal{M}$:= manifold with $dim(\mathcal{M}) = 10 - n$) and a supersymmetric CFT in $n - 1$ space-time dimensions. It is worthy of mention that the gauge group of the CFT depend on the choice of the manifold $\mathcal{M}$. The prototype of this correspondence [1] is the exact equivalence between type IIB string theory compactified on $AdS_5 \times S^5$, and four-dimensional $\mathcal{N} = 4$ supersymmetric Yang Mills theory. This was first worked out and proposed by Juan Maldacena in 1997.

Quit general, AdS_d/CFT_{d-1} relates a d-dimensional theory of gravity on AdS_d to a $d-1$-dimensional conformal quantum field theory on its boundary. This will now be explained in the special case of an AdS_3 space.

2.1 Anti-de Sitter space

The Anti-de Sitter space is a maximally symmetric solution of the Einstein equations with a negative cosmological constant. The AdS_3-space which was used by us is a three dimensional manifold

$$-X_0 - X_1 + X_2 + X_3 = L^2$$

embedded in a four-dimensional space with metric,

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2$$

The boundary ofAdS_3 is a 2-dimensional Minkowski space-time of an anti-de Sitter space is an example of a warped space: in a suitable local coordinate system,

$$ds^2 = L^2(dr^2 + e^{2r}(n_{\mu\nu}dx^\mu dx^\nu)),$$

where r is the radial direction and x^μ ($\mu = 0, 1$) the coordinate on the boundary. In the above equation the factor of L is just a scale factor. The boundary of the anti-de Sitter space could be reached by the limit $r \to \infty$. Here the warp factor e^{2r} blows up. On this boundary we expect the boundary field theory to live.

2.2 Maldacenas conjecture

To derive the AdS/CFT correspondence like Maldacena did it in [1] one has to understand the notion of Dp-branes, where the p stands for the (spatial) dimension of the brane. A Dp-brane is a $p+1$ dimensional hyperplane where open strings end with Dirichlet boundary conditions.

For his conjecture Maldacena used a D3-Brane. This has a description both in terms of open and closed strings. In the limit of vanishing string length $l_s \to 0$, the open strings on the D3 generate a $\mathcal{N} = 4$ super Yang Mills theory with the gauge group $U(N)$ and gauge coupling constant g_{YM}^2. On the other hand the D3-branes also emit closed strings, which generate an $AdS_5 \times S^5$ space. For our consideration we identify the following parameters:

$$g_s = g_{YM}^2, \qquad (L/l_s)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda$$

Hence the AdS/CFT correspondence is an open-closed string duality.

2.3 WZW-model of Lie Groups

In our case we are looking at a $AdS_3 \times S^3 \times T^4$ string theory, which we will describe by using the WZW-formulation (cf. [5]), where WZW stands for Wess-Zumino-Witten. For the construction of the WZW-model we will follow [6] and [2]. The Lagrangian formulation of the WZW model is given by

$$S = -\frac{k}{8\pi} \int \mathcal{K}\left(\gamma^{-1}\partial^\mu \gamma, \gamma^{-1}\partial_\mu \gamma\right) dx^2 + 2\pi \tilde{S}(\gamma) \tag{2.1}$$

In this Lagrangian $\gamma : \mathbb{C} \to G$ is a G-valued function which lives on the complex plane. In this report we are only looking at Lie Groups G . We are also use the partial derivative $\partial_\mu = \frac{\partial}{\partial x_\mu}$ in the Euclidean metric the Killing form of the Lie Algebra $\mathfrak{g}$ of G.

$$\mathcal{K}(x, y) = \frac{Tr(ad_x ad_y)}{2h^\vee} \quad x, y \in \mathfrak{g}. \tag{2.2}$$

If $\mathfrak{g}$ is an Lie Algebra then $ad : \mathfrak{g} \to End(\mathfrak{g})$ is the adjoint map, and is defined by $ad_x y := [x, y] \in \mathfrak{g}$. The number $h^\vee$ which arises in eq. (2.2) is an integer which is fixed by the algebra $\mathfrak{g}$ and is called Coexter number of $\mathfrak{g}$. The second term of the action (2.1) is the Wess-Zumino term,

$$\tilde{S}(\gamma) = \frac{-1}{48\ \pi^2} \int_{B^3 = \partial S^2} d^3y\ \epsilon^{ijk} \mathcal{K}\left(\gamma^{-1}\frac{\partial \gamma}{\partial y^i}, \left[\gamma^{-1}\frac{\partial \gamma}{\partial y^j}, \gamma^{-1}\frac{\partial \gamma}{\partial y^k}\right]\right) \tag{2.3}$$

where y^i, $i = 1, 2, 3$ denote the coordinates of B. These models are exactly solvable by affine Lie algebras which are given by the currents. If one determines the conserved currents by make use of the Noether theorem one obtains

$$J := -k\partial_z \gamma\gamma^{-1} \quad , \quad \bar{J} := k\gamma^{-1}\partial_{\bar{z}}\gamma$$

Expanding $J = \sum_a t^a J^a$ one obtains that the singular part of the operator product expansion (OPE) of two currents is equal to

$$J^a(z)J^b(w) \sim \frac{\delta^{ab} k}{(z-w)^2} + \frac{i f^{abc}\ J^c(w)}{(z-w)} \tag{2.4}$$

This OPE is equivalent to an affine Lie algebra. To see this we expand the currents by the Laurent series

$$J^a(z) = \sum_{n\in\mathcal{Z}} (z-w)^{-1-n} J^a_n(w),$$
$$J^a_n(z) = \oint_z dw\ (w-z)^n J^a(w).$$

Using this and the operator product expansion one obtains for the commutation relation of two $J^a_n(w)$

$$\left[J^a_n(z), J^b_m(z)\right] = if^{abc}J^c_{n+m} + kn\delta_{m+n,0}. \tag{2.5}$$

One could do the same for the current operators $\bar{J}$ and obtains that the WZW-model of a compact Lie group G is solved by a direct sum of two $\hat{\mathfrak{g}} \bigotimes \hat{\mathfrak{g}}$ affine lie algebras.

2.4 The $AdS_3 \times S^3 \times T^4$ case

Now we are able to construct the $\mathcal{N} = 1$ supersymmetric worldsheet theory for string theory on $AdS_3 \times S^3 \times T^4$. We may forget about the T^4 since it is only four times the well known free boson/fermion on a circle, a free $U(1)^4$ theory. In this review of the string theory we will follow [6].

The supersymmetric $SL(2,R)_k$ model (where k is the level of the affine algebra) has symmetries generated by the currents $\psi^A + \theta J^A$, $A = 1,2,3$. The relevant operator product expansions are

$$J^A(z)J^B(w) \sim \frac{\delta^{AB}\frac{k}{2}}{(z-w)^2} + \frac{i\epsilon^{ABC}\ J^C(w)}{(z-w)},$$
$$J^A(z)\psi^B(w) \sim \frac{i\epsilon^{ABC}\ \psi^C(w)}{(z-w)},$$
$$\psi^A(z)\psi^B(w) \sim \frac{\frac{k}{2}\eta^{AB}}{(z-w)}.$$

In the above equations $\epsilon^{123} = 1$ is the structure constant, and $\eta^{AB} = diag(1,1,-1)$. The second model is the one of the supersymmetric $SU(2)_k$ which is constructed by the supercurrents $\chi^a + \theta K^a$, $a = 1,2,3$ with the ope's

$$K^a(z)K^b(w) \sim \frac{\delta^{ab}\frac{k}{2}}{(z-w)^2} + \frac{i\epsilon^{abc}\ K^c(w)}{(z-w)},$$
$$K^a(z)\chi^b(w) \sim \frac{i\epsilon^{abc}\ \chi^c(w)}{(z-w)},$$
$$\chi^a(z)\chi^b(w) \sim \frac{\frac{k}{2}\delta^{ab}}{(z-w)}.$$

Since we will use them later we show how the WZW currents J^A, K^a are split into bosonic and fermionic currents

$$J^A = j^A + \hat{j}^A = j^A - \frac{i}{k}e^{ABC} : \psi^B\psi^C : ,$$
$$K^a = k^a + \hat{k}^a = k^a - \frac{i}{k}e^{abc} : \chi^b\chi^c : ,$$

where $: AB :$ is the normal ordering of the operator product of A and B which is defined by

$$: AB : (z) = \frac{1}{2\pi i} \oint_z \frac{dw}{w-z} A(w)B(z). \qquad (2.6)$$

It is also usefull to rewrite the current operator by

$$\begin{aligned} J^{\pm} &= J^1 \pm iJ^2, \qquad & \psi^{\pm} &= \psi^1 \pm i\psi^2, \\ K^{\pm} &= K^1 \pm iK^2, \qquad & \chi^{\pm} &= \chi^1 \pm i\chi^2. \end{aligned}$$

For more information on how these models are constructed take a look for $SU(2)$ at [28] and for $SL(2,R)$ at [27],[26],[6] or [3].

The chiral primary operators of $SL(2)$ and $SU(2)$ are denoted by $\Phi_{h,m,\bar{m}}(z)$ and $\Phi'_{j,m,\bar{m}}(z)$. Sometimes it is usefull to rewrite the primary fields in the notation of mode expansion

$$\begin{aligned} \Phi_h(x,z) &= \Phi_h^{SL(2)}(x,z) = \sum_{m,\bar{m}} \Phi^{SL(2)}_{h,m,\bar{m}} x^{-h-m} x^{-h-\bar{m}}, \\ \Phi'_j(y,z) &= \Phi_j^{SU(2)}(y,z) = \sum_{m,\bar{m}} \Phi^{SU(2)}_{j,m,\bar{m}} y^{j+m} y^{h+\bar{m}}. \end{aligned}$$

These satisfy the equations

$$\begin{aligned} J^a_n \Phi_j &= \hat{J}^a_n \Phi_j = 0 \quad for\ n > 0 \\ J^2_0 \Phi_j &= \eta^{ab} J^a_0 J^b_0 \Phi_j = j(j+1)\Phi_j \\ \bar{J}^2_0 \Phi_j &= \eta^{ab} \bar{J}^a_0 \bar{J}^b_0 \Phi_j = j(j+1)\Phi_j \end{aligned}$$

Similarly, there is a mode expansion for the currents

$$\begin{aligned} J(x,z) &= -J^+(z) + 2xJ^0(z) - x^2 J^-(z) \\ \psi(x,z) &= -\psi^+(z) + 2x\psi^0(z) - x^2\psi^-(z) \\ K(x,z) &= -K^+(z) + 2yK^0(z) - y^2 K^-(z) \\ \chi(x,z) &= -\chi^+(z) + 2x\chi^0(z) - x^2\chi^-(z) \end{aligned}$$

2.5 Spectrum of chiral operators

In the following we summarize the chiral primaries of the worldsheet theory [23, 22, 6]. Therefore we drop the z- and $\bar{z}$-dependence of the holomorphic and anti-holomorphic operators respectively, because every operator has this dependence. The chiral operators are constructed from the conformal dimension zero operator

$$\mathcal{O}_j(x,y) = \Phi_h(x)\Phi'_j(y) \qquad with \quad h = j+1, \quad j = 0, 1/2, \ldots, \frac{k-2}{2} \qquad (2.7)$$

The fact that $\mathcal{O}$ has conformal scale dimension zero follows immediately from $\Delta(j) = j(j+1)/k$ and

$$\Delta(h) = -h(h-1)/k \underbrace{=}_{h=j+1} -j(j+1)/k.$$

In eq. (2.7) $\Phi_h(x)$ and $\Phi'_j(y)$ are the mode expanded Primaries of the $SL(2)$ and $SU(2)$ respectively. Quit generally, in any string theory [30] there exist two sectors, NS and RR.

In the holomorphic sector there are three families of chiral primaries. In the -1 ($-1/2$) picture of the NS (R) sector they are obtained by multiplying $\mathcal{O}_j(x,y)$ by any of the operators $e^{-\phi}\ \psi(x)$, $e^{-\phi}\ \chi(x)$, $e^{-\phi/2}\ s^a(x,y)$ $a=1,2$. The $s(x,y)$ operators appear, because of spin operators which we need for the construction of the R-sector primaries (c.f [3]). Since we do not need the R sector chiral primaries we will not explain how they could be constructed and why they are in the spectrum. Below we will list all chiral primaries of the theory and in which picture they are :

$$\begin{array}{lll}
\left.\begin{array}{lll}
Pic. & Op. & Expansion \\
-1 & \mathcal{O}^{(0)}_j & = e^{-\phi}\mathcal{O}_j(x,y)\psi(x) \\
0 & \tilde{\mathcal{O}}^{(0)}_j & = \left((1-h)\hat{j} + j + \frac{k}{2}\psi(x)\chi_a(y)P^a_y\right)\mathcal{O}_j(x,y)
\end{array}\right\} & (NS\ sector) & \\
\left.\begin{array}{lll}
-\frac{1}{2} & \mathcal{O}^{(a)}_j & = e^{-\phi/2}\mathcal{O}_j(x,y)s^a_-(x,y) \\
-\frac{3}{2} & \tilde{\mathcal{O}}^{(a)}_j & = -\sqrt{k}(2h-1)^{-1}e^{-\phi 3/2}\mathcal{O}_j(x,y)s^a_+(x,y)
\end{array}\right\} & (R\ sector) & \\
\left.\begin{array}{lll}
-1 & \mathcal{O}^{(2)}_j & = e^{-\phi}\mathcal{O}_j(x,y)\chi(y) \\
0 & \tilde{\mathcal{O}}^{(2)}_j & = \left(h\hat{k} + k + \frac{k}{2}\chi(y)\psi_A(y)D^A_x\right)\mathcal{O}_j(x,y)
\end{array}\right\} & (NS\ sector) & (2.8)
\end{array}$$

To complete the discussion about the chiral primaries we have to multiply the anti-holomorphic and holomorphic operators,

$$\mathcal{O}^{(0,\bar{0})}_j(x,\bar{x};y,\bar{y}) = \mathcal{O}^{(0)}_j(x,y,z)\mathcal{O}^{(\bar{0})}_j(\bar{x},\bar{y},\bar{z}).$$

The picture of the operators are important, because the correlators have to be in -2 picture. This means that we have to take care that the sum of all picture numbers is equal to -2.

3 Extremal four-point correlator

In this section we calculate a particular four point correlation function of chiral primaries. This correlator was not calculated before, but was expected to be equal to the corresponding correlator in the boundary conformal field theory (bcft). The computation follows closely that in [3]. In the boundary theory, this correlator is given by [6]

$$\langle O^{(2,2)\dagger}_{n_4} O^{(0,0)}_{n_3} O^{(0,0)}_{n_2} O^{(0,0)}_{n_1}\rangle = \frac{1}{N}\frac{n_4^{1/2}}{\sqrt{n_1 n_2 n_3 n_4}}, \tag{3.1}$$

where $O^{(0,0)}_n$ are the chiral primary operator's in the bcft. Roughly speaking, we identify the operator O_n in the field theory with those in the worldsheet theory by

$$O_n(x) = \int dz^2 \mathbb{O}_j(x,z),$$

where $n = 2j+1$, . In order to get (3.1) from the worldsheet, we need to show

$$\begin{aligned}
\mathbb{G}^{(0,0)}_4(x,\bar{x}) &= g_s^{-2}\int d^2z \left\langle \mathbb{O}^{(2,2)}_{j_4}(x_4,y_4)\mathbb{O}^{(0,0)}_{j_3}(x_3,y_3)\tilde{\mathbb{O}}^{(0,0)}_{j_2}(x_2,y_2)\tilde{\mathbb{O}}^{(0,0)}_{j_1}(x_1,y_1)\right\rangle \\
&= \frac{1}{N}\frac{(2(j_1+j_2)+1)}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}},
\end{aligned} \tag{3.2}$$

where g_s/k is identified with N by

$$\frac{g_s}{k} = \frac{1}{N^{1/2}}.$$

3.1 Preliminary

We begin by computing the correlator

$$g_s^{-2} \int d^2 z_1 \int d^2 z_2 \int d^2 z_3 \int d^2 z_4$$
$$\times \left\langle \mathcal{O}_{j_4}^{(2,2)}(x_4, y_4) \mathcal{O}_{j_3}^{(0,0)}(x_3, y_3) \tilde{\mathcal{O}}_{j_2}^{(0,0)}(x_2, y_2) \tilde{\mathcal{O}}_{j_1}^{(0,0)}(x_1, y_1) \right\rangle$$

Using the modular invariance and sending $z_{4,3,2,1} = \infty, 1, z, 0$ as well as $x_{4,3,2,1} = \infty, 1, x, 0$ the above equation becomes

$$G_4(x, \bar{x}) = g_s^{-2} \int d^2 z \left\langle \mathcal{O}_{j_4}^{(2,2)}(\infty, y_4) \mathcal{O}_{j_3}^{(0,0)}(1, y_3) \tilde{\mathcal{O}}_{j_2}^{(0,0)}(x, y_2) \tilde{\mathcal{O}}_{j_1}^{(0,0)}(0, y_1) \right\rangle \tag{3.3}$$

For simplification we only look at the holomorphic part of the correlation function and take at the end the modulus squared of the equation. The holomorphic part of the chiral primaries which appear in the four point function (3.3) are

$$\begin{aligned}
\mathcal{O}_j^{(2)} &= e^{-\phi(z)} \mathcal{O}_j(x, y) \chi(y) \\
\mathcal{O}_j^{(0)} &= e^{-\phi(z)} \mathcal{O}_j(x, y) \psi(x) \\
\tilde{\mathcal{O}}_j^{(0)} &= \left((1-h)\hat{j}(x) + j(x) + \frac{2}{k} \psi(x) \chi_a P_y^a \right) \mathcal{O}_j(x, y)
\end{aligned}$$

By looking at the list (2.8) one obtains that the total ghost number of the correlator is $-1-1+0+0=-2$. Now it is possible to decompose the correlator (3.3) into objects which could separately be calculated. So that we could write (3.3) as

$$\begin{aligned}
G_4(x, \bar{x}) &= g_s^{-2} \int d^2 z \left\langle \mathcal{O}_{j_4}^{(2,2)}(x_4, y_4) \mathcal{O}_{j_3}^{(0,0)}(x_3, y_3) \tilde{\mathcal{O}}_{j_2}^{(0,0)}(x_2, y_2) \tilde{\mathcal{O}}_{j_1}^{(0,0)}(x_1, y_1) \right\rangle \\
&\overbrace{=}^{holom.} \left\langle e^{-\phi_1} e^{-\phi_2} \right\rangle \langle \mathcal{O}_{j_4}(x_4, y_4) \chi(y_4) \mathcal{O}_{j_3}(x_3, y_3) \psi(x_3) \\
&\quad + \left((1-h_2)\hat{j}(x_2) + j(x_2) + \frac{2}{k} \psi(x_2) \chi_a P_{y_2}^a \right) \mathcal{O}_{j_2}(x_2, y_2) \\
&\quad + \left((1-h_1)\hat{j}(x_1) + j(x_1) + \frac{2}{k} \psi(x_1) \chi_a P_{y_1}^a \right) \mathcal{O}_{j_1}(x_1, y_1) \Big\rangle
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
G_4(x,\bar{x}) = g_s^{-2} \int d^2z \left\langle e^{-\phi_1} e^{-\phi_2} \right\rangle [\\
&+ \frac{(1-h_2)}{k} \left\langle \psi(x_3)\hat{j}(x_2)\psi(x_1) \right\rangle [\langle \chi(y_4)\chi(y_1)\rangle \partial_{y_1} - j_1 \partial_{y_1} \langle \chi(y_4)\chi(y_1)\rangle] \left\langle \prod_{i=1}^{n=4} \mathcal{O}_{j_i}(x_i,y_i) \right\rangle \\
&+ \frac{1}{k} \langle \psi(x_3)\psi(x_1)\rangle [\langle \chi(y_4)\chi(y_1)\rangle \partial_{y_1} - j_1 \partial_{y_1} \langle \chi(y_4)\chi(y_1)\rangle] \left\langle j(x_2) \prod_{i=1}^{n=4} \mathcal{O}_{j_i}(x_i,y_i) \right\rangle \\
&+ \frac{(1-h_1)}{k} \left\langle \psi(x_3)\hat{j}(x_1)\psi(x_2) \right\rangle [\langle \chi(y_4)\chi(y_2)\rangle \partial_{y_2} - j_2 \partial_{y_2} \langle \chi(y_4)\chi(y_2)\rangle] \left\langle \prod_{i=1}^{n=4} \mathcal{O}_{j_i}(x_i,y_i) \right\rangle \\
&+ \frac{1}{k} \langle \psi(x_3)\psi(x_2)\rangle [\langle \chi(y_4)\chi(y_2)\rangle \partial_{y_2} - j_2 \partial_{y_2} \langle \chi(y_4)\chi(y_2)\rangle] \left\langle j(x_1) \prod_{i=1}^{n=4} \mathcal{O}_{j_i}(x_i,y_i) \right\rangle]
\end{aligned}
\tag{3.5}
$$

In the above equation we use the fact that [3]

$$
2\chi_a P^a_{y_i} = \chi(y_i)\partial_{y_i} - j_i \partial_{y_i}\chi(y_i) \tag{3.6}
$$

as well as the fact that correlation functions which include an odd number of ψ or χ fields vanish.

3.2 Correlators and simplifications

To give the reader a better understanding how we calculate a correlation function we calculate every summand separately. Before we are able to do this we have to list the four point correlators of the $SL(2)$ and the $SU(2)$. The four point correlator of the $SL(2)$ is given by

$$
\begin{aligned}
\left\langle \prod_{i=1}^{n=4} \Phi_{h_i}(x_i,z_i;\bar{x}_i,\bar{z}_i) \right\rangle &= |x_{24}|^{-4(h_2)} |x_{14}|^{2(h_3+h_2-h_4-h_1)} |x_{34}|^{2(h_1+h_2-h_4-h_3)} |x_{13}|^{2(h_4-h_3-h_2-h_1)} \\
&\times |z_{24}|^{-4(\Delta_2)} |z_{14}|^{2(\nu_1)} |z_{34}|^{2(\nu_2)} |z_{13}|^{2(\nu_3)} \mathcal{F}_{SL(2)}(x,z;\bar{x},\bar{z})
\end{aligned}
\tag{3.7}
$$

In the above equation we used the following notation for the exponent of the z and the x and z.

$$
\begin{gathered}
\nu_1 = \Delta_2 + \Delta_3 - \Delta_1 - \Delta_4 \quad \nu_2 = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 \quad \nu_3 = \Delta_4 - \Delta_2 - \Delta_1 - \Delta_3 \\
x = \frac{x_{12}x_{34}}{x_{13}x_{24}} \quad z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad x_{ij} = x_i - x_j \quad z_{ij} = z_i - z_j
\end{gathered}
$$

The function $\mathcal{F}_{SL(2)}(x,z;\bar{x},\bar{z})$ which appears in our ansatz is expressed by an integral over all possible values of the representation [27].

$$
\mathcal{F}_{SL(2)}(x,z;\bar{x},\bar{z}) = \int_{\frac{1}{2}+i\mathbb{R}} dh \mathcal{C}(h) \left|F_h(x,z)\right|^2 \tag{3.8}
$$

It is worthy of mention that the function $F_h(x,z)$ could be expressed in terms of a series in x (cf. [3, 18, 27]). For simplification we use the notation of $u = z/x$. Maldacena derived in [18] the following expansion for $F_h(x,z)$.

$$
F_h(x,z) = x^{(\Delta(h)-\Delta(h_1)-\Delta(h_2)+h-h_1-h_2)} u^{(\Delta(h)-\Delta(h_1)-\Delta(h_2))} \sum_{n=0}^{\infty} g_m(u) x^m
$$

It was also shown by Teschner [27] that $g_0(u)$ is a solution of the hypergeometric equation. The solution of the hypergeometric equation is a series in u,

$$g_0 = F(a, b, c|u) = F_h(u) = \sum_{n=0}^{\infty} \mathcal{H}(a, b, c, n) u^n, \tag{3.9}$$

where we use the function

$$\mathcal{H}(a, b, c, n) = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)},$$
$$a = h_1 + h_2 - h, \quad b = h_3 + h_4 - h, \quad c = k - 2h.$$

In the same manner as got the four point correlator for the $SL(2)$ we get the four point correlator for the $SU(2)$ chrial primaries with the difference that representations of the $SU(2)$ are finite dimensional. We will skip the derivation of the four point function and refer this to [3].

$$\begin{aligned}\left\langle \prod_{i=1}^{n=4} \Phi'_{j_i}(y_i, z_i; \bar{y}_i, \bar{z}_i) \right\rangle &= |y_{24}|^{4(j_2)} |y_{14}|^{2(j_4+j_1-j_3-j_2)} |y_{34}|^{2(j_4+j_3-j_1-j_2x)} |x_{13}|^{2(j_1+j_2+j_3-j_4)} \\ &\times |z_{24}|^{-4(\Delta'_2)} |z_{14}|^{2(\nu'_1)} |z_{34}|^{2(\nu'_2)} |z_{13}|^{2(\nu'_3)} \\ &\times \sum_{j',n'} \frac{|z|^{2(\Delta(j')-\Delta(j_1)-\Delta(j_2)+n')}}{|y|^{2(j-j_1-j_2+n')}} C'(j') \mathcal{D}(j_1, j_2, J) \mathcal{D}(j_3, j_4, J),\end{aligned} \tag{3.10}$$

where we use the limits $y \ll 1$ and $z \ll 1$ as well as we make use of the notation

$$\nu_1 = \Delta'_1 + \Delta'_3 - \Delta'_2 - \Delta'_4, \quad \nu_2 = \Delta'_1 + \Delta'_2 - \Delta'_3 - \Delta'_4, \quad \nu_3 = \Delta'_4 - \Delta'_2 - \Delta'_1 - \Delta'_3,$$
$$y = \frac{y_{12}y_{34}}{y_{13}y_{24}}, \quad z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

3.3 Calculation of single correlators

In this section we are interested in some correlators which we need for the computation of the whole correlator $G_4(x, \bar{x})$. We will first derive all correlators and then make use of the modular invariance to fix x, y. The first correlator we look at is

$$\left\langle \psi(x_3) \hat{j}(x_2) \psi(x_1) \right\rangle = \sum_{i=1, i\neq 2}^{n=3} \mathcal{D}_{2i}^{(-1)} \left\langle \psi(x_3)\psi(x_1) \right\rangle = \frac{2k x_{12} x_{13} x_{23}}{z_{12} x_{23}} \tag{3.11}$$

For this computation we used (A.3). The next factor of the first summand is this $2\chi_a P^a_{y_i}$ which acts on the $SU(2)$ four point function by (3.6). Before we are able to give the result we multiply this by (3.11) and make use of the modular invariance $(y, z, x)_{1,2,3,4} = 0, x, 1, \infty$ and a small cross ratios $y, z, x \ll 1$. Therefore it follows that the first term of

(3.5) is

$$\frac{(1-h_2)}{k}\left\langle\psi(x_3)\hat{j}(x_2)\psi(x_1)\right\rangle\left[\langle\chi(y_4)\chi(y_1)\rangle\,\partial_{y_1}-j_1\partial_{y_1}\langle\chi(y_4)\chi(y_1)\rangle\right]\left\langle\prod_{i=1}\Phi'_{j_i}(y_i,z_i;\bar{y}_i,\bar{z}_i)\right\rangle$$
$$=\Bigg(\frac{2k(h_2-1)(j'(y-1)+j_1+j_2+y(j_3-j_4)+(y-1)n'}{yz}+$$
$$\frac{2k(h_2-1)(j'(y-1)+j_1+j_2+y(j_3-j_4)+(y-1)n'}{y}+O(z^1)+O(x^2)\Bigg)$$
$$\times\left\langle\prod_{i=1}^{n=4}\Phi'_{j_i}(y_i,z_i;\bar{y}_i,\bar{z}_i)\right\rangle, \tag{3.12}$$

where we made use of the $SL(2)$ and $SU(2)$ two point functions (A.1) and (A.2) respectively. With the aid of an interchange of the indices $2\leftrightarrow 1$ we find the third term of (3.5).

$$\frac{(1-h_1)}{k}\left\langle\psi(x_3)\hat{j}(x_1)\psi(x_2)\right\rangle\left[\langle\chi(y_4)\chi(y_2)\rangle\,\partial_{y_2}-j_2\partial_{y_2}\langle\chi(y_4)\chi(y_2)\rangle\right]\left\langle\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle$$
$$=\left(\frac{2k(h_1-1)(j'-j_1-j_2+n')}{yz}+O(z^1)+O(x^2)\right)\left\langle\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle \tag{3.13}$$

In the remaining terms we need the following correlator.

$$\left\langle j(x_2)\prod_{i=1}^{n=4}\Phi_{h_i}(x_i,y_i)\right\rangle$$

For the calculation of the above correlator we need the derivation (A.3). The remaining factors of the second term are known from the calculation of the first term. Hence (3.14) leads to

$$\frac{1}{k}\langle\psi(x_3)\psi(x_1)\rangle\left[\langle\chi(y_4)\chi(y_1)\rangle\,\partial_{y_1}-j_1\partial_{y_1}\langle\chi(y_4)\chi(y_1)\rangle\right]\left\langle j(x_2)\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle$$
$$=\left(\frac{k(h-n-h_3-h_4)(j'(y-1)+j_1+j_2+y(j_3-j_4)+(y-1)n')}{y}+O(z^1)+O(x^1)\right)$$
$$\times\left\langle\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle. \tag{3.14}$$

In the same way as we did it in (3.14), by interchanging of the indices $2\leftrightarrow 1$, we get the fourth term of (3.5), which is :

$$\frac{1}{k}\langle\psi(x_3)\psi(x_2)\rangle\left[\langle\chi(y_4)\chi(y_2)\rangle\,\partial_{y_2}-j_2\partial_{y_2}\langle\chi(y_4)\chi(y_2)\rangle\right]\left\langle j(x_1)\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle$$
$$=\left(\frac{-k(j-n-h_3-h_4)(J-j_1-j_2+n')}{y}+O(z^1)+O(x^1)\right)\left\langle\prod_{i=1}^{n=4}\mathcal{O}_{j_i}(x_i,y_i)\right\rangle. \tag{3.15}$$

3.4 The correlator and the intermediate channel

To get further with the computation of the four point correlation function we have to think about the chiral operators in the intermediate channel. From the fusion rules [6]

$$(0,0) \times (0,0) = (0,0) + (2,2)$$

we conclude that there are two possible c configurations of chiral operators which could be in the intermediate channel. Hence our sum over j' has two summands one with $j = j_1 + j_2$ and one with $j = j_1 + j_2 - 1$. This follows also from, the fusion rules. From the first constraint it follows that the integrand of (3.3) is equal to

$$\frac{2k(j_1+j_2)}{u} + 2j_3 + O(u) + O(x) \tag{3.16}$$

We need to investigate this formula further. What we did first was to sum up all parts of the integrand. After that we use $u = z/x$, the extremal condition $j_4 = j_1 + j_2 + j_3 - 1$, the relation between h and j $h_i = j_i + 1$ and the constraint from the fusion rules $j = j_1 + j_2$. As well as the above relations we only take care about the first summands. This follows from the limit of small x and u. In the last step we skip all descendants by choosing $n' = 0$. We will call this summand of the integrand the 1-particle $(0,0)$ contribution. It was shown in [18] that by make use of the resticted integration $\int_{|u|<\epsilon}$ the only contribution is that of the one particle state in the intermediate channel. For further investigations of $G_4(x,\bar{x})$ we forgot about the second term. So that (3.5) becomes

$$\begin{aligned} G_4^{(0,0)}(x,\bar{x}) = g_s^{-2}(2(j_1+j_2)+1)^2 k^2 \int du^2 \int dh \mathcal{C}(h)\mathcal{C}'(j) \left| \frac{\mathcal{H}(a,b,c|0)}{u} \right|^2 \\ \times |x|^{2(\Delta(h)-\Delta(j))} u^{2(\Delta(h)-\Delta(j))} \end{aligned} \tag{3.17}$$

The integration *Before we do the integral over h we integrate over u. Therefore the relation listed below would be helpful*

$$\int_{|u|<\epsilon} du |u|^{2(\lambda-1)} = \frac{\pi}{\lambda}\epsilon^{2\lambda}$$

For further computations we refer to the residue theorem. So it is possible to calculate the integral over h by make use of

$$\oint dh f(h) = 2i\pi \sum_i Res(f, h_0)$$

Here is h_0 a pole of the function $f(h)$. We consider only the pole $\lambda = 0$. The argumentation why we could do this is given in [3] and [18]. Therefore we only have to determine the residuum of the function $f(h)$. This could be done by

$$Res(f, h_0) = \frac{\pi \epsilon^{2(\lambda(h_0))}}{\partial_h \lambda(h)|_{h=h_0}}$$

Now it is possible to calculate $G_4^{(0,0)}(x,\bar{x})$ by taking the limit $\epsilon \to 0$ and with the identification $\lambda = \Delta(h) - \Delta(j)$. Before we present the result of the two integrations it is worthy to mention that the product structure constants $\mathcal{C}(h)$ and $\mathcal{C}'(j)$ of the $SL(2)$ and the $SU(2)$ respectively, is equal $\frac{c_\mu^{1/2}}{2\pi}\prod_{i=1}^{n=3}\sqrt{B(h_i)}$. Therefore $G_4^{(0,0)}(x,\bar{x})$ is given by

$$G_4^{(0,0)}(x,\bar{x}) = g_s^{-2}(2j+1)^2k^2\underbrace{|\mathcal{H}(a,b,c|0)|^2}_{=1}\frac{c_\nu}{4\pi^2}\prod_{i=1}^{n=4}\sqrt{B(j_i+1)}\frac{2\pi^2}{\partial_h\lambda(h)|_{h=h_0}}\Bigg|_{j=j_1+j_2} \tag{3.18}$$

In the above formula $c_\nu = \frac{1}{2\pi^4k^3}$ and $2\pi^2/(\partial_h\lambda(h)|_{h=h_0}) = 2\pi^2k/(2j+1)$. Before we are able to compare the four point function of AdS_3 theory with the corresponding one of the bcft we have to appropriately rescale the operators. The correct normalization of the two point functions are [3]

$$\mathbb{O}_{j_1}^{(0,\bar{0})}(x,\bar{x}) = \frac{\sqrt{2\pi^2}}{\sqrt{kB(h)(2h-1)}}g_s\mathcal{O}_{j_1}^{(0,\bar{0})}(x,\bar{x}) \tag{3.19}$$

and

$$\mathbb{O}_{j_1}^{(2,\bar{2})}(x,\bar{x}) = \frac{\sqrt{2\pi^2}}{\sqrt{kB(h)(2h-1)}}g_s\mathcal{O}_{j_1}^{(2,\bar{2})}(x,\bar{x}) \tag{3.20}$$

Subsituting this normalization into (3.18) we are allowed to give a result of (3.3) which could be compared to the bcft once.

$$\begin{aligned}
\mathbb{G}_4^{(0,0)}(x,\bar{x}) &= g_s^{-2}\int d^2z\left\langle \mathbb{O}_{j_4}^{(2,2)}(x_4,y_4)\mathbb{O}_{j_3}^{(0,0)}(x_3,y_3)\tilde{\mathbb{O}}_{j_2}^{(0,0)}(x_2,y_2)\tilde{\mathbb{O}}_{j_1}^{(0,0)}(x_1,y_1)\right\rangle \\
&= \frac{4\pi^4}{\sqrt{k^4\prod_{i=1}^{n=4}B(j_i+1)(2j_i+1)}}g_s^4 \\
&\times\left\langle \mathcal{O}_{j_4}^{(2,2)}(x_4,y_4)\mathcal{O}_{j_3}^{(0,0)}(x_3,y_3)\tilde{\mathcal{O}}_{j_2}^{(0,0)}(x_2,y_2)\tilde{\mathcal{O}}_{j_1}^{(0,0)}(x_1,y_1)\right\rangle \\
\Rightarrow \mathbb{G}_4^{(0,0)}(x,\bar{x}) &= \frac{1}{N}\frac{(2(j_1+j_2)+1)}{\sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)}}
\end{aligned} \tag{3.21}$$

If we now look at (3.21) we see that this is exactly what we expected. But it is worthy of mention that we take a few limit to derive the correspondence. We worked in a limit in which $x \ll 1$ as well as that we only integrate about a very small area of the z-plane.

4 Operator product expansion on the worldsheet

In this section we generalize the computation to arbitrary p-point correlation functions. In particular, we derive a recursion relation for the p point correlator using appropriate operator product expansions. Later, we will compare ourresults with a similar recursion relation found in [20], in the bcft.

4.1 Worldsheet operator product expansion

In the next two subsections we derive an operator product expansion of two normalized chiral primaries by argue that the four point function reduced to an product of two three point functions (cf. [3]). So that we could make an ansatz by relating the four point function to the three point function. The most general form of a worldsheet OPE is, in the limit $z \to 0$, [19]:

$$\mathbb{O}_{j_1}(0)\mathbb{O}_{j_2}(x,\bar{x},z,\bar{z}) = \sum_j \int dh \int_{\mathbb{C}} d^2x' \frac{F(j_i,j,h_i,h)\mathbb{O}_j^{(0,0)}(x,0)}{|x_1|^\alpha |x-x'|^\beta |x'|^\gamma} |z|^{2(\Delta(h,j)-\Delta_1-\Delta_2)} + \text{desc.}$$

The d^2x integral simplifies by the assumption that x is very small and by the substitution $y = x'/|x|$.

$$= \sum_j \int dh \frac{F(j_i,j,h_i,h)|z|^{2(\Delta(h,j)-\Delta_1-\Delta_2)}}{|x|^{2(\alpha+\beta+\gamma-2)}}$$

$$\times \int_{\mathbb{C}} d^2y \frac{\mathbb{O}_j^{(0,0)}(|x|y,0)}{|1-y|^\beta |y|^\gamma} + \text{desc.}$$

The expected x and z dependence of the operator product expansion is immediately determind by performing a scale transformation on both sides of the equation. Hence we get

$$\alpha + \beta + \gamma - 2 = h_1 + h_2 + h$$

To evaluate the d^2y integral we perform a Taylor expansion of the operator $\mathbb{O}_j^{(0,0)}$.

$$= \sum_j \int dh \frac{F(j_i,j,h_i,h)|z|^{2(\Delta(h,j)-\Delta_1-\Delta_2)}}{|x|^{2(\alpha+\beta+\gamma-2)}} \mathbb{O}_j^{(0,0)}(0,0)$$

$$\times \int_{\mathbb{C}} d^2y \frac{1}{|1-y|^\beta |y|^\gamma} + \text{desc.}$$

where $\Delta(h,j)$ denotes the weight of the operator $\mathbb{O}_j^{(0,0)}$. The dependence on z and x is completely determined by conformal invariance. For the moment, we ignore the contribution of descendants. Let us consider the OPE of the chiral primary operator $\mathbb{O}_{j_1} = \mathbb{O}_{j_1}^{(0,0)}$, which is in the -1 picture, and the corresponding operator $\mathbb{O}_{j_2} = \tilde{\mathbb{O}}_{j_2}^{(0,0)}$ in 0 picture. Their worldsheet conformal weights are $\Delta(1) = \Delta(2) = 1$. For further simplification of the OPE, we take the limit of small x. In this limit, we have

$$\mathbb{O}_{j_1}^{(0,0)}(0)\tilde{\mathbb{O}}_{j_2}^{(0,0)}(x,\bar{x},z,\bar{z}) = \sum_j \int dh\, |z|^{2(\Delta(h,j)-1)} |x|^{2(h-h_1-h_2+1)} F(h_i,h)\mathbb{O}_j^{(0,0)}(0) \quad (4.1)$$

where

$$F(h_i,h) = \frac{g_s}{2\pi^2 k^2}(h+h_1+h_2-2)^2 \frac{(2h-1)^{1/2}}{(2h_1-1)^{1/2}(2h_2-1)^{1/2}} \quad (4.2)$$

and $\Delta(h,j) = \Delta(h) + \Delta(j)^*$. After the argumentation and the proof that the p point function reduce to the product of a three point function and $p-1$ point function we will show how we could exactly compute this operator product expansion.

*x-dep.: $h^{(0)} - h_1^{(0)} - h_2^{(0)}$. Use $h^{(0)} = h - 1$.

4.2 Specifically relating three- to four point functions

As a first application we will substitute the ope into the four point function for relating it to a three point function. This allows us to simplify the proof of the algorithm of reducing a p-point function to the product of a three-point function and a $p-1$-point function. We also know from [2] and [3] that the three- and four-point functions, respectively, of chiral primaries of the $AdS_3 \times S^3 \times T^4$ string-theory are equal to the chiral operator correlator of the bcft. Since that and the fact that Packman et.al. showed how to calculate the n-point function on the bcft side it is possible to proof the identity of the p-point correlators of chiral primary operators.
But before this is done consider the four point function and reduce it to the product of two three point functions as Kirsch did it in [3, Sec. 4.3]. The four point function which is considered here is

$$\mathbb{G}_4(x,\bar{x}) = g_s^{-2} \int d^2z \left\langle \tilde{\mathbb{O}}_{j_4}(\infty)\mathbb{O}_{j_3}(1)\tilde{\mathbb{O}}_{j_2}(x,\bar{x};z,\bar{z})\mathbb{O}_{j_1}(0)\right\rangle \tag{4.3}$$

The substitution of eq. (4.1) into the above equation leads to

$$\begin{aligned}\mathbb{G}_4(x,\bar{x}) = g_s^{-2} \sum_j \int_{\frac{1}{2}+i\mathbb{R}} dh \int d^2z |z|^{2(\Delta(h,j)-1)}|x|^{2(h-h_1-h_2+1)} \\ \times \frac{g_s}{2\pi^2k^2} \frac{(h+h_1+h_2-2)^2\sqrt{(2h-1)}}{\sqrt{(2h_1-1)(2h_2-1)}} \left\langle \mathbb{O}_{j_4}(\infty)\mathbb{O}_{j_3}(1)\mathbb{O}_j(0)\right\rangle\end{aligned} \tag{4.4}$$

The z- and h-integral are computed exactly as shown in section 3. Hence $\mathbb{G}_4(x,\bar{x})$ is

$$\begin{aligned}\mathbb{G}_4(x,\bar{x}) &\underbrace{=}_{h=j+1} \frac{1}{g_sk} \sum_j |x|^{2(j-j_1-j_2)} \frac{(j+j_1+j_2+1)^2}{\sqrt{(2j_1+1)(2j_2+1)(2j+1)}} \\ &\times \left\langle \mathbb{O}_{j_4}(\infty)\mathbb{O}_{j_3}(1)\mathbb{O}_j(0)\right\rangle \\ &= g_s^{-2} \left\langle \mathbb{O}_{j_4}(\infty)\tilde{\mathbb{O}}_{j_3}(1)\mathbb{O}_j(0)\right\rangle \left\langle \tilde{\mathbb{O}}_{j_2}(x,\bar{x})\mathbb{O}_{j_1}(0)\mathbb{O}_j(\infty)\right\rangle \\ &= \mathbb{G}_3(x,\bar{x}) \left\langle \tilde{\mathbb{O}}_{j_2}(x,\bar{x})\mathbb{O}_{j_1}(0)\mathbb{O}_j(\infty)\right\rangle,\end{aligned} \tag{4.5}$$

where $\mathbb{G}_3(x,\bar{x})$ is given by

$$\mathbb{G}_3(x,\bar{x}) = \frac{g_s}{k} \frac{(j_1+j_2+j+1)^2}{\sqrt{(2j_2+1)(2j_1+1)(2j+1)}}|x|^{-2j_{12}} \tag{4.6}$$

The last step requires the knowledge of the three point function. We therefore refer to [2]. As well as we make use of the three point function we used the fact that only operators with $j = j_1 + j_2$ could stay on the intermediate channel.

4.3 Recursion relation for worldsheet p-point functions

We consider the worldsheet p-point function of the (rescaled) operators $\mathbb{O}_j = \mathbb{O}_j^{(0,0)}$ (in the following we drop the index $(0,0)$) given by

$$\mathbb{G}_p = g_s^{-2} \int d^2z \left\langle \tilde{\mathbb{O}}_{j_p}(\infty)\mathbb{O}_{j_{p-1}}(1) \left(\prod_{i=3}^{p-2} \int d^2z_i\, \tilde{\mathbb{O}}_{j_i}(x_i,\bar{x}_i;z_i,\bar{z}_i)\right) \tilde{\mathbb{O}}_{j_2}(x,\bar{x};z,\bar{z})\mathbb{O}_{j_1}(0)\right\rangle. \tag{4.1}$$

Modular invariance has been used to to fix three of the p worldsheet points as $z_{1,p-1,p} = 0, 1, \infty$. Similarly, the continuous $SL(2)$ representation labels are chosen as $x_{1,p-1,p} = 0, 1, \infty$. The x labels will later be identified with the complex coordinates in the boundary conformal field theory [24]. The correlator $\mathbb{G}_p$ involves $p-2$ ghost number zero and 2 ghost number -1 operators, $\tilde{\mathbb{O}}_j^{(0,0)}$ and $\mathbb{O}_j^{(0,0)}$, respectively. Note that the total ghost number of a correlator on a genus-g surface must be $-\chi = -(2-2g)$, which is -2 on the sphere.
Since it is known that there is a certain identity for the two, three and the four point function of the $AdS_3 \times S^3 \times T^4$ and a 2 dimensional bcft it was a big goal to calculate n point functions. A few lines below we will show exactly the following identity, with the aid of our ansatz (4.1) :

$$\mathbb{G}_p = \left\langle \mathbb{O}_j(\infty)\tilde{\mathbb{O}}_{j_2}(1)\mathbb{O}_{j_1}(0) \right\rangle \mathbb{G}_{p-1} \,. \tag{4.2}$$

It is interesting to compare this relation with the corresponding relation in the bcft found in [20],

$$C_p \frac{\tilde{n}}{n_p} = \left\langle O_{\tilde{n}}^{(0,0)\dagger}(\infty) O_{n_2}^{(0,0)}(1) O_{n_1}^{(0,0)}(0) \right\rangle C_{p-1} \,, \tag{4.3}$$

where $\tilde{n} = n_1 + n_2 - 1$. Both recursion relations are identical, apart from the additional factor $\frac{\tilde{n}}{n_p}$ in (4.3). This factor is coming from two-particle contributions in the intermediate channel. To obtain this factor, we would have to integrate z from 0 to ∞. At present its not known how to compute z over this region. We therefore can only compute the contribution from one-particle states in the intermediate channel.

Proof: Substituting (4.1) into $\mathbb{G}_p$, we obtain

$$\begin{aligned}\mathbb{G}_p =& g_s^{-2} \sum_j \left\langle \tilde{\mathbb{O}}_{j_p}(\infty)\mathbb{O}_{j_{p-1}}(1) \left(\prod_{i=3}^{p-2} \int d^2 z_i \, \tilde{\mathbb{O}}_{j_i}(x_i, \bar{x}_i; z_i, \bar{z}_i) \right) \mathbb{O}_j(0) \right\rangle \\ &\times \int d^2 z \int dh \, \frac{g_s}{2\pi^2 k^2} (h + h_1 + h_2 - 2)^2 \frac{(2h-1)^{1/2}}{(2h_1 - 1)^{1/2}(2h_2-1)^{1/2}} \\ &\times |z|^{2(\Delta(h,j)-1)} |x|^{2(h-h_1-h_2+1)}\end{aligned}$$

The next step requires the evaluation of the z and the h integral which was allready been done in section 4.2. Hence we only will mention the result of the integration and use the three point function from [2].

$$\begin{aligned}&= \sum_j g_s^{-2} \int d^2 z_{p-2} \left\langle \tilde{\mathbb{O}}_{j_p}(\infty)\mathbb{O}_{j_{p-1}}(1) \left(\prod_{i=3}^{p-3} \int d^2 z_i \, \tilde{\mathbb{O}}_{j_i}(x_i, \bar{x}_i; z_i, \bar{z}_i) \right) \right. \\ &\quad \left. \tilde{\mathbb{O}}_{j_i}(x_{p-2}, \bar{x}_{p-2}; z_{p-2}, \bar{z}_{p-2}) \mathbb{O}_j(0) \right\rangle \left\langle \tilde{\mathbb{O}}_{j_2}(x,\bar{x}) \mathbb{O}_{j_1}(0) \mathbb{O}_j(\infty) \right\rangle \\ &= \sum_j \mathbb{G}_{p-1}(x_i, \bar{x}_i; z_i, \bar{z}_i; j_i, j) \left\langle \tilde{\mathbb{O}}_{j_2}(x,\bar{x}) \mathbb{O}_{j_1}(0) \mathbb{O}_j(\infty) \right\rangle \\ &= \mathbb{G}_{p-1}(j_i, j = j_1 + j_2) \; \mathbb{G}_3(j_1, j_2, j = j_1 + j_2)\end{aligned} \tag{4.4}$$

4.4 Exact computations

To verify the ansatz we do a direct computation of the operator product expansion of the $SL(2)$ and the $SU(2)$ WZW model. The OPE of two $SL(2)$ primaries is [27]

$$\Phi_{h_2}(x_2,z_2)\Phi_{h_1}(x_1,z_1) = \int_{\frac{1}{2}+i\mathbb{R}} dh \frac{\mathcal{C}(h_2,h_1,h)|z_{21}|^{2(-\Delta_{21})}\gamma(-2h)}{(-\pi)\gamma(h_2-h_1-h)\gamma(h_1-h_2-h)B(h)} \times \int_{\mathbb{C}} d^2x_3|x_{21}|^{-2(h_1+h_2+h+1)}|x_{23}|^{-2(h_2-h_1-h-1)} \times |x_{13}|^{-2(h_1-h_2-h-1)}\Phi_h(x_3,z_1). \tag{4.5}$$

This operator product requires some attention. A significant distinction between the ope which was used by Teschner and the one of Eq. (4.5) is that we do not care about descendents. The $\mathcal{C}$ arising in the above equation is the structure constant of the three point function which we had seen already during the calculation of the four point function. The second functions which appears is $B(h)$ it is structure constant of the two point function. We also need a closer look to the γ function, since we use some of its properties. There exist an invers of the γ function

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \Rightarrow \gamma(x)^{-1} = \gamma(1-x) \tag{4.6}$$

This property can be used to show that the γ functions which appear in the ope (4.5) are exactly the invers of the integral over x. Therefore we consider only the x integral

$$I = \int_{\mathbb{C}} d^2x_3|x_{21}|^{-2(h_1+h_2+h+1)}|x_{23}|^{-2(h_2-h_1-h-1)}|x_{13}|^{-2(h_1-h_2-h-1)}\Phi_h(x_3,z_1).$$

To derive the result we have to perform some substitutions. The first we use is $y' = x_{13}$. Hence our integration measure becomes $dy' = dx_3$ and

$$I = |x_{21}|^{-2(h_1+h_2+h+1)} \int_{\mathbb{C}} d^2y'|x_{12}-y'|^{-2(h_2-h_1-h-1)}|y'|^{-2(h_1-h_2-h-1)}\Phi_h(x_1-y',z_1)$$

The second substitution which we make use of is $y' = y/|x_{12}|$. Therefore our integration measure becomes $dy|x_{12}| = d'y$. We also use the Taylor expansion $\Phi_h(x_1 - y|x_{12}|, z_1) \approx \Phi_h(x_1,z_1) + O(x_1)$ for small $|x_{12}| \ll 1$. Hence we finally find that the integral about dx_3 is equal to

$$I = |x_{12}|^{-2(h_1+h_2-h)}\Phi_h(x_1,z_1)\int_{\mathbb{C}} d^2y|y|^{-2(h_2-h_1-h-1)}|1-y|^{-2(h_1-h_2-h-1)}$$

This integral can be evaluated using [27]

$$\int_{\mathbb{C}} d^2t|t|^{2a}|1-t|^{2b} = -\pi\frac{\gamma(-1-a-b)}{\gamma(-a)\gamma(-b)}$$

This shows that the dx_3 integral is the inverse of the factor of the three γ functions and the $-\pi$. If we put everything together (4.5) leads to

$$\Phi_{h_2}(x_2,z_2)\Phi_{h_1}(x_1,z_1) = \int_{\frac{1}{2}+i\mathbb{R}} dh \frac{\mathcal{C}(h_2,h_1,h)|z_{21}|^{2(-\Delta_{21})}|x_{12}|^{-2(h_1+h_2-h)}}{B(h)}\Phi_h(x_1,z_1) \tag{4.7}$$

The next step towards an operator product expansion of the (normalized) chiral primary operators is to consider the ope of the $SU(2)$ primaries. This was first proposed in [29].

$$\Phi'_{j_2}(y_2, z_2)\Phi'_{j_1}(y_1, z_1) = \sum_j \mathcal{C}'(j)|z_{12}|^{-2(\Delta'_{12})}|y_{12}|^{2j_{12}}\Phi'_j(y_1, z_1)$$

Again putting everything together gives an operator product expansion of the chiral primaries with zero conformal scaling dimensions.

$$\begin{aligned}
&\mathcal{O}_{j_2}(x_2, y_2)\mathcal{O}_{j_1}(x_1, y_1) \\
&= \sum_j \int_{\frac{1}{2}+i\mathbb{R}} dh \frac{\mathcal{C}'(j)\mathcal{C}(h)|z_{12}|^{-2(\Delta'_{12}+\Delta_{21})}|y_{12}|^{2j_{12}}}{B(h)|x_{12}|^{2(h_1+h_2-h)}}\Phi_h(x_1, z_1)\Phi'_j(y_1, z_1) \\
&= \sum_j \int_{\frac{1}{2}+i\mathbb{R}} dh \frac{\mathcal{C}'(j)\mathcal{C}(h)|z_{12}|^{2(\Delta(h)+\Delta(j))}|y_{12}|^{2j_{12}}}{B(h)|x_{12}|^{2(h_1+h_2-h)}}\mathcal{O}_{j,h}(x_1, y_1)
\end{aligned} \tag{4.8}$$

The above operator product expansion seems to be almost that what we expected. The distinction between (4.8) and (4.2),(4.1) is that we are looking for an ope which consists of "dressed" and normalized operators (in sense of the two point function). The "dressed" chiral primaries are given by (2.8). Inserting the operators, we are interested in, leads to

$$\begin{aligned}
&\tilde{\mathcal{O}}^{(0)}_{j_2}(y_2, x_2)\mathcal{O}^{(0)}_{j_1}(x_1, y_1) \\
&= \left((1-h_2)\hat{j}(x_2, z_2) + j(x_2, z_2) + \frac{k}{2}\psi(x_2, z_2)\chi_a(z_2)P^a_{y_2}\right)\mathcal{O}_{j_2}(x_2, y_2)e^{-\phi}\psi(x_1)\mathcal{O}_{j_1}(x_1, y_1)
\end{aligned}$$

Before we are allowed to make use of our operator product expansion we have to determine the action of the "dressings" on the primaries. Therefore we refer to the appendix (B.5)-(B.9).

$$\begin{aligned}
&\tilde{\mathcal{O}}^{(0)}_{j_2}(y_2, x_2)\mathcal{O}^{(0)}_{j_1}(x_1, y_1) \\
&= (1-h_2)e^{-\phi}\left[\hat{j}(x_2, z_2)\psi(x_1)\right]\mathcal{O}_{j_2}(x_2, y_2)\mathcal{O}_{j_1}(x_1, y_1) \\
&+ e^{-\phi}\psi(x_1)\mathcal{O}_{j_2}(x_2, y_2)\left[j(x_2, z_2)\mathcal{O}_{j_1}(x_1, y_1)\right] \\
&+ \frac{k}{2}e^{-\phi}\left[\psi(x_2, z_2)\psi(x_1, z_1)\right]\left[(\chi(y_2)\partial_{y_2} - j_2\partial_{y_2}\chi(y_2))\,\mathcal{O}_{j_2}(x_2, y_2)e^{-\phi}\psi(x_1)\mathcal{O}_{j_1}(x_1, y_1)\right]
\end{aligned}$$

Where $[\dots]$ means, make use of the singular terms of the ope's. Hence

$$\begin{aligned}
&\tilde{\mathcal{O}}^{(0)}_{j_2}(y_2, x_2)\mathcal{O}^{(0)}_{j_1}(x_1, y_1) \\
&= \left((1-h_2)e^{-\phi}\left(\mathcal{D}^{(-1)}_{21}\psi(x_1)\right) + e^{-\phi}\psi(x_1)\mathcal{D}^{(h_1)}_{21} + \frac{k^2}{2}e^{-\phi}\frac{x_{12}^2}{z_{12}}\left(\chi(y_2)\partial_{y_2} - j_2\partial_{y_2}\chi(y_2)\right)\right) \\
&\times \mathcal{O}_{j_2}(x_2, y_2)\mathcal{O}_{j_1}(x_1, y_1) \\
&= 2(h_1+h_2-1)\frac{x_{12}}{z_{12}}e^{-\phi}\psi(x_1)\mathcal{O}_{j_2}(x_2, y_2)\mathcal{O}_{j_1}(x_1, y_1) \\
&+ \frac{x_{12}^2}{z_{12}}\partial_{x_1}\mathcal{O}_{j_2}(x_2, y_2)\mathcal{O}_{j_1}(x_1, y_1) + O(x_{21}^2)
\end{aligned} \tag{4.9}$$

It is now possible to construct the operator product expansion of the dressed and normalized chiral primaries which we are looking for.

$$\begin{aligned}
&\tilde{\mathbb{O}}^{(0,\bar{0})}_{j_1}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}^{(0,\bar{0})}_{j_2}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&= \frac{2\pi^2}{k\sqrt{B(h_1)(2h_1-1)B(h_2)(2h_2-1)}} g_s^2 \mathcal{O}^{(0,\bar{0})}_{j_1}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathcal{O}^{(0,\bar{0})}_{j_1}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&\overbrace{=}^{|x_{12}|\ll 1} \sum_j \int_{\frac{1}{2}+i\mathbb{R}} dh \frac{|z_{12}|^{2(\Delta(h,j)-1)}|y_{12}|^{2j_{12}}}{|x_{12}|^{2(h_{12}+1)}} \frac{(h_1+h_2+h-2)^2(2h-1)}{\sqrt{(2h-1)(2h_2-1)(2h_1-1)}}\\
&\times \frac{g_s\sqrt{2\pi^2}\ \mathcal{C}'(j)\mathcal{C}(h)}{\sqrt{k\ B(h_1)B(h_2)B(h)}}\mathbb{O}^{(0,\bar{0})}_{j,h}(x_1,\bar{x}_1;y_1,\bar{y}_1). \qquad (4.10)
\end{aligned}$$

If we make use of the identity

$$\left\langle \tilde{\mathbb{O}}^{(0,\bar{0})}_{j_1}(\infty)\mathbb{O}^{(0,\bar{0})}_{j_2}(1)\mathbb{O}^{(0,\bar{0})}_{j_3,h_3}(0)\right\rangle = \frac{\mathcal{C}(h)\mathcal{C}'(j)\ (2\pi)^2}{\sqrt{B(h_1)B(h_2)B(h_3)\ c_\nu}} \left\langle \tilde{\mathbb{O}}^{(0,\bar{0})}_{j_1}(\infty)\mathbb{O}^{(0,\bar{0})}_{j_2}(1)\mathbb{O}^{(0,\bar{0})}_{j_3}(0)\right\rangle_{h_3=j_3+1},$$

where $h_i = j_i + 1, \quad (i=1,2)$, then the OPE (4.10) is given by

$$\begin{aligned}
&\tilde{\mathbb{O}}^{(0,\bar{0})}_{j_1}(x_2,\bar{x}_2;y_2,\bar{y}_2)\mathbb{O}^{(0,\bar{0})}_{j_2}(x_1,\bar{x}_1;y_1,\bar{y}_1)\\
&= \sum_j \int_{\frac{1}{2}+i\mathbb{R}} dh\ \frac{(2h-1)}{2\pi^2 k} \frac{|z_{12}|^{2(\Delta(h,j)-1)}|y_{12}|^{2j_{12}}}{|x_{12}|^{2(h_{12}-1)}} g_s^{-2}\left\langle \tilde{\mathbb{O}}^{(0,\bar{0})}_{j_1}(\infty)\mathbb{O}^{(0,\bar{0})}_{j_2}(1)\mathbb{O}^{(0,\bar{0})}_{j,h}(0)\right\rangle\ \mathbb{O}^{(0,\bar{0})}_{j,h}(x_1,\bar{x}_1;y_1,\bar{y}_1).
\end{aligned} \qquad (4.11)$$

The above equation is not what we expected, but if we take a closer look to the integrand we will find that it is the same in the case of $h = j+1$. Because in that case we can make use of the identity

$$\mathcal{C}'(j)\mathcal{C}(h) = \frac{c_\nu^{1/2}}{2\pi}\sqrt{B(h)B(h_1)B(h_2)},$$

where $c_\nu = 1/(2\pi^4 k^3)$. Hence the ope becomes what we expected. Therefore the prove of the recursion formula get its justification, because by make use of the above ope we get a pole at $h = j+1$.

5 Conclusion

In this report I showed how one could compute a four point function in string theory on AdS background as well as how to compute extremal p point functions. Instead of computing a p point function by a direct computation of the correlator we proved an algorithm for reducing every correlator to a product of a three-point and a $p-1$-point function. This relation is given by (4.11), which is the main result of this project. Since Gaberdiel and Krisch showed that there is a correspondence of the two, three and four point functions the above mentioned reduction leads directly to the correspondence of all p point functions of the AdS string theory and the bcft.

There is still one open problem. Since Pakman showed that there are also two particle contributions in the intermediate channel the correlators are equal up to a known factor. Currently we are not in the position to calculate this factor. Therefore we refer our correspondence test to the one particle contributions.

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A Correlators

Here we list all correlator which emerge often in this kind of calculations. For the most of them we are refer to [3]

$$\langle \psi(x_1)\psi(x_3)\rangle = k\frac{x_{12}^2}{z_{12}} \tag{A.1}$$

$$\langle \chi(y_1)\chi(y_3)\rangle = k\frac{y_{12}^2}{z_{12}} \tag{A.2}$$

$$\langle j(x_k)\psi(x_1)\psi(x_3)\rangle = \sum_{i=1,i\neq k}^{n=3} \mathcal{D}_{ki}^{(-1)} \langle \psi(x_1)\psi(x_3)\rangle \tag{A.3}$$

$$\left\langle j(x_k)\prod_{i=1}^{n}\Phi_{h_i}(x_i,y_i)\right\rangle = \sum_{i=1,i\neq k}^{n} \mathcal{D}_{ki}^{(h_i)} \left\langle \prod_{i=1}^{n}\Phi_{h_i}(x_i,y_i)\right\rangle \tag{A.4}$$

$$\mathcal{D}_{ki}^{(h_i)} = \frac{1}{z_{ki}}\left(x_{ki}^2\partial_{x_i} - h_i 2x_{ki}\right)$$

B Singular terms of ope's

As well as the correlators the singular parts of the ope's of different currents with primaries, other currents, etc. are important. Hence we will list them here and refer to [3, 27] for more informations

$$\begin{aligned} j(x_k)\Phi_{h_i}(x_i) &= (-j^+ + 2x_k j^3 - x_k^2 j^-)\Phi_{h_i}(x_i) \\ &\sim \frac{1}{z_{ik}}\left(-D_{x_i}^+ + 2x_k D_{x_i}^3 - x_k^2 D_{x_i}^-\right)\Phi_{h_i}(x_i) \\ &= \frac{1}{z_{ik}}\left(-x_i^2\partial_{x_i} - 2h_i x_i + 2x_k(x_i\partial_{x_i} + h_i) - x_k^2\partial_{x_i}\right)\Phi_{h_i}(x_i) \\ &= \mathcal{D}_{ki}^{(h_i)}\Phi_{h_i}(x_i) \end{aligned} \tag{B.5}$$

$$j(x_1)j(x_2) \sim (k+2)\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}_{12}^{(-1)} j(x_2)\,, \tag{B.6}$$

$$\hat{\jmath}(x_1)\hat{\jmath}(x_2) \sim -2\frac{x_{12}^2}{z_{12}^2} + \mathcal{D}_{12}^{(-1)}\hat{\jmath}(x_2)\,, \tag{B.7}$$

$$\hat{\jmath}(x_1)\psi(x_2) \sim \mathcal{D}_{12}^{(-1)}\psi(x_2)\,, \tag{B.8}$$

$$\psi(x_2)\psi(x_1) \sim k\frac{x_{12}^2}{z_{12}} \tag{B.9}$$

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