Matrix Model Duality

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Abstract

In this report we review a duality between three different Matrix Models. We will prove the duality between the Kontsevich-Penner (W_∞) model and a model which we will call the F model, where F is an arbitrary complex matrix. Both models describe tachyon scattering in $c = 1$ non-critical string theory. Before the proof we develop the necessary tools of group and representation theory.

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1 Introduction

In the 20th century there were great improvements in understanding the structure of the universe and the nature of the fundamental interactions. Two extremely successful theories, *General Relativity* and *Quantum Theory*, were developed and have ever since been experimentally verified to high precision. It is clear that both of these theories describe their separate domains of physical reality in great accuracy.

Since the advent of these theories physicists and mathematicians have tried to unite quantum theory with relativity, but the attempts have not succeeded so far. String theory attempts to solve this problem in a mathematically consistent way. A simplified setting to study string theory is the non-critical $c = 1$ string with two-dimensional target space. The $c = 1$ string theory already exhibits one of the most interesting features of string theory: gauge-gravity duality. It can be expressed by a variety of matrix models, three of which we will look into.

These models are interesting due to the existence of dualities between the different types of models, reflected in their Feynman diagrams. In some models the closed string insertions appear as vertices (a 'V-type' model), while in others they are associated to faces (an 'F-type' model). Another example of this type of duality is topological 2d gravity, where the two models in question are the double-scaled Hermitian matrix model (V-type) and the Konsevich matrix model [12] (F-type).

Applications of these matrix models also arise in 4d $\mathcal{N} = 4$ Super Yang-Mills theory. There the Z model is a generating function for correlation functions of holomorphic and antiholomorphic half-BPS operators built from a single complex scalar transforming in the adjoint of the gauge group $U(N)$ [6], [13]. Local operators in $\mathcal{N} = 4$ Super Yang-Mills correspond to vertices of the Z model. On the other hand, when considered as expressing the $c = 1$ non-critical string at the self-dual radius the Z model is a generating function for the scattering matrix of tachyons with positive and negative integer momenta.

2 Matrix Model Generalities

In this chapter we start by reviewing some generalities of matrix models and their Feynman graph expansions. We will take a closer look at one example of a correlation function of trivalent vertices. In the second section we'll consider three different matrix models in more detail.

2.1 Matrix models and their Feynman graph expansions [1]

Matrix models are the simplest kind of quantum gauge theories, namely ones in zero dimensions. The basic field of the theory is a Hermitian $N \times N$ field M. Consider an action of the form

$$W(M) = \mathrm{tr}(M^2) + \sum_{p \geq 3} \mathrm{tr}(M^p) \tag{1}$$

An action like this has the gauge symmetry

$$M \to UMU^\dagger \tag{2}$$

with U a $U(N)$ matrix. The partition function of the theory is

$$\mathcal{Z} = \frac{1}{\mathrm{vol}(U(N))} \int dM \, e^{-W(M)} \tag{3}$$

The factor $\mathrm{vol}(U(N))$ is the volume of the unitary group and this arises from fixing the gauge and so we're in fact considering a gauged matrix model. The measure $[dM]$ is given by

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1 \leq i < j \leq N} d\,\Re M_{ij} \, d\,\Im M_{ij} \ . \tag{4}$$

Partition functions with general actions of the type(1) can be computed considering perturbations around the Gaussian point [14].

One can visualize perturbations of the free energy

$$F = \ln Z \tag{5}$$

with fatgraphs, in the double-line notation used by 't Hooft [17]. The reason that we want to do this is that in $U(N)$ gauge theories there are apart from gauge coupling parameters another parameter, namely the rank of the gauge group, N, and the perturbative expansion can be done in powers of N. In general a single Feynman diagram gives rise to a polynomial in N, where the N dependence comes from the group factors associated to the Feynman graph. However, the N dependence is hard to see in single-line Feynman graphs, and so we choose to use the double-line notation, as introduced by 't Hooft. Thus the Feynman diagrams become "fatgraphs".

In order to demonstrate this method, we take (1), and consider just the cubic case, that is, picking just the $p = 3$ term from the sum. The field M_{ij} is in the adjoint representation, which is defined as the tensor product of the fundamental and antifundamental representations. Therefore one can consider i as being and index of the fundamental and j as an index of the antifundamental representation. This index notation can be visualized in the double line notation as in Figure 1

Interpreting this as one would a Feynman graph, one can see that the kinetic term of the theory is given by [2]

$$\mathrm{tr}\left(M^2\right) = M_{ij} M_{ji} \tag{6}$$

[1] In this section the normalization and gauge coupling constants will mostly be omitted for clarity.

[2] Henceforth we will implicitly assume Einstein summation, whenever it simplifies notation. Sometimes we will write the sums out explicitly, in order to clarify the ideas presented.

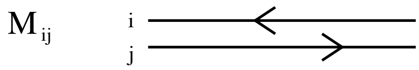

Figure 1: The index structure of the field M_{ij} can be visualized as a double line (figure from [14])

Hence the propagator is

$$\langle M_{ij}\, M_{kl} \rangle = \delta_{il}\, \delta_{jk} \tag{7}$$

This is illustrated in Figure 2

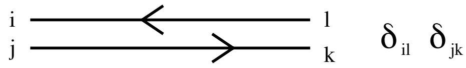

Figure 2: The propagator visualized in the double line notation (figure from [14])

The vertices of the theory can be drawn as p double lines joined together for a vertex of pth order. In the cubic potential under consideration we have a third-order vertex

$$\mathrm{tr}\left(M^3\right) = M_{ij} M_{jk} M_{ki} \tag{8}$$

which is displayed in Figure 3.

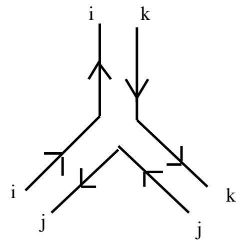

Figure 3: The cubic vertex $\mathrm{tr}\big(M^3\big)$ (figure from [14])

The Feynman diagram can give rise to many different fatgraphs, depending on which terms of the Wick contraction are considered. Contracting two trivalent vertices can be done in two ways; in a way that renders just one face that corresponds to a factor N in the perturbative expansion, another giving rise to a three-face diagram, corresponding to a term with N^3. This calculation is done in Appendix A.

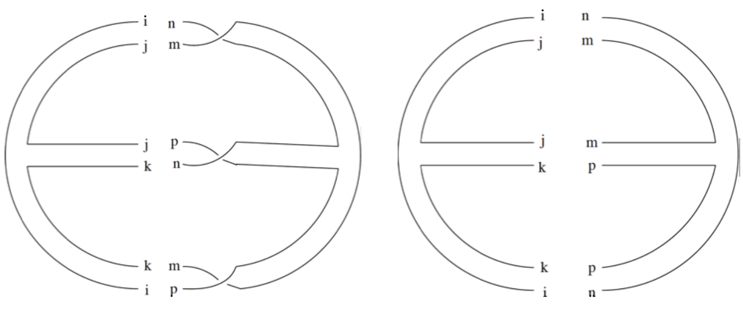

Figure 4: Two different ways of contracting two trivalent vertices; on the left the loop diagram corresponding to a factor of N, the diagram on the right corresponds to a factor N^3 (figure from [14]) .

We see the same correlation function is split into two fatgraphs. The closed loops are given by the way in which the contraction is performed, and looking at Figure 4 one sees that the power of N associated with a fatgraph is actually equal to the number of closed loops in the graph. The graph on the right in Figure 4 is planar, and can be drawn without intersections on a sphere, a Riemann surface of the lowest possible genus. The graph on the left, on the other hand, is not planar and can only be drawn without intersections on a Riemann surface of a genus one higher than the sphere, that is, the two-torus. Generally, a fatgraph is characterized topologically by the number of propagators or edges E, the number of vertices V and the number of closed loops h. This implies that a fatgraph can be regarded as a Riemann surface with holes so that each closed loop represents the boundary of a hole. Formally the genus g of such a surface is related to the above mentioned parameters by the topological formula

$$2g - 2 = E - V - h \tag{9}$$

Generally, the fatgraph corresponding to the highest factor of N is the planar graph of the expansion. Then, computing all the different contractions one gets terms of falling order in N^2. For each suppression by $\frac{1}{N^2}$ the corresponding fatgraph can be drawn without intersecting on a Riemann surface whose genus is one higher than that of the previous one.

It follows that we can organize the computation of the different quantities in the matrix model in terms of fatgraphs. This is called the *genus expansion* of the free energy of the matrix model.

2.2 Specifics of Different Matrix Models

The first model we consider is the Z model, which describes tachyon scattering for the $c = 1$ string at self-dual radius[1]. The map from tachyons $\mathcal{T}_p$ with integer momentum p to the matrix variables Z, $Z^\dagger$ is for $k > 0$

$$\mathcal{T}_k \to \mathrm{tr}(Z^k) \qquad \mathcal{T}_{-k} \to \mathrm{tr}(Z^{\dagger k}) \tag{10}$$

And the individual tachyon correlation functions

$$\left\langle \mathcal{T}_{k_1} \cdots \mathcal{T}_{k_p} \mathcal{T}_{-\bar{k}_1} \cdots \mathcal{T}_{-\bar{k}_q} \right\rangle_{c=1} = \left\langle \mathrm{tr}(Z^{k_1}) \cdots \mathrm{tr}(Z^{k_p}) \mathrm{tr}(Z^{\dagger k_1}) \cdots \mathrm{tr}(Z^{\dagger k_q}) \right\rangle \tag{11}$$

On the righthand side the correlation function is taken using the complex matrix model with Gaussian action $\mathrm{tr}(ZZ^\dagger)$. The correlator is computed by Wick-contracting with the propagator

$$\left\langle Z_{ij}\, Z^\dagger_{kl} \right\rangle = \delta_{il}\delta_{jk} \tag{12}$$

The partition function of the Z model is

$$\mathcal{Z}_Z(\{t\}, \{\bar{t}\}) = \int [dZ]_{N\times N}\, e^{-\mathrm{tr}(ZZ^\dagger) + \sum_{k=1}^\infty t_k \mathrm{tr}(Z^k) + \sum_{k=1}^\infty \bar{t}_k \mathrm{tr}(Z^{\dagger k})} \tag{13}$$

whence one sees that the model has two infinite sets of couplings $\{t_k\}$ and $\{\bar{t}_k\}$. These couplings can be rewritten in terms of Kontsevich-Miwa transform

$$t_k = \sum_{i=1}^n \frac{1}{k a_i^k} = \frac{1}{k}\mathrm{tr}(A^{-k}) \qquad \bar{t}_k = \sum_{j=1}^n \frac{1}{k b_j^k} = \frac{1}{k}\mathrm{tr}(B^{-k}) \tag{14}$$

where $A = \mathrm{diag}(a_1, a_2, ...a_i)$ and $B = \mathrm{diag}(b_1, b_2, ..., b_j)$.

Another model in literature is the Kontsevich-Penner, or W_∞ model, where just the couplings corresponding to tachyons with positive momenta are rewritten as $t_k = \frac{1}{k}\mathrm{tr}(A^{-k})$. The W_∞

model is defined for Hermitian positive semi-definite $N \times N$ matrices M and is in its general form given by [16]

$$\mathcal{Z}_{W_\infty}(\{t\},\{\bar{t}\}) = \int [dM]^{H^+}_{N\times N}\, e^{\mathrm{tr}M-(i\mu+N)\mathrm{tr}\,\log M+\sum_{k=1}^{\infty}\bar{t}_k \mathrm{tr}(MA^{-1})^k} \tag{15}$$

In sections to come we will set $\nu = -i\mu = N$, where μ is the cosmological constant, so that

$$\mathcal{Z}_{W_\infty}(\{t\},\{\bar{t}\}) = \int [dM]^{H^+}_{N\times N}\, e^{\mathrm{tr}\left(-M+\sum_{k=1}^{\infty}\bar{t}_k (MA^{-1})^k\right)} \tag{16}$$

This model has the propagator [8]

$$\langle \hat{m}_{i_1 j_1}\hat{m}_{i_2 j_2}\rangle_{W_\infty} = \frac{1}{\nu}\, A^{-1}_{i_1 j_2} A^{-1}_{i_2 j_1} \tag{17}$$

where the variable $\hat{m}$ is introduced because the quantum action $\Gamma(\hat{M}) = -\nu N + \nu\ \mathrm{tr}\hat{M}A - \nu\,\mathrm{tr}\log\hat{M}$ leads to equations of motion such that the expectation value of the field is $\langle M\rangle = A^{-1}$. One then shifts $\hat{M}$ by $\hat{m}$ around this expectation value, so that $\hat{M} = A^{-1} + \hat{m}$.

A generic Hermitian matrix can be written as a product of unitary matrices and a diagonal matrix [15] $M \to UDU^\dagger$, with U Unitary, and D diagonal, $D = diag(m_1, m_2, ..., m_N)$, where the $\{m_i\}$ are the eigenvalues of M. This is convenient, because the action has the gauge symmetry (2), and it enables us to reduce the integral to eigenvalues.

The metric on the matrix space after this gauge transformation can be calculated as in [2]

$$ds^2 = \mathrm{tr}(dMdM) = dm_1^2 + dm_2^2 + ... + dm_N^2 + \sum_{i\neq j}(m_i - m_j)|dU_{ik}U^\dagger_{kj}|^2 \tag{18}$$

The Jacobian of the transformation is given by the square root of the determinant of the above expression

$$\sqrt{\det(ds^2)} = \prod_{i<j}(m_i - m_j)^2 \tag{19}$$

And so

$$d^{N^2}M = (d^{N^2-N}U)\prod_{i=1}^{N} dm_i \prod_{i<j}(m_i - m_j)^2 \equiv [dU]\,(d^N m_i)\Delta^2(m_i) \tag{20}$$

Here Δ is the Vandermonde determinant.

The third and final model we consider is the F model, in which also the couplings corresponding to negative momenta are rewritten as in (14). This model has the partition function

$$\begin{aligned}\mathcal{Z}_F(\{t\},\{\bar{t}\}) &= \int [dF]^{\mathbb{C}}_{N\times N}\, e^{-\mathrm{tr}(FF^\dagger)-N\,\mathrm{tr}\log(1-A^{-1}FB^{-1}F^\dagger)} &(21)\\ &= \int [dF]^{\mathbb{C}}_{N\times N}\, e^{-\mathrm{tr}(FF^\dagger)-N\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}\left[(A^{-1}FB^{-1}F^\dagger)^k\right]} &(22)\end{aligned}$$

And the propagator

$$\left\langle F^i_j\, {F^\dagger}^k_l \right\rangle_F = \frac{\delta^i_l \delta^j_k}{(1 - N a_i^{-1} b_j^{-1})} \tag{23}$$

Now we see that the first model considered, the Z model, is in fact symmetric in the two couplings corresponding to tachyons with positive and negative momenta.

Both the Z and F models are defined for complex matrices $Z, F \in \mathbb{C}(N)$. Since the analysis is the same for both models, we'll in the following just consider one, say the F model. A generic compex matrix can be written as a product of two (different) unitary matrices and a diagonal matrix [15]

$$F \to W\sqrt{D}\,V^\dagger \qquad V, W \in U(N) \tag{24}$$

Here D is diagonal with eigenvalues of $FF^\dagger$. Performing the change of variables (24) gives for the Gaussian density just $\mathrm{tr}(FF^\dagger) = \mathrm{tr}(D)$ since $FF^\dagger = WD\,W^\dagger$. It is clear that the measure transforms in the same way as in the above considered case for the W_∞ model, with the difference that now we have two unitary matrices instead of just one, so we'll get measures $[dW]$ and $[dV]$, each with half the dimension of that of $[dU]$.

3 Proof of duality between the W_∞ and F models

In the first section of this chapter we review some definitions of representation theory, to be used in the proof of the second section. In proving the duality between the Kontsevich-Penner and the Z models we use methods of character expansion and the Itzykson-Zuber integral [9].

3.1 Methods of group and representation theory

We want to be able to expand the correlators in terms of characters of the representation, and so we will need some tools from group and representation theory.

As seen in the previous sections, the matrix models considered are symmetric under the gauge group $U(N)$. Representations of $U(N)$ are parametrized by the partitions μ with the size $|\mu| = \sum_i \mu_i$ and length $l(\mu) \leq N$ such that

$$\mu_1 \geq \mu_2 \geq ... \geq \mu_{l(\mu)} > 0 = \mu_{l(\mu)+1} = \; ... \tag{25}$$

For the symmetric group S_n the irreducible representations and hence the conjugacy classes are in one-to-one correspondence with Young tableaux containing n boxes. For S_n equation (25) can also be interpreted as describing the length of the rows of the Young diagram μ, and now the row lengths μ_i correspond to the possible partitions of n (see Appendix B).

To prove the duality between the two matrix models we want to expand the exponent of the W_∞ model, and in this expansion one gets a product of traces of matrices to different powers. These product are often referred to as multi-trace operators and we write each product using a conjugacy class element of S_k where k is the sum of the powers of the matrix, say Z. Write the conjugacy class as a partition $[\mu_1, \mu_2, ..., \mu_p]$ of k, where each μ_i is a μ_i -cycle in S_k. Thus we need to define [4]

$$\mathrm{tr}(Z^{\mu_1})\mathrm{tr}(Z^{\mu_2}) \cdots \mathrm{tr}(Z^{\mu_p}) = Z^{e_1}_{e_{\alpha(1)}} Z^{e_2}_{e_{\alpha(2)}} \cdots Z^{e_k}_{e_{\alpha(k)}} \equiv \mathrm{tr}(\alpha\, Z^{\otimes k}) \tag{26}$$

where α is in the conjugacy class $[\mu_1, \mu_2, ..., \mu_p]$. For a partition μ of k (written $\mu \vdash k$) we write a representative of the corresponding conjugacy class $[\mu]$ as $\alpha_\mu \in [\mu] \subset S_k$. The size of a conjugacy class is $|\,[\mu]\,| = \frac{k!}{|\mathrm{Sym}(|\mu|)|} = \frac{k!}{\prod_{p=1}^k p^{i_p(\mu)} i_p(\mu)!}$. Here $\mathrm{Sym}(\alpha_\mu) = \{\rho \in S_k \mid \rho\alpha_\mu\rho^{-1} = \alpha_\mu\}$, and we wish to distinguish $|\,[\mu]\,|$ from the size of the partition $|\mu| = \sum_i \mu_i = k$. For the factors in the product of the expression of $\mathrm{Sym}(\alpha_\mu)$, $i_p(\mu)$ represents the length of a row with p boxes in a Young diagram. The multi-trace operator can also be written in terms of these as

$$\mathrm{tr}(\alpha_\mu\, Z^{\otimes k}) \equiv \prod_{p=1}^{k} [\,\mathrm{tr}(Z^p)\,]^{i_p(\mu)} \tag{27}$$

Further, it can be shown that [4] this operator can be written in terms of characters as

$$\mathrm{tr}(\alpha_\mu\, Z^{\otimes k}) = \sum_{\lambda \vdash k} \chi_\lambda(\alpha_\mu)\, \chi_\lambda(Z) \tag{28}$$

3.2 Proof of Duality

Now that we've developed the tools required, we take the W_∞ model (16)

$$\mathcal{Z}_{W_\infty}(\{t\},\{\bar{t}\}) = \int [dM] e^{-\mathrm{tr}(M)+\sum_{k=1}^{\infty}\bar{t}_k \mathrm{tr}\left[(MA^{-1})^k\right]} \tag{29}$$

And expand it in terms of characters with the methods described in the previous section. We begin by expanding the exponent in a series

$$\mathcal{Z}_{W_\infty}(\{t\},\{\bar{t}\}) = \int [dM] e^{-\mathrm{tr}(M)} \sum_{k=1}^{\infty} \prod_{p=1}^{k} \sum_{n} \frac{1}{n!} \left[\bar{t}_p \mathrm{tr}\left[(MA^{-1})^p\right]\right]^n \tag{30}$$

$$= \int [dM] e^{-\mathrm{tr}(M)} \sum_{k=1}^{\infty} \sum_{\mu\vdash k} \prod_{p=1}^{k} \frac{1}{p^{i_p(\mu)} i_p(\mu)!} \left[\mathrm{tr}(B^{-p})\, \mathrm{tr}\left[(MA^{-1})\right]^p\right]^{i_p(\mu)} \tag{31}$$

$$= \int [dM] e^{-\mathrm{tr}(M)} \sum_{k=1}^{\infty} \sum_{\mu\vdash k} \frac{|\,[\mu]\,|}{|\mu|!} \left[\mathrm{tr}\left(\alpha_\mu (B^{-1})^{\otimes|\mu|}\right) \mathrm{tr}\left(\alpha_\mu (MA^{-1})^{\otimes|\mu|}\right)\right]^{i_p(\mu)} \tag{32}$$

$$= \int [dM] e^{-\mathrm{tr}(M)} \sum_{k=1}^{\infty} \sum_{\mu\vdash k} \frac{|\,[\mu]\,|}{|\mu|!} \sum_{\lambda} \chi_\lambda(\alpha_\mu)\chi_\lambda(B^{-1}) \sum_{\lambda'} \chi_{\lambda'}(\alpha_\mu)\chi_{\lambda'}(MA^{-1}) \tag{33}$$

$$= \int [dM] e^{-\mathrm{tr}(M)} \sum_{l(\lambda)\le N} \chi_\lambda(B^{-1})\, \chi_\lambda(MA^{-1}) \tag{34}$$

In the second step we've used the Kontsevich-Miwa transform (14).The last step follows from the orthonormality of the characters.

In order for this integral to be well defined it is required that the eigenvalues of M be positive semi-definite. This requirements is automatically fulfilled by the F model [5]. Performing now the change of variables given by (2)

$$\mathcal{Z}_{W_\infty} = \int [\mathrm{d}U] \prod_{i=1}^{N} \mathrm{d}m_i\, \Delta^2(m_i)\, e^{-\mathrm{tr}(D)} \sum_{l(\lambda)\le N} \chi_\lambda(B^{-1})\, \chi_\lambda(UDU^\dagger A^{-1}) \tag{35}$$

The integral over U can be calculated using the formula of Itzykson and Zuber [9]

$$\int [dU]^{U}_{N\times N}\, \chi_\lambda(UXU^\dagger Y) = \frac{\chi_\lambda(X)\chi_\lambda(Y)}{\mathrm{dim}_N\lambda} \tag{36}$$

In our case this leads to

$$\mathcal{Z}_{W_\infty} = \int \prod_{i=1}^{N} \mathrm{d}m_i\, \Delta^2(m_i)\, e^{-\mathrm{tr}(D)} \sum_{l(\lambda)\le N} \frac{1}{\mathrm{dim}_N\lambda} \chi_\lambda(B^{-1})\, \chi_\lambda(A^{-1})\, \chi_\lambda(D) \tag{37}$$

It is further known that [4]

$$\int \prod_{i=1}^{N} dm_i\, \Delta^2(m_i)\, e^{-\mathrm{tr}(D)} \chi_\lambda(D) = \left[\mathrm{dim}_N\lambda\right]^2 \tag{38}$$

And so we finally get

$$\mathcal{Z}_{W_\infty} = \sum_{l(\lambda)\le N} \mathrm{dim}_N\lambda\, \chi_\lambda(B^{-1})\, \chi_\lambda(A^{-1}) \tag{39}$$

In order to get to the F model, where F is a generic $n \times n$ complex matrix, we now make a change of variables in (22) such that $F \to W\sqrt{D}\,V^\dagger$, where W, V are unitary and D is diagonal with eigenvalues of F as entries. The measure is calculated in exactly the same way as in (18),(19) and so

$$\mathcal{Z}_F(\{t\},\{\bar{t}\}) = \int [dF]^{\mathbb{C}}_{n\times n}\, e^{-\mathrm{tr}(FF^\dagger)+N\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}\left[(A^{-1}FB^{-1}F^\dagger)^k\right]} \tag{40}$$

$$= \int [dV]\,[dW] \prod_{i=1}^{N} dm_i\, \Delta^2(m_i) e^{-\mathrm{tr}(D)+N\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{tr}\left[(A^{-1}W\sqrt{D}\,V^\dagger B^{-1}V\sqrt{D}\,W^\dagger)^k\right]} \tag{41}$$

Performing a character expansion as in (31) - (34)

$$\mathcal{Z}_F = \int [dV]\,[dW] \prod_{i=1}^{n} dm_i\, \Delta^2(m_i) e^{-\mathrm{tr}(D)} \sum_{l(\lambda)\leq N} \chi_\lambda(\mathbb{I}_N)\, \chi_\lambda(A^{-1}W\sqrt{D}\,V^\dagger B^{-1}V\sqrt{D}\,W^\dagger) \tag{42}$$

Now both the integrals over $[dV]$ and $[dW]$ can be evaluated using Equation (36). Integrating first over $[dW]$ and writing $\chi_\lambda(\mathbb{I}_N) = \dim_N\lambda$

$$\mathcal{Z}_F = \int [dV] \prod_{i=1}^{n} dm_i\, \Delta^2(m_i) e^{-\mathrm{tr}(D)} \sum_{l(\lambda)\leq N} \frac{\dim_N\lambda\; \chi_\lambda(A^{-1})\; \chi_\lambda(\sqrt{D}\,V^\dagger B^{-1}V\sqrt{D})}{\dim_n\lambda} \tag{43}$$

Integration over $[dV]$ yields

$$\mathcal{Z}_F = \int \prod_{i=1}^{n} dm_i\, \Delta^2(m_i) e^{-\mathrm{tr}(D)} \sum_{l(\lambda)\leq N} \frac{\dim_N\lambda\; \chi_\lambda(A^{-1})\; \chi_\lambda(D)\; \chi_\lambda(B^{-1})}{\dim_n\lambda\, \dim_n\lambda} \tag{44}$$

This is in agreement with (37), if we set $n = N$. However, as a final step we can use (38) to get the result

$$\mathcal{Z}_F = \sum_{l(\lambda)\leq N} \dim_N\lambda\; \chi_\lambda(A^{-1})\; \chi_\lambda(B^{-1}) \tag{45}$$

Which is exactly the same as in (39).

4 Conclusions

The matrix models studied in this report have correlation functions that correspond to both tachyon scattering for the $c = 1$, $R = 1$ non-critical string and to a half-BPS sector in the (free) 4d $\mathcal{N} = 4$ super Yang-Mills theory [10]. There are three models considered here: the Z model, where tachyons appear as vertices [1], the F model [4] with closed string insertions associated with faces of the Feynman diagrams and the W_∞ model [8]. The duality between the Z and W_∞ models was shown by Mukherjee and Mukhi [16]; the duality between the Z and F models was shown by Brown [4]; in this report we have closed the triangle and shown the duality of the F and W_∞ models. We have used methods of character expansion and Itzykson-Zuber integral [9].

These results are exciting since this matrix model duality could provide a prototype for understanding AdS/CFT duality on a microscopical scale [3]. There is an idea that [4] it may be possible to rewrite $\mathcal{N} = 4$ super Yang-Mills as a dual theory, and that local operators and vertices from $\mathcal{N} = 4$ SYM could correspond to faces of the dual Feynman graphs.

5 Acknowledgements

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Appendices

A Calculating the correlation function of trivalent vertices

The correlator between two trivalent vertices, as considered graphically in Section 2.1, is

$$\left\langle \mathrm{tr}(M^3)\,\mathrm{tr}(M^3)\right\rangle = \left\langle \left(\mathrm{tr}(M^3)\right)^2 \right\rangle = \left\langle M^i_j M^j_k M^k_i M^l_m M^m_n M^n_l \right\rangle$$

where M is an $N \times N$ field. This can be split into propagators between first order vertices as in Equation (7)

$$\left\langle M^i_j M^k_l \right\rangle = \delta^i_l \delta^k_j$$

The complete correlator between the third order vertices is given by the sum over all possible ways of breaking it down to propagators between first order vertices.

Consider contracting the propagator like

$$M^i_j M^j_k M^k_i M^l_m M^m_n M^n_l$$

to get the decomposition

$$\left\langle M^i_j M^j_k M^k_i M^l_m M^m_n M^n_l \right\rangle = \left\langle M^i_j M^l_m \right\rangle \left\langle M^j_k M^m_n \right\rangle \left\langle M^k_i M^n_l \right\rangle = \delta^i_m\ \delta^l_j\ \delta^j_n\ \delta^m_k\ \delta^k_l\ \delta^n_i = \delta^n_m\ \delta^l_n\ \delta^m_l = N$$

Contracting in another way yields

$$M^i_j M^j_k M^k_i M^l_m M^m_n M^n_l \implies \left\langle M^i_j M^n_l \right\rangle \left\langle M^j_k M^k_i \right\rangle \left\langle M^l_m M^m_n \right\rangle = \delta^i_l \delta^n_j \delta^j_i \delta^k_k \delta^l_n \delta^m_m = N^2 \delta^i_l \delta^n_i \delta^l_n = N^3$$

Taking into account all the possible permutations of contractions we get the final answer

$$\left\langle \mathrm{tr}(M^3)\,\mathrm{tr}(M^3)\right\rangle = 12N^3 + 3N$$

B The symmetric group

The symmetric group S_n consists of the permutations of n objects (or their labels), and it is of order $n!$. A general permutation can always be written as a product of disjoint cycles, and so it is convenient to group the permutations of S_n by cycle structure, a cycle being e.g. (123). An arbitrary element has k_j j-cycles, where

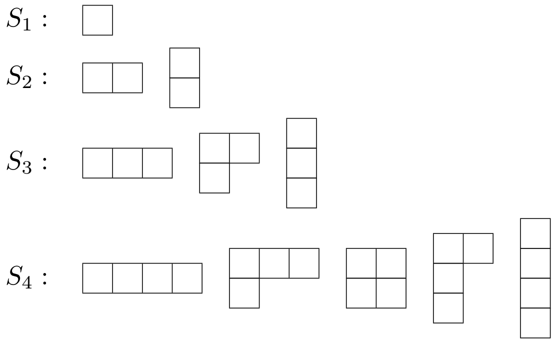
$$\sum_{j=1}^{n} j k_j = n$$

The conjugacy classes consist of all possible permutations with a particular cycle structure. The number of different permutations in the conjugacy class is [7]

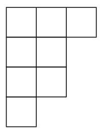
$$\frac{n!}{\prod_j j^{k_j} k_j!}$$

since each permutation of number 1 to n gives a permutation in the class, but cyclic order doesn't matter within a cycle and order doesn't matter at all between cycles of the same length.

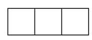
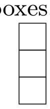
This group has a fundamental importance in the theory of finite groups, since, by virtue of Cayley's theorem, *every* finite group of order n is isomorphic to a subgroup of S_n [11]. The representations of S_n are labeled by Young diagrams with n boxes [3]:



Each column of boxes of length j represents a j-cycle. The first diagram in each of the above groups is the identity element, which is always a conjugacy class by itself. A tableau like



is a 4-cycle, a 3-cycle and a 1-cycle, which is an irreducible representation of S_8. Since each tableau represents a conjugacy class, the tableaux are in one-to-one correspondence with the irreducible representations [7].

[3]Each tableau can also be thought of as representing a particular process of symmetrization and antisymmetrization of a tensor to produce a tensor transforming according to some irreducible representation of $SU(N)$. A tensor with n indices corresponds to a Young tableau with n boxes. E.g. for a tensor with three indices the tableau  gives completely symmetrized states and a tableau  gives completely antisymmetrized states.

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