DESY Summer Student Program 2010

Representation of an Arbitrary Beam Transfer Matrix and Solution of the Beam Matching Problem Using Equally Spaced Thin–Lens Quadrupoles

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Hamburg, 2010

Contents

1 Introduction

Since invention of alternating–gradient focusing there are still several questions which do not have strict mathematical answers yet:

1. Is it possible to obtain an arbitrary block–diagonal 4 by 4 beam transfer matrix using only drifts and quadrupoles?

2. If the answer to the question above is positive, could it be done with the number of quadrupoles which does not depend on input beam transfer matrix? Note, that from practical point of view this second question is even more important than the first one, because even in the case of positive answer on the first question there is a possibility, such that for every natural n there exists an input transfer matrix which cannot be represented with less than n quadrupoles. What is the practical meaning of the positive answer on the first question, if , for example, the needed beam transfer matrix can not be constructed with less then 100 quadrupoles?

3. If again the answer to the question above is positive, what is the minimum number of quadrupoles required and could we get analytical formulas for quadrupole strengths and lengths of field free spaces (drifts)?

Surprisingly, the answers to these questions are not known not only for quadrupoles but also for the situation when instead of quadrupoles their thin–lens approximations are used.

In this work we will try to answer these questions in the case when thin–lens approximation for quadrupole focusing matrices is used.

2 Problem Formulation

Let M be an arbitrary real four by four block diagonal matrix of the form

$$M = \begin{pmatrix} M_x & 0 \\ 0 & M_y \end{pmatrix}, \tag{1}$$

constrained additionally by symplecticity requirement

$$\det(M_x) = \det(M_y) = 1, \tag{2}$$

and let $Q_2(k)$ and $D_2(l)$ be 2 by 2 matrices of thin lens quadrupole and drift space respectively:

$$Q_2(k) \ = \ \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \qquad D_2(l) \ = \ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}. \tag{3}$$

The problem of representation of given beam transfer matrix is: are there a natural number n, real numbers $k_1, \ldots, k_n$, and non–negative reals $l_1, \ldots, l_n$, such that the following equality is held:

$$\begin{cases} Q_2(k_1)D_2(l_1)\ldots Q_2(k_n)D_2(l_n) = M_x, \\ Q_2(-k_1)D_2(l_1)\ldots Q_2(-k_n)D_2(l_n) = M_y. \end{cases} \tag{4}$$

There are several papers, whose authors tried to find solution of this problem in the form of exact analytical formulas, with most advanced probably being [1] and [2]. For example, authors of the paper [1] give formulas for k–values and drift lengths in the case of three thin lenses and three variable field free spaces, i.e., in the case of the equations:

$$\begin{cases} Q_2(k_1)D_2(l_1)Q_2(k_2)D_2(l_2)Q_2(k_3)D_2(l_3) = M_x, \\ Q_2(-k_1)D_2(l_1)Q_2(-k_2)D_2(l_2)Q_2(-k_3)D_2(l_3) = M_y. \end{cases} \tag{5}$$

Unfortunately, the given solution contains roots and denominators, and it is clearly visible that these denominators can be equal to zero for some input matrices and, due to presence of roots, some solutions can become unphysical (complex roots). And actually, as it is shown in the appendix A, three drifts and three thin lenses are insufficient for representation of an arbitrary beam transfer matrix.

2.1 Our Problem and Systems of Polynomial Equations

After multiplying all matrices on the left–hand side of the equations (4) we obtain the following system of eight equations (plus 2 symplecticity conditions for the matrix M):

$$\begin{cases} f_1(k_1, l_1, \ldots, k_n, l_n) - m_{11}^x = 0, \\ \ldots \\ f_8(k_1, l_1, \ldots, k_n, l_n) - m_{22}^y = 0. \end{cases} \tag{6}$$

Here the functions $f_1, \ldots, f_8$ are polynomials of variables k_i and l_i of degree not more than $2n$, i.e., we have to solve the system of non–linear polynomial equations.

2.2 What is Known about Solutions of the Polynomial Systems

Let us consider first the linear system (system of first order polynomial equations):

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = b_1, \\ \ldots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m. \end{cases} \tag{7}$$

For linear systems there is a complete theory, which in algorithmic way allows to answer the questions about the number of its solutions (unique solution, many solutions, solution does not exist at all). This algorithmic way, based on the process which is called Gaussian elimination, allows to construct triangular system, which is equivalent to the original one and can be easily analyzed (and solved).

Surprisingly, for general polynomial system there exists the similar algorithm, which allows to construct the equivalent system, which is called Gröbner basis (see [3]), and it is an analogy of triangular form of linear system. So, for general polynomial system one can also answer in algorithmic way about absence or existence and number of its solutions. Nowadays, the reduction of original system to its Gröbner form can be done with the help of computers using such formula manipulators as Maple or Mathematica.

Unfortunately, this equivalence of original system and its Gröbner form takes place only over the field of complex numbers and appears to be not very useful for us, as far as we are interested in physical (real) solutions.

Nevertheless, the use of computer assistance is not completely useless in our problem. Because the absence of complex solutions means also the absence of real solutions, computer helped us to construct examples of particular matrices, which can not be represented with certain number of thin lenses.

3 Equally Spaced Quadrupoles

As far as the hope to give the main job to the computer failed, the only way left to get some insides into the problem is to make hand calculations. For this purpose we consider the system constructed from equally spaced thin lenses. Such systems, as we will see, have very pleasant symmetry, which significantly simplifies hand calculations.

3.1 Quadrupole Thin–Lens Sandwiched Between Two Equal Drift Spaces

Let us introduce some notations first. As usual, we describe the action of a thin–lens quadrupole on the four dimensional transverse phase space (x, p_x, y, p_y) by means of a block–diagonal four–by–four matrix

$$Q_4(k) \;=\; \begin{pmatrix} Q_2(k) & 0 \\ 0 & Q_2(-k) \end{pmatrix} \tag{8}$$

and use the transfer matrix

$$D_4(l) \;=\; \begin{pmatrix} D_2(l) & 0 \\ 0 & D_2(l) \end{pmatrix} \tag{9}$$

for the field free space (drift) of the length l.

As an elementary building block for representation of arbitrary beam transfer matrices we will use the matrix of the quadrupole thin–lens sandwiched between two equal drift spaces of the length l, i.e. we will use the matrix

$$B_4(k, l) = D_4(l) Q_4(k) D_4(l). \tag{10}$$

Using the identity

$$B_2(k, l) \equiv D_2(l) Q_2(k) D_2(l) = S_2(l) P_2(a) S_2^{-1}(l), \tag{11}$$

where

$$S_2(l) \;=\; \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{l} & -\sqrt{l} \\ 1/\sqrt{l} & 1/\sqrt{l} \end{pmatrix},\; P_2(a) \;=\; \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix},\; a = 2 + 2kl, \tag{12}$$

we can represent the matrix B_4 in the form

$$B_4(k, l) \;=\; \begin{pmatrix} S_2(l) & 0 \\ 0 & S_2(l) \end{pmatrix} \begin{pmatrix} P_2(a) & 0 \\ 0 & P_2(4-a) \end{pmatrix} \begin{pmatrix} S_2(l) & 0 \\ 0 & S_2(l) \end{pmatrix}^{-1}. \tag{13}$$

Denote

$$S_4(l) = \begin{pmatrix} S_2(l) & 0 \\ 0 & S_2(l) \end{pmatrix}, \ P_4(a) = \begin{pmatrix} P_2(a) & 0 \\ 0 & P_2(4-a) \end{pmatrix} \tag{14}$$

In notation (14) we can rewrite (13) as

$$B_4(k, l) = S_4(l) P_4(a) S_4^{-1}(l). \tag{15}$$

3.2 Representation of an Arbitrary Beam Transfer Matrix by Equally Spaced Thin Lenses

Substituting representation (15) into formulas (4), we obtain

$$P_4(a_1) P_4(a_2) \ldots P_4(a_n) = S_4^{-1}(l) M S_4(l) = \bar{M}. \tag{16}$$

If we know elements of the matrix M, the elements of the matrix $\bar{M}$ can be calculated as follows. Let us denote

$$M_x = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \bar{M}_x = \begin{pmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{pmatrix}. \tag{17}$$

In these notations the connection between elements of the matrices M_x and $\bar{M}_x$ takes the following form

$$\begin{gathered} M_x = \frac{1}{2} \begin{pmatrix} \bar{m}_{11} - \bar{m}_{21} - \bar{m}_{12} + \bar{m}_{22} & l(\bar{m}_{11} - \bar{m}_{21} + \bar{m}_{12} - \bar{m}_{22}) \\ (\bar{m}_{11} + \bar{m}_{21} - \bar{m}_{12} - \bar{m}_{22})/l & \bar{m}_{11} + \bar{m}_{21} + \bar{m}_{12} + \bar{m}_{22} \end{pmatrix}, \\ \bar{M}_x = \frac{1}{2l} \begin{pmatrix} l m_{11} + l^2 m_{21} + m_{12} + l m_{22} & -l m_{11} - l^2 m_{21} + m_{12} + l m_{22} \\ -l m_{11} + l^2 m_{21} - m_{12} + l m_{22} & l m_{11} - l^2 m_{21} - m_{12} + l m_{22} \end{pmatrix}, \end{gathered} \tag{18}$$

with formulas for M_y and $\bar{M}_y$ being completely analogous. Note, that in the following for simplification of notations we will skip bar on the top of the matrix $\bar{M}$, which should not lead to any essential confusions, i.e., now our problem takes the form $P_4(a_1) P_4(a_2) \ldots P_4(a_n) = M$ for complete 2D case and the form $P_2(a_1) P_2(a_2) \ldots P_2(a_n) = M_x$ for its 1D version.

4 Three and Four Lens Solutions of One–Dimensional Problem

Let us consider first the one dimensional problem, when we are interested, for example, only in horizontal motion. This consideration is useful not only from methodological point of view but we will use it later for the solution of complete 2D problem.

4.1 Three Lens Solution and its Incompleteness

We start from consideration of three lens system and show that 3 lenses are insufficient for representation of an arbitrary 2 by 2 matrix with unit determinant. Note, that instead of matrices $P_2(a)$ in (12) we will consider matrices of the form

$$P_2(a,c) = \begin{pmatrix} a & c \\ -\dfrac{1}{c} & 0 \end{pmatrix}, \tag{19}$$

where $c \neq 0$ is an additional parameter. This generalization we will need later when we will solve 2D problem by its reduction to two uncoupled 1D problems.

Let us consider a system of equations with three such matrices:

$$P_2(a_1,c_1)P_2(a_2,c_2)P_2(a_3,c_3) = M = (m_{ij}), \quad i,\, j = \overline{1,\, 2}. \tag{20}$$

After multiplying all matrices on the left–hand side of (20) we obtain

$$\begin{pmatrix} a_1a_2a_3 - a_3\dfrac{c_1}{c_2} - a_1\dfrac{c_2}{c_3} & a_1a_2c_3 - \dfrac{c_1c_3}{c_2} \\ -\dfrac{a_2a_3}{c_1} + \dfrac{c_2}{c_1c_3} & -\dfrac{a_2c_3}{c_1} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \tag{21}$$

The complete solution of the system (21) is

$$
\left[\begin{array}{lll}
1. & m_{22} \neq 0, & \left\{\begin{array}{l}
a_3 = \dfrac{\left(m_{21} - \dfrac{c_2}{c_1 c_3}\right) c_3}{m_{22}}, \\[2ex]
a_2 = -\dfrac{m_{22} c_1}{c_3}, \\[2ex]
a_1 = -\dfrac{m_{12} + \dfrac{c_1 c_3}{c_2}}{m_{22} c_1},
\end{array}\right. \\[8ex]
2. & m_{22} = 0 \text{ and } m_{21} = \dfrac{c_2}{c_1 c_3}, & \left\{\begin{array}{l}
a_2 = 0, \\[2ex]
a_3 \dfrac{c_1}{c_2} + a_1 \dfrac{c_2}{c_3} = -m_{11},
\end{array}\right. \\[8ex]
3. & m_{22} = 0 \text{ and } m_{21} \neq \dfrac{c_2}{c_1 c_3}, & \varnothing,
\end{array}\right. \tag{22}
$$

and one sees that independently from the values of c_1, c_2 and c_3 there are three possibilities: unique solution, one–parameter family of solutions, or solution does not exist at all. The presence of the third variant (absence of solution) shows us that 3 thin lenses are insufficient for representation of an arbitrary 2 by 2 symplectic matrix.

4.2 Four Lens Solution

In the previous subsection we have shown that 3 lenses are insufficient for existence of solution of 1D problem. One more free parameter is required. It could be the distance between lenses l or, as we prefer in this paper, one additional lens.

Let us now consider the system

$$
P_2(a_1, c_1) P_2(a_2, c_2) P_2(a_3, c_3) P_2(a_4, c_4) = M = (m_{ij}). \tag{23}
$$

By multiplying both sides by matrix $P_2^{-1}(a_4, c_4)$ we obtain

$$P_2(a_1,c_1)P_2(a_2,c_2)P_2(a_3,c_3) = \begin{pmatrix} \dfrac{m_{12}}{c_4} & -m_{11}c_4 + m_{12}a_4 \\ \dfrac{m_{22}}{c_4} & -m_{21}c_4 + m_{22}a_4 \end{pmatrix} = \begin{pmatrix} \bar{m}_{11} & \bar{m}_{12} \\ \bar{m}_{21} & \bar{m}_{22} \end{pmatrix} \tag{24}$$

Because m_{21} and m_{22} can not be equal to zero simultaneously, we can choose a_4 in such a way that $\bar{m}_{22} \equiv -m_{21}c_4 + m_{22}a_4 \neq 0$ and therefore can use formulas (22) with m_{ik} replaced by $\bar{m}_{ik}$ as solution of our problem.

Because a_4 satisfying the condition $\bar{m}_{22} \neq 0$ is non–unique, the solution of 1D problem with 4 lenses is also non–unique, and that can be used, for example, for minimization of the lens strengths.

5 Solution for an Arbitrary Block–Diagonal Four by Four Beam Transfer Matrix

Though 1D problem can be solved by hand, the complete 2D problem still remains too complicated for the hand solution. It is mainly connected with the fact that thin lens can not act in two planes independently; if it focuses beam horizontally, then it defocuses beam vertically and vice versa.

The way which we have found for solution of 2D problem is its decomposition into two 1D problems with subsequent usage of solution of 1D problem described in the previous section. Such problem decomposition can be made in different ways. First, we present 7 lens block combination which in one transverse plane is equal to a single thin lens and in the other plane produces constant matrix equal to 2 by 2 symplectic unit matrix J_2. And second, we introduce 4 lens block which could act independently in both transverse planes. And though its action in each plane is not exactly the action of a single thin lens, this block still allows to find solution of complete 2D problem.

5.1 Block from 7 Thin Lenses

Let us denote by

$$R_x = P_2(\bar{a}_1)P_2(\bar{a}_2)\ldots P_2(\bar{a}_7), \tag{25}$$

$$R_y = P_2(4-\bar{a}_1)P_2(4-\bar{a}_2)\ldots P_2(4-\bar{a}_7) \tag{26}$$

the block constructed from 7 lenses with parameters $\bar{a}_k$ chosen as follows

$$\left\{\begin{array}{l} \bar{a}_1 = 4 + 2\sqrt{3} - (7 + 4\sqrt{3})\bar{a}_4 - \bar{a}_7, \\ \bar{a}_2 = 2 - \sqrt{3}, \\ \bar{a}_3 = 2 + \sqrt{3}, \\ \bar{a}_5 = 2 + \sqrt{3}, \\ \bar{a}_6 = 2 - \sqrt{3}. \end{array}\right. \tag{27}$$

Then we obtain

$$R_x \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J_2, \quad J_2^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \tag{28}$$

$$R_y = \begin{pmatrix} 28 - 16\sqrt{3} + 8\sqrt{3}\bar{a}_4 & 1 \\ -1 & 0 \end{pmatrix} = P_2(w), \tag{29}$$

where $w = 28 - 16\sqrt{3} + 8\sqrt{3}\bar{a}_4$.

Let us denote our 7 lens block as

$$\bar{R}(w) = \begin{pmatrix} R_x & 0 \\ 0 & R_y \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ \text{-1} & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} w & 1 \\ \text{-1} & 0 \end{matrix} \end{array}\right), \tag{30}$$

and let us consider quadrupole system of the form:

$$\bar{\bar{R}}(w_1, w_2, w_3, w_4)\bar{\bar{P}}(a_1, a_2, a_3, a_4), \tag{31}$$

where

$$\bar{\bar{R}}(w_1, w_2, w_3, w_4) = \bar{R}(w_1)\bar{R}(w_2)\bar{R}(w_3)\bar{R}(w_4) \tag{32}$$

is a combination of four 7 lens blocks and

$$\bar{\bar{P}}(a_1, a_2, a_3, a_4) = P_4(a_1)P_4(a_2)P_4(a_3)P_4(a_4) \tag{33}$$

is a combination of our usual elementary matrices.

Introducing notation

$$\bar{\bar{P}}_x(a_1, a_2, a_3, a_4) = P_2(a_1)P_2(a_2)P_2(a_3)P_2(a_4), \tag{34}$$

$$\bar{\bar{P}}_y(a_1, a_2, a_3, a_4) = P_2(4 - a_1)P_2(4 - a_2)P_2(4 - a_3)P_2(4 - a_4), \tag{35}$$

we can rewrite our system in the form

$$\left\{\begin{array}{l} \underbrace{\bar{\bar{R}}_x(w_1,\dots,w_4)}_{=I}\bar{\bar{P}}_x(a_1,\dots,a_4) = M_x, \\ \bar{\bar{R}}_y(w_1,\dots,w_4)\bar{\bar{P}}_y(a_1,\dots,a_4) = M_y. \end{array}\right. \tag{36}$$

At first we solve the first equation of the system (36) with respect to unknowns $a_1,\dots,a_4$ using formulae (22) with $c_1 = c_2 = c_3 = c_4 = 1$. Because $\bar{\bar{R}}_x \equiv I$, the system (36) becomes equivalent to the second equation

$$\bar{\bar{R}}_y(w_1,\dots,w_4) = M_y\bar{\bar{P}}_y^{-1}(a_1,\dots,a_4). \tag{37}$$

Here $\bar{\bar{P}}_y^{-1}(a_1,\dots,a_4)$ is a known matrix and $\bar{\bar{R}}_y$ is equivalent to "usual four lens combination" and therefore formulas (22) are still applicable. Hence we have obtained solution with 32 quadrupoles (28 for $\bar{\bar{R}}$ and 4 for $\bar{\bar{P}}$) for representation of an arbitrary beam transfer matrix.

5.2 Four Lens Block

The solution presented in the previous subsection solves the question of possibility to represent an arbitrary beam transfer matrix by finite number of thin lenses by complete reduction of 2D problem to two identical four–lens 1D problems. In this subsection we introduce new four–lens block which allows to make an essential reduction of the number of thin lenses needed.

Let us consider 4 lens combination

$$\left\{\begin{array}{l} R_x = P(a_1)P(a_2)P(a_3)P(a_4), \\ R_y = P(4-a_1)P(4-a_2)P(4-a_3)P(4-a_4), \end{array}\right. \tag{38}$$

where $a_2 = 2-\sqrt{3}$ and $a_3 = 2+\sqrt{3}$. By multiplying matrices on the left–hand side of (38) we obtain

$$\left\{\begin{array}{l} R_x = \begin{pmatrix} 1-(2-\sqrt{3})a_1-(2+\sqrt{3})a_4 & -(2+\sqrt{3}) \\ 2-\sqrt{3} & 0 \end{pmatrix} = \begin{pmatrix} r_x & -2-\sqrt{3} \\ -\dfrac{1}{-2-\sqrt{3}} & 0 \end{pmatrix}, \\ \\ R_y = \begin{pmatrix} -15+(2+\sqrt{3})a_1+(2-\sqrt{3})a_4 & -(2-\sqrt{3}) \\ 2+\sqrt{3} & 0 \end{pmatrix} = \begin{pmatrix} r_y & -2+\sqrt{3} \\ -\dfrac{1}{-2+\sqrt{3}} & 0 \end{pmatrix}, \end{array}\right. \tag{39}$$

where R_x and R_y are matrices of the form (19) and parameters r_x and r_y can be varied independently because the determinant of the Jacobian matrix of the coordinate transformation from a_1, a_4 to r_x, r_y is non–degenerated:

$$\begin{vmatrix} -2+\sqrt{3} & -2-\sqrt{3} \\ 2+\sqrt{3} & 2-\sqrt{3} \end{vmatrix} = -(2-\sqrt{3})^2 + (2+\sqrt{3})^2 = 8\sqrt{3} \neq 0 \tag{40}$$

Now consider the system

$$\begin{cases} R_x(r_x^1)R_x(r_x^2)R_x(r_x^3)P_2(a_4) = M_x, \\ R_y(r_y^1)R_y(r_y^2)R_y(r_y^3)P_2(4-a_4) = M_y. \end{cases} \tag{41}$$

We can rewrite (41) in the form

$$\begin{cases} R_x(r_x^1)R_x(r_x^2)R_x(r_x^3) = M_x P_2^{-1}(a_4), \\ R_y(r_y^1)R_y(r_y^2)R_y(r_y^3) = M_y P_2^{-1}(4-a_4). \end{cases} \tag{42}$$

If it will be possible to find such a_4 that matrices in the right sides of (42) have non–zero m_{22}^x and m_{22}^y simultaneously then we can use our 1D solution (22) for solution of complete 2D problem. Consider right–hand sides of (42):

$$\begin{gathered} \begin{pmatrix} m_{11}^x & m_{12}^x \\ m_{21}^x & m_{22}^x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_4 \end{pmatrix} = \begin{pmatrix} * & * \\ * & m_{22}^x a_4 - m_{21}^x \end{pmatrix} \\ \begin{pmatrix} m_{11}^y & m_{12}^y \\ m_{21}^y & m_{22}^y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 4-a_4 \end{pmatrix} = \begin{pmatrix} * & * \\ * & -m_{21}^y + (4-a_4)m_{22}^y \end{pmatrix} \end{gathered} \tag{43}$$

Now we want to show that it is possible to choose a_4 such that

$$\begin{cases} m_{22}^x a_4 - m_{21}^x \neq 0, \\ -m_{21}^y + (4-a_4)m_{22}^y \neq 0. \end{cases} \tag{44}$$

System (44) is equivalent to

$$\begin{cases} a_4 m_{22}^x \neq m_{21}^x, \\ a_4 m_{22}^y \neq 4m_{22}^y - m_{21}^y. \end{cases} \tag{45}$$

Since $(m_{22}^x)^2 + (m_{21}^x)^2 \neq 0$ and $(m_{22}^y)^2 + (m_{21}^y)^2 \neq 0$ due to the symplecticity of the matrices M_x and M_y, the necessary a_4 can always be found.

So, we have obtained solution, which requires 13 quadrupoles (12 for $R_x(r_x^i)$ and $R_y(r_y^i)$ and 1 for $P_2(a_4)$ and $P_2(4-a_4)$), and setting of 6 of them is fixed and does not depend on input matrices M_x and M_y. Although the total number of lenses used in this solution is probably still not the minimal possible, we have simple explicit formulas for the lens strengths as functions of the coefficients of the input transfer matrix. And as concerning number of variable parameters in our system (7 parameters), it is the minimum possible value because we have an example of transfer matrix which can not be represented by using six lenses with fixed distance between them.

5.3 Example of Transfer Matrix which can not be Represented by Less Than Seven Equally Spaced Thin Lenses with Fixed Distance between Them

We used Maple 13 and Mathematica 7.0 for calculations of the Gröbner bases for the matrix

$$M_x = M_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{46}$$

These bases were constructed for different number of quadrupoles. For all systems with less than 7 quadrupoles their Gröbner bases contained constant and it means that system is inconsistent and has no solution. And for 7 quadrupoles Gröbner basis (in terms of matrices $P_2(a_i)$ and $P_2(4-a_i)$) is:

$$\begin{aligned}
f_1 &= -239 + 2220a_6 - 5547a_6^2 + 2240a_6^3 - 120a_6^4 - 48a_6^5 + 4a_6^6, \\
f_2 &= -1726 + 33a_5 + 7126a_6 - 4402a_6^2 + 467a_6^3 + 100a_6^4 - 10a_6^5, \\
f_3 &= -10030 + 594a_4 + 44821a_6 - 28252a_6^2 + 3002a_6^3 + 640a_6^4 - 64a_6^5, \\
f_4 &= 1198 + 99a_3 - 6994a_6 + 4402a_6^2 - 467a_6^3 - 100a_6^4 + 10a_6^5, \\
f_5 &= 3188 + 99a_2 - 14285a_6 + 8804a_6^2 - 934a_6^3 - 200a_6^4 + 20a_6^5, \\
f_6 &= -4286 + 594a_1 - 1669a_6 + 1840a_6^2 - 200a_6^3 - 40a_6^4 + 4a_6^5 + 594a_7.
\end{aligned} \tag{47}$$

Equation $f_1 = 0$ has real roots (checked with Maple) with respect to variable a_6. After fixing value of a_6 and substituting it into remaining equations (47) we obtain a linear system for unknowns a_1, a_2, a_3, a_4, a_5, a_7, which has a triangular form and is solvable. So our matrix (46) can be represented with 7 lenses and can not be represented with smaller number of them.

6 Beam Matching Problem and Its Solution

In this section we will consider the beam matching problem and will find the solution using equally spaced thin lenses.

6.1 Problem Formulation

Let us assume that we have two sets of Twiss parameters

$$\Sigma_x^i = \begin{pmatrix} \beta_x^i & -\alpha_x^i \\ -\alpha_x^i & \gamma_x^i \end{pmatrix}, \quad \Sigma_y^i = \begin{pmatrix} \beta_y^i & -\alpha_y^i \\ -\alpha_y^i & \gamma_y^i \end{pmatrix}, \quad i = 1, 2 \tag{48}$$

given in the two points of a beam line. The beam matching problem is the problem of finding symplectic transfer matrix $A = \operatorname{diag}(A_x, A_y)$, such that the following identity holds:

$$\begin{cases} \Sigma_x^2 = A_x \Sigma_x^1 A_x^T, \\ \Sigma_y^2 = A_y \Sigma_y^1 A_y^T. \end{cases} \tag{49}$$

6.2 General Solution

The first question is if this problem has symplectic solution at all for arbitrary two sets of Twiss parameters. Because the system (49) is uncoupled between horizontal and vertical degrees of freedom, let us consider only one equation of the form

$$\Sigma_2 = A \Sigma_1 A^T, \tag{50}$$

in which subscripts x and y are omitted. Because the matrices Σ_i are symplectic and positive definite, we can make their symplectic Cholesky decomposition

$$\Sigma_i = L_i L_i^T, \quad i = \overline{1, 2}, \tag{51}$$

where L_i are lower triangular symplectic matrices

$$L_i = \begin{pmatrix} \sqrt{\beta}_i & 0 \\ -\alpha_i / \sqrt{\beta}_i & 1/\sqrt{\beta}_i \end{pmatrix}, \quad i = \overline{1, 2}. \tag{52}$$

By substituting this decomposition into our equation (50) we obtain:

$$L_2 L_2^T = A L_1 L_1^T A^T. \tag{53}$$

We can rewrite system (53) in the form

$$L_2^T (A^T)^{-1} (L_1^T)^{-1} = L_2^{-1} A L_1, \tag{54}$$

which is equivalent to the equation

$$(L_2^{-1} A L_1)(L_2^{-1} A L_1)^T = I. \tag{55}$$

The equation (55) shows that the matrix $L_2^{-1} A L_1$ is orthogonal, and because this matrix has determinant equal to one, it is a rotation matrix:

$$L_2^{-1} A L_1 = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}. \tag{56}$$

Let us denote

$$R(\mu) = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}. \tag{57}$$

Then our system (50) is equivalent to the equation

$$L_2^{-1} A L_1 = R(\mu), \tag{58}$$

where μ is an arbitrary real parameter. So, we see that matching problem always has symplectic solution given by the following formula

$$A = L_2 R(\mu) L_1^{-1}. \tag{59}$$

And in terms of the original system (49) its symplectic solution is

$$\begin{cases} A_x = L_{x2} R(\mu_x) L_{x1}^{-1}, \\ A_y = L_{y2} R(\mu_y) L_{y1}^{-1}, \end{cases} \tag{60}$$

where μ_x and μ_y are two arbitrary real parameters, which are called phase advances.

Because our previous solution for an arbitrary transfer matrix certainly gives also the solution of the matching problem, our current goal is to solve matching problem with less thin lenses using freedom in choosing the phase advances. We will use the same scheme as before and will find first the number of thin lenses required for solution of 1D problem, and then we will use our four–lens blocks for solution of complete 2D problem.

6.3 Matching Problem in New Variables

Representing matching matrix A as a product of matrices of equally spaced thin lenses

$$A = S_4(l)P_4(a_1)P_4(a_2)\dots P_4(a_n)S_4^{-1}(l) \tag{61}$$

and substituting this representation in the system (50) we obtain

$$\underbrace{S_4^{-1}(l)\Sigma_2(S_4^{-1}(l))^T}_{\bar{\Sigma}_2} = \underbrace{P_4(a_1)\dots P_4(a_n)}_{\bar{A}}\underbrace{S_4^{-1}(l)\Sigma_1(S_4^{-1}(l))^T}_{\bar{\Sigma}_1}(P_4(a_1)\dots P_4(a_n))^T. \tag{62}$$

Hence, after change of variables we obtain system of the same form

$$\bar{\Sigma}_2 = \bar{A}\bar{\Sigma}_1\bar{A}^T. \tag{63}$$

So, further we will try to solve the system (63) and for simplification of notations we will skip all bars on the tops of $\bar{\Sigma}_i$ and $\bar{A}$. And for transfer between solutions of (63) and (50) there exists formulas (62) and (61).

6.4 One Dimensional Solution

Again for simplification of notations let us use the equation (50) as a formulation of 1D problem. Denote

$$L_2^{-1}AL_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \tag{64}$$

Because the matrix on the right–hand side of (64) must be a rotation matrix, the matching problem is equivalent to the system

$$\begin{cases} g_{11} = g_{22}, \\ g_{12} = -g_{21}, \end{cases} \tag{65}$$

and substituting into (65) the elements of the matrices $A = (a_{ij})$ and L we obtain

$$\begin{cases} \beta_1 a_{11} - (\alpha_1 + \alpha_2)a_{12} - \beta_2 a_{22} = 0, \\ \alpha_2\beta_1 a_{11} + \beta_1\beta_2 a_{21} - \alpha_1\alpha_2 a_{12} - \alpha_1\beta_2 a_{22} + a_{12} = 0. \end{cases} \tag{66}$$

In general, one has to add to this system symplecticity condition for the matrix A, but because we will try to find this matrix as a product of our elementary symplectic matrices, the matrix A will be automatically symplectic.

Because, as before, later on we will use solution of 1D problem for 2D case, let us consider matrix A as a product of matrices $P_2(a_k, c_k)$ depending on additional parameters c_k. Substituting $A = P_2(a_1, c_1)P_2(a_2, c_2)$ into (66) we obtain

$$\begin{cases} c_1c_2\beta_1a_1a_2 - c_1c_2^2(\alpha_1 + \alpha_2)a_1 - c_1^2\beta_1 + c_2^2\beta_2 = 0, \\ c_1c_2\alpha_2\beta_1a_1a_2 - c_1c_2^2\alpha_1\alpha_2a_1 - c_2\beta_1\beta_2a_2 + c_1c_2^2a_1 + c_2^2\alpha_1\beta_2 - c_1^2\alpha_2\beta_1 = 0 \end{cases} \tag{67}$$

After multiplying first equation by $-\alpha_2$ and summing it with the second one we have:

$$-c_1c_2^2(\alpha_2^2 + 1)a_1 + c_2\beta_1\beta_2a_2 + c_2^2\alpha_2\beta_2 - c_2^2\alpha_1\beta_2 = 0. \tag{68}$$

Remembering that $\beta_2\gamma_2 - \alpha_2^2 = 1$ and after dividing this equation by $c_2\beta_2$ we obtain:

$$-c_1c_2\gamma_2a_1 + \beta_1a_2 + c_2\alpha_2 - c_2\alpha_1 = 0. \tag{69}$$

From this it follows that

$$\beta_1a_2 = c_2(\alpha_1 - \alpha_2) + c_1c_2\gamma_2a_1. \tag{70}$$

Substituting it in the first equation of the system (67) and making simplification we obtain quadratic equation for the unknown a_1

$$(c_1c_2)^2\gamma_2a_1^2 - 2c_1c_2^2\alpha_2a_1 - c_1^2\beta_1 + c_2^2\beta_2 = 0. \tag{71}$$

This equation has real solution

$$a_1 = (c_2\alpha_2 \pm \sqrt{c_1^2\beta_1\gamma_2 - c_2^2})/(c_1c_2\gamma_2), \tag{72}$$

if $\beta_1\gamma_2 \geq c_2^2/c_1^2$, and does not have real solution otherwise. If there is a real solution for a_1 then a_2 is given by the following formula

$$a_2 = (c_2\alpha_1 \pm \sqrt{c_1^2\beta_1\gamma_2 - c_2^2})/\beta_1. \tag{73}$$

Again we see that we one more parameter is needed in order to match two arbitrary sets of Twiss parameters. And again we can choose between adding one more thin lens or using as additional parameter the distance l between thin lenses. Here we also prefer the first variant.

Let us add one more lens to the matrix A, i.e., let $A = P_2(a_1, c_1)P_2(a_2, c_2)P_2(a_3, c_3)$. Writing the equation (50) in the form:

$$\Sigma_2 = \underbrace{P_2(a_1, c_1)P_2(a_2, c_2)}_{\bar{A}} \underbrace{P_2(a_3, c_3)\Sigma_1 P_2(a_3, c_3)^T}_{\bar{\Sigma}_1} (P_2(a_1, c_1)P_2(a_2, c_2))^T \quad (74)$$

where the matrix $\bar{A}$ is constructed from two lenses and

$$\bar{\Sigma}_1 = \begin{pmatrix} \bar{\beta}_1 & -\bar{\alpha}_1 \\ -\bar{\alpha}_1 & \bar{\gamma}_1 \end{pmatrix} = \begin{pmatrix} \beta_1 a_3^2 - 2c_3\alpha_1 a_3 + c_3^2\gamma_1 & -\beta_1 a_3/c_3 + \alpha_1 \\ -\beta_1 a_3/c_3 + \alpha_1 & \beta_1/c_3^2 \end{pmatrix} \quad (75)$$

is the new set of incoming Twiss parameters, we will try to choose a_3, such that the inequality $\bar{\beta}_1\gamma_2 - c_2^2/c_1^2 > 0$ is satisfied. In terms of β_1, α_1, γ_1 and a_3 this inequality takes the form

$$\beta_1 a_3^2 - 2c_3\alpha_1 a_3 + c_3^2\gamma_1 - c_2^2/(c_1^2\gamma_2) \geq 0, \quad (76)$$

where the left–hand side is quadratic polynomial with respect to the variable a_3. Because the leading coefficient of this polynomial is positive, with appropriate choice of a_3 the inequality (76) always can be satisfied.

So, for solution of 1D matching problem 3 thin lenses are sufficient.

6.5 Two–Dimensional Problem

For solution of complete 2D problem we will use, as before, two four–lens blocks plus one additional lens for satisfying the conditions $\beta_x^1\gamma_x^2 \geq 1$ and $\beta_y^1\gamma_y^2 \geq 1$. In terms of strength a of this additional lens, these conditions can be rewritten in the form:

$$\begin{cases} \beta_x^1 a^2 - 2\alpha_x^1 a + \gamma_x^1 - 1/\gamma_x^2 \geq 0, \\ \beta_y^1(4-a)^2 - 2\alpha_y^1(4-a) + \gamma_y^1 - 1/\gamma_y^2 \geq 0. \end{cases} \quad (77)$$

Because the leading coefficients of quadratic polynomials on the left–hand sides of (77) are positive, it is clear that such a always can be found.

So we have obtained the solution of the beam matching problem which uses 9 thin lenses, 4 of them are set to the constant values and only 5 parameters depend on Twiss functions. And in the following subsection we will give an example which proves that 5 variable parameters is the minimum number needed in order to match arbitrary two sets of Twiss parameters.

6.6 Example of Twiss Parameters which can not be Matched by Less Than Five Equally Spaced Thin Lenses

We found two sets of Twiss parameters

$$\Sigma_1^x = \Sigma_1^y = \begin{pmatrix} 1/3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \Sigma_2^x = \Sigma_2^y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{78}$$

which can not be matched with less than five thin lenses. We used Gröbner bases in Maple program for proving it. For all systems with less than 4 quadrupoles Gröbner bases contained constant and it means that system is inconsistent. And for 4 quadrupoles the Gröbner basis is (in terms of matrices $P_2(a_i)$ and $P_2(4 - a_i)$):

$$f_1 = 1659952448 - 2884692736a_4 + 3307890560a_4^2 - 2760384096a_4^3 + 1834149068a_4^4 - 1002491264a_4^5 + 460305056a_4^6 - 179294144a_4^7 + 59520572a_4^8 + 4010200a_4^{10} - 797664a_4^{11} - 16807392a_4^9 + 129332a_4^{12} - 16464a_4^{13} + 1548a_4^{14} - 96a_4^{15} + 3a_4^{16},$$

$$f_2 = -572928628940044301120 + 99410699439079666032a_4 - 117374598008235424952a_4^2 + 100177210244188013684a_4^3 - 65550381943519831456a_4^4 + 34551266471131239024a_4^5 - 14966720248682032760a_4^6 + 5374237484360774884a_4^7 - 1601922925204511280a_4^8 - 79161365231907112a_4^{10} + 12678342805492540a_4^{11} + 394492844152254648a_4^9 - 1573549041137424a_4^{12} + 144855092875104a_4^{13} - 9035783133030a_4^{14} + 301192771101a_4^{15} + 41450786744450428 80a_3,$$

$$f_3 = 30280529753705085702 24a_4 + 62176180116675643200a_2 - 37888134011316334011 44a_4^2 + 33764705087455575171 08a_4^3 - 23384477628221160512 32a_4^4 + 13114527963622077389 28a_4^5 - 6078412428392366538 80a_4^6 + 2354050146591064400 68a_4^7 - 763087719213689301 60a_4^8 - 45930187974003681 04a_4^{10} + 8257692793170991 00a_4^{11} + 206204309977090312 56a_4^9 - 11549987341095652 8a_4^{12} + 11822189594425488a_4^{13} - 790702154027910a_4^{14} + 26356738467597a_4^{15} - 144050437314435276 5984,$$

$$f_4 = -33865454271352153152a_4 + 10362696686112607 20a_1 + 394711919252068782 32a_4^2 - 33611734595500272644a_4^3 + 22608437332356479296a_4^4 - 12471408080055248064a_4^5 + 5739763288953333800a_4^6 - 2224533690630644044a_4^7 + 725549734134995280a_4^8 + 44682334346370952a_4^{10} - 8151956031330700a_4^{11} - 198031802810244168a_4^9 + 1156768131825384a_4^{12} - 119808491147964a_4^{13} + 8069722409430a_4^{14} - 268990746981a_4^{15} + 14124996950565758432.$$

One can check that the first equation $f_1 = 0$, which depends only on a_4, does not have real solutions (we have checked that with Maple). It means that we cannot match our two sets of Twiss parameters (78) by less than 5 thin lenses.

7 Summary

It is proven that an arbitrary beam transfer matrix and the problem of matching of arbitrary Twiss parameters can be solved with finite number of thin lenses.

We have found a solution for representation of transfer matrix, which uses 13 thin lenses (or 12 lenses plus variable space between them). From this 13 parameters 6 are independent from the elements of the input matrix. We also have found an example of the matrix, which can not be represented with less than 7 parameters.

We have found a solution of matching problem, which uses 9 thin lenses (or 8 lenses plus variable space between them). From this 9 parameters 4 are independent from the elements of the input matrix. We also have found an example of Twiss parameters, which can not be matched with less than 5 parameters.

8 Acknowledgments

Being a summer student I would like to thank all members of the Machine Physics Group for their hospitality, support, and enormous help with solving many organizational questions.

I would like to thank the head of DESY Summer Student Program Professor Meyer for giving me the opportunity to come here and get such great experience not only from professional point of view.

Last but not least, I would like to thank my supervisor V.V. Balandin for his wonderful thoughts and fantastical support in all questions that I had, for many hours of patient explanations, corrections and clarifications, as well as for guidance throughout the whole summer student program.

A Example of Transfer Matrix which can not be Represented by Three Thin Lenses and Three Arbitrary Drift Spaces

Consider the matrix:

$$M_x = M_y = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{79}$$

This matrix can not be represented with less than 5 thin lenses and 4 variable drift spaces (in total, 9 parameters). This example was found with the help of Maple and checked once more using Mathematica program. This example clearly shows that the simple degree–of–freedom count not always leads to the correct answers and three thin lenses with variable distances between them are insufficient for representation of an arbitrary beam transfer matrix.

References

[1] Y–Chiu Chao and John Irwin, "Solution of a three–thin–lens system with arbitrary transfer properties", SLAC-PUB-5834, October 1992.

[2] Napoly, "Thin lens telescopes for final focus systems", CERN/LEP–TH/89-69, CLIC Note 102, November 1989.

[3] Adams W.W., Loustaunau P., "An introduction to Gröbner Bases", AMS, Graduated Studies in Math. Vol. III, 1994.