Thermodynamic limit of the AdS_4/CFT_3 Bethe ansatz equations.

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Abstract

The AdS_4/CFT_3 duality which was proposed recently implies a correspondence between a certain gauge theory and a string theory in $AdS_4 \times CP^3$. A set of Bethe ansatz equations describing the spectrum of that string theory for any value of the 't Hooft coupling was conjectured in [3]. In the present work we study the thermodynamic limit of those equations. We derive the Thermodynamic Bethe Ansatz (TBA) equations for this theory and prove that they can be written in a compact form known as the Y-system. After standard modifications, our TBA equations could be used to obtain energies of some excited states of the mirror theory in finite volume.

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Introduction

Recently, a conjecture was proposed that there exists a correspondence, similar to AdS_5/CFT_4, between a string theory and a gauge field theory. This duality relates a three dimensional superconformal Chern-Simons theory with a string theory on $AdS_4 \times S^7/Z_k$. A consequence of this duality is a correspondence between a planar gauge theory and a string theory in $AdS_4 \times CP^3$ (see e.g. [3] and references in that work). In particular, a correspondence was proposed between anomalous dimensions of certain operators in the gauge theory and string state energies. In addition, integrability properties have been discovered for those two theories, as well as for AdS_5/CFT_4. A set of Bethe equations, yielding the spectrum of anomalous dimensions/string states, was put forward in [3].

In this work, we study the thermodynamic limit of those equations. We show that the corresponding Thermodynamic Bethe Ansatz (TBA) equations can be written in the form known as the Y-system, as is the case for AdS_5/CFT_4 and several other integrable theories. Standard modification of the AdS_5/CFT_4 Y-system allowed to find exact energies of certain mirror theory states in finite volume [4]. Future work will include making similar modifications to the AdS_4/CFT_3 Y-system. However, solutions of this Y-system provide information only about the mirror theory, which is the theory obtained from the original one by a double Wick rotation. Our eventual goal is to obtain the TBA equations for the mirror AdS_4/CFT_3, as those equations give exact finite volume energies for the physical theory.

This paper is organized as follows. In Section 1 we review the algebraic Bethe ansatz for the Heisenberg spin chain. Section 2 contains a review of the Thermodynamic Bethe Ansatz method for a simple integrable field theory. Section 3 contains discussion of the Bethe equations for AdS_4/CFT_3. In Section 4 we present the main results of this work.

1 The Heisenberg spin chain.

In this section we show how Bethe ansatz can be used to solve the eigenvalue problem for a quantum-mechanical system. In other parts of this work we will discuss applications of this method in field theory. A review of the Bethe ansatz technique for the Heisenberg spin chain can be found in [1].

Consider a chain of L spins $\frac{1}{2}$, with periodic boundary conditions. The space of states of this quantum mechanical system is $\bigotimes_{i=1}^{L} h_i$, where $h_i \cong \mathbb{C}^2$. Consider the eigenvalue problem for the Heisenberg Hamiltonian

$$\hat{H} = \sum_{n=1}^{L} \frac{1}{2} \left(1 - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\right), \tag{1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. We will describe a way to solve this problem which uses the integrable structure of the Heisenberg Hamiltonian. This approach is known as the algebraic Bethe ansatz, and it provides all the eigenvectors and eigenstates of the Hamiltonian. The Bethe equations we will obtain can also be used to investigate the limit $L \to \infty$.

Denote by V_1 and V_2 two auxiliary $\mathbb{C}^2$ spaces, and by $\hat{P}_{i,j}$ the permutation operator. Introduce the $R - matrix$

$$R_{1,2}(u) = u\hat{1}_{V_1,V_2} + i\hat{P}_{V_1,V_2} \quad R_{1,2}(u) : V_1 \otimes V_2 \to V_1 \otimes V_2 \tag{2}$$

and the Lax operator

$$L_{n,1}(u) = \left(u - \frac{i}{2}\right)\hat{1}_{n,V_1} + i\hat{P}_{n,V_1}, \quad L_{n,1}(u) : h_n \otimes V_2 \to h_n \otimes V_2. \tag{3}$$

It can be shown that the Yang-Baxter equation

$$R_{1,2}(u-v)L_{n,1}(u)L_{n,2}(v) = L_{n,2}(v)L_{n,1}(u)R_{1,2}(u-v) \tag{4}$$

holds for these operators. Next, define the transfer matrix

$$T_1(u) = L_{L,1}(u)\dots L_{1,1}(u) \quad \in End\,(h_1 \otimes h_2 \otimes \dots \otimes h_L \otimes V_1) \tag{5}$$

and the monodromy matrix

$$T(u) = \mathrm{tr}_{V_1} T_1(u) \quad \in End\,(h_1 \otimes h_2 \otimes \dots \otimes h_L\,)\,. \tag{6}$$

It can be shown that the TTR relation

$$R_{1,2}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{1,2}(u-v) \tag{7}$$

holds, with the use of which the following identity can be proved:

$$[T(u), T(v)] = 0\,. \tag{8}$$

The Heisenberg Hamiltonian can be written in terms of the operators we have introduced:

$$\hat{H} = L - i\,\frac{dT}{du}T^{-1}\bigg|_{u=i/2}. \tag{9}$$

Therefore,

$$[\hat{H}, T(v)] = 0 \tag{10}$$

for any complex number v. This relation provides $L-2$ independent conserved quantities. Together with the Hamiltonian and a component of the total spin (e.g. $\hat{S}_3$) they form a set of L conserved quantities. This is a manifestation of the integrable structure behind the Heisenberg spin chain.

Using the fact that the Heisenberg Hamiltonian has the form (9), we can construct its eigenvectors in the following way. Any vector $|\Psi\rangle$ satisfying

$$T(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle, \quad \forall u \in \mathbb{C} \tag{11}$$

is also an eigenvector of our Hamiltonian, with the eigenvalue

$$E = L - i\,\frac{d}{du}\log\Lambda(u)\bigg|_{u=i/2}. \tag{12}$$

Such vectors $|\Psi\rangle$ can be constructed as follows. The operator $T_1(u)$ can be written as

$$T_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{13}$$

where $A, B, C, D \in End\,(h_1 \otimes h_2 \otimes \cdots \otimes h_L)$. It can be shown that the vector

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_J)|\uparrow\uparrow\uparrow\uparrow\dots\rangle \tag{14}$$

is an eigenvector of $T(u)$ if the complex numbers $u_1, u_2, \dots, u_J$ satisfy the Bethe ansatz equations (BAEs)

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k\neq j}\frac{u_j - u_k + i}{u_j - u_k - i}. \tag{15}$$

If equations (15) hold, then

$$T(u)|\Psi\rangle = \left[\prod_{j=1}^{J}\frac{u - u_k - i}{u - u_k}\left(u + \frac{i}{2}\right)^L + \prod_{j=1}^{J}\frac{u - u_k + i}{u - u_k}\left(u - \frac{i}{2}\right)^L\right]|\Psi\rangle. \tag{16}$$

Thus, each solution of the Bethe equations corresponds to an eigenvector of the Heisenberg Hamiltonian. It can be shown that all the eigenvectors and eigenvalues can be obtained in this way. Moreover, to solve the eigenvalue problem completely it is sufficient to consider only such solutions of BAEs in which all roots are distinct.

The Bethe equations make it possible to study the thermodynamic limit $L \to \infty$. It can be shown [1] that if the total number of roots is fixed, the roots form complexes called strings, in which the spacing between roots tends to i as $L \to \infty$. The elements of an n-root complex can be written as

$$u_{n,a} = u + ia, \quad a = -\frac{n-1}{2},\ -\frac{n-3}{2},\ \dots\ ,\frac{n-1}{2}, \tag{17}$$

where $u \in \mathbb{R}$ is the center of the complex and n is integer. It turns out that complexes appear also as solutions of Bethe equations for integrable field theories. Examples for AdS_4/CFT_3 are given in section 3.

2 Thermodynamic limit of Bethe equations for an integrable field theory.

The techniques we review in this section are described in Ref. [2].

Consider a QFT in 1+1 dimensions on a flat torus defined by two orthogonal geodesic circles C and B, with circumferences R and L. For the theory on C with B as its time domain, the partition function has the form

$$Z(R, L) = \mathrm{Tr}_{\mathcal{C}}\left(\mathrm{e}^{-LH_C}\right), \tag{18}$$

and in the limit $L \to \infty,\ L \gg R$ we have

$$Z(R, L) \sim \mathrm{e}^{-E(R)L}, \tag{19}$$

where $E(R)$ is the ground state energy of the theory. On the other hand, we can perform a double Wick rotation and find that the function Z from (18) is also the partition function for a so-called mirror theory, which is defined on B with C as its time domain:

$$Z(R, L) = \mathrm{Tr}_{\mathcal{B}}\left(\mathrm{e}^{-LH_B}\right). \tag{20}$$

If our QFT is Lorentz-invariant, the initial and the mirror theories coincide.

Denote by $f(R)$ the bulk free energy for the theory on B at temperature $1/R$. Then,

$$-LRf(R) = \ln Z(R, L). \tag{21}$$

Therefore, in the thermodynamic limit $L \to \infty$ we have

$$E(R) = Rf(R). \tag{22}$$

This relation can be used to find the ground state energy $E(R)$ for arbitrary R.

In the remainder of this section, we will describe a technique for calculating the free energy $f(R)$ for a simple integrable quantum field theory. Here, by integrability we mean the existence of an infinite number of conserved charges. We do not specify the theory in question, but assume that it is Lorentz-invariant and includes only one type of particles - neutral particles of mass m. We assume that the following properties of the scattering process in the theory follow from integrability:

- factorization of scattering
- conservation of the number of particles in each scattering event
- the sets of momenta for the initial and final states of a scattering process are identical (though individual momenta may get redistributed between the particles).

Consider the case $L \gg R_c$, where R_c is a characteristic radius of interactions between particles. In general, it is not appropriate to describe with a wavefunction a system of particles in quantum field theory. However, for $L \gg R_c$ there exist regions in the configuration space where the particles are strongly separated. We call those regions free regions. In such a region, we can introduce coordinates x_i and momenta p_i of the particles. The criterion of strong separation is $|x_i - x_j| \gg R_c$. Properties of the scattering process (see above) imply that in each free region the number of particles is the same. The particles in a free region can be described by a wavefunction, and have well-defined energies and momenta, which can be written in terms of rapidities β_i:

$$e(\beta_i) = m \cosh \beta_i, \quad p(\beta_i) = m \sinh \beta_i. \tag{23}$$

Denote the pair scattering amplitude by $S(\beta)$. For a system on a circle of circumference L, the matching conditions between free regions lead [2] to quantization equations for the particles' momenta. Those equations are the Bethe ansatz equations for our theory:

$$\mathrm{e}^{ip_iL} \prod_{j \neq i} S(\beta_i - \beta_j) = 1, \quad \text{for all } i. \tag{24}$$

Note that the BAEs for the Heisenberg spin chain provide the exact spectrum of the Hamiltonian. However, equations (24) are only exact in the infinite volume limit ($L \to \infty$).

For $S(\beta) = \mathrm{e}^{i\alpha(\beta)}$ we have, from (24):

$$mL \sinh \beta_i + \sum_{j \neq i} \alpha(\beta_i - \beta_j) = 2\pi n_i. \tag{25}$$

Let the number of particles N increase also as L tends to infinity, so that $N \sim L$. In this case, we can introduce a rapidity density of particles $\rho(\beta)$. Eq. (25) takes the form

$$mL \sinh \beta_i + \int \alpha(\beta_i - \beta')\rho(\beta')d\beta' = 2\pi n_i. \tag{26}$$

We can introduce the total density of rapidity levels in the following way. Consider a set of N rapidities $\{\beta_i\}$ (we call this set a quantum state) which satisfy, with the particle density corresponding to this set, Eq. (26). In case the l.h.s of (26) is a montonously increasing function of β_i, we can put a rapidity value β_j in correspondence to *each* integer number, and not only those integers which correspond to the solution $\{\beta_i\}$. In this way we define the rapidity levels β_j. The distance between consecutive levels is of order $1/(mL)$ and tends to zero as $L \to \infty$. Therefore, we can introduce the density of rapidity levels. We denote this density by $\rho(\beta) + \bar{\rho}(\beta)$. From (26) we find

$$2\pi(\rho(\beta) + \bar{\rho}(\beta)) = mL \cosh \beta + \int \frac{\partial \alpha(\beta - \beta')}{\partial \beta} \rho(\beta')d\beta'. \tag{27}$$

The energy of our system can be written as

$$H_B = \int d\beta m \cosh \beta \rho(\beta). \tag{28}$$

A large number of quantum states correspond to each consistent pair of densities $\rho(\beta),\ \bar{\rho}(\beta)$, because the densities are not sensitive to local redisributions of particles between rapidity levels. Consider the case when each rapidity level can be occupied by no more than one particle (see [2] for details). Then, it can be shown that for a macroscopic state described by ρ and $\bar{\rho}$, the entropy can be written as

$$\mathcal{S}\left[\rho,\ \bar{\rho}\right] = \int d\beta (\rho \ln(1 + \bar{\rho}/\rho) + \bar{\rho} \ln(1 + \rho/\bar{\rho})). \tag{29}$$

The minimal value of the expression $-RH_B[\rho] + \mathcal{S}\left[\rho,\ \bar{\rho}\right]$ with respect to the densities $\rho, \bar{\rho}$, which are constrained by (27), is equal to $-RLf(R)$. The extremum conditions for this functional, together with the Bethe equations, provide a set of equations called the Thermodynamic Bethe Ansatz (TBA) equations. Solutions of those equations provide the bulk free energy $f(R)$ and, from Eq. (22), the ground state energy $E(R)$.

For the AdS_4/CFT_3 theory, considered in the next section, the mirror ground state energy is known to be zero, because the vacuum is invariant with respect to supersymmetry. However, for AdS_5/CFT_4 the techniques presented in the current section can be modified [4] so as to describe certain excited states as well. A similar procedure could be applied in the AdS_4/CFT_3 case.

3 Bethe ansatz equations for AdS_4/CFT_3.

The Bethe ansatz technique for a field theory, presented in the previous section, allows to obtain the theory's spectrum in the infinite volume limit. A set of Bethe equations describing the spectrum of string states/anomalous dimensions in AdS_4/CFT_3 for any value of the 't Hooft coupling λ was conjectured in [3]. The object of the present work was to show that in the TD limit the corresponding TBA equations can be written in a compact form known as the Y-system. A similar result for AdS_5/CFT_4 was obtained in [4]. The initial Bethe equations are only valid in the infinite volume limit. However, in the AdS_5/CFT_4 case solutions of the modified Y-system for the mirror theory allowed to obtain *exact* finite volume energies of certain states of the initial theory. Similar modifications could be made to our Y-system, to obtain finite volume energies for the mirror AdS_4/CFT_3 theory.

The Bethe equations from [3] (see Eq. (1.7) in that work) have the following form:

$$
\begin{aligned}
1 &= \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/\left(x_{1,k}x_{4,j}^{+}\right)}{1-1/\left(x_{1,k}x_{4,j}^{-}\right)} \prod_{j=1}^{K_{\bar 4}} \frac{1-1/\left(x_{1,k}x_{\bar 4,j}^{+}\right)}{1-1/\left(x_{1,k}x_{\bar 4,j}^{-}\right)}\,, \\
1 &= \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}}\,, \\
1 &= \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x_{4,j}^{+}}{x_{3,k}-x_{4,j}^{-}} \prod_{j=1}^{K_{\bar 4}} \frac{x_{3,k}-x_{\bar 4,j}^{+}}{x_{3,k}-x_{\bar 4,j}^{-}} \\
\left(\frac{x_{4,k}^{+}}{x_{4,k}^{-}}\right)^{L} &= \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{K_1} \frac{1-1/\left(x_{4,k}^{-}x_{1,j}\right)}{1-1/\left(x_{4,k}^{+}x_{1,j}\right)} \prod_{j=1}^{K_3} \frac{x_{4,k}^{-}-x_{3,j}}{x_{4,k}^{+}-x_{3,j}} \times \\
&\times \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{4,k},u_{4,j}) \prod_{j=1}^{K_{\bar 4}} \sigma_{\mathrm{BES}}(u_{4,k},u_{\bar 4,j})\,, \\
\left(\frac{x_{\bar 4,k}^{+}}{x_{\bar 4,k}^{-}}\right)^{L} &= \prod_{j=1}^{K_{\bar 4}} \frac{u_{\bar 4,k}-u_{\bar 4,j}+i}{u_{\bar 4,k}-u_{\bar 4,j}-i} \prod_{j=1}^{K_1} \frac{1-1/\left(x_{\bar 4,k}^{-}x_{1,j}\right)}{1-1/\left(x_{\bar 4,k}^{+}x_{1,j}\right)} \prod_{j=1}^{K_3} \frac{x_{\bar 4,k}^{-}-x_{3,j}}{x_{\bar 4,k}^{+}-x_{3,j}} \times \\
&\times \prod_{j\neq k}^{K_{\bar 4}} \sigma_{\mathrm{BES}}(u_{\bar 4,k},u_{\bar 4,j}) \prod_{j=1}^{K_4} \sigma_{\mathrm{BES}}(u_{\bar 4,k},u_{4,j})\,,
\end{aligned}
\tag{30}
$$

and only solutions satisfying the zero momentum condition

$$
1 = \prod_{j=1}^{K_4} \frac{x_{4,j}^{+}}{x_{4,j}^{-}} \prod_{j=1}^{K_{\bar 4}} \frac{x_{\bar 4,j}^{+}}{x_{\bar 4,j}^{-}} \tag{31}
$$

should be considered. Here, $u_1, u_2, u_3, u_4, u_{\bar 4}$ are the Bethe roots, and L is a positive integer. The function $x(u)$ is defined by

$$
x + \frac{1}{x} = \frac{u}{h(\lambda)}, \quad |x(u)| \geq 1, \tag{32}
$$

where $h(\lambda)$ is a function of the 't Hooft coupling λ (see [3] for details), and the following general notation is used throughout this work:

$$
f^{\pm}(u) \equiv f(u \pm i/2), \quad f^{[+a]} \equiv f(u + ia/2). \tag{33}
$$

The above set of Bethe equations describes the string state energies for a string theory in $AdS_4 \times CP^3$, as well as anomalous dimesions in the dual gauge field theory. These quantities are given by

$$
E = h(\lambda)\mathcal{Q}_2\,, \tag{34}
$$

where

$$\mathcal{Q}_n = \sum_{j=1}^{K_4} \mathbf{q}_n(u_{4,j}) + \sum_{j=1}^{K_4} \mathbf{q}_n(u_{\bar{4},j}) \quad , \quad \mathbf{q}_n = \frac{i}{n-1}\left(\frac{1}{(x^+)^{n-1}} - \frac{1}{(x^-)^{n-1}}\right) . \tag{35}$$

The only roots which carry energy or momentum are u_4, $u_{\bar{4}}$. For a single root, those quantities are given by

$$p = \frac{1}{i}\log\frac{x^+}{x^-}, \quad \epsilon = \frac{1}{2} + h(\lambda)\left(\frac{i}{x^+} - \frac{i}{x^-}\right) . \tag{36}$$

Further discussion will be based on the following hypothesis: in the large L limit, each Bethe root from a set of roots which satisfies (30) is part of a complex of roots; the complexes and notation we use for them are given by the table below.

$u_4 = u + ij, \quad j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$:	string of roots :	$\bullet_n$
$u_{\bar{4}} = u + ij, \quad j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$:	string of roots :	$\odot_n$
$u_2 = u + ij, \quad j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}$:	string of roots :	$\circ_n$
$u_3 = u + ij, \quad j = -\frac{n-1}{2}, \dots, \frac{n-1}{2}$ $u_2 = u + ij, \quad j = -\frac{n-2}{2}, \dots, \frac{n-2}{2}$: $u_1 = u + ij, \quad j = -\frac{n-3}{2}, \dots, \frac{n-3}{2}$	trapezia :	$\triangle_n$
$u_1 = u$:	single fermion root :	$\oplus$
$u_3 = u$:	single fermion root :	$\otimes$

Here, u denotes the real center of a complex. We use indices A, B to label the complexes. Denote by ϵ_A (resp. p_A) the sum of energies (resp. momenta) of the roots in a complex. For a string of n u_4 roots we have

$$p_{\bullet_n} = \frac{1}{i}\log\frac{x^{[+n]}}{x^{[-n]}}, \quad \epsilon_{\bullet_n} = \frac{n}{2} + h(\lambda)\left(\frac{i}{x^{[+n]}} - \frac{i}{x^{[-n]}}\right) . \tag{37}$$

The same expression is correct for a $u_{\bar{4}}$ string. For other complexes, p_A and ϵ_A are zero.

Multiplying the Bethe equations for all roots in a complex, we obtain the equations for the density of complexes of that type. Those equations have the form

$$\bar{\rho}_A(u) + \rho_A(u) = -\frac{1}{2\pi}\frac{dp_A(u)}{du} - K_{BA}(v,u) * \rho_B(v) , \tag{38}$$

where $K(v,u) * f(v) \equiv \int dv K(v,u) f(v)$. Here, we use the following normalization:

$$\int du \rho_A(u) = \frac{\text{total number of complexes of type } A}{L}. \tag{39}$$

Since our theory includes fermions, we must work with the Witten index instead of the thermal partition function to compute energies of the mirror states in finite volume [4]. The Witten index is defined as

$$W = \sum_n (-1)^{f_n} \mathrm{e}^{-E_n/T}, \tag{40}$$

where f_n is the number of fermions in the state labelled by n. Introduce $h_A = i\pi = \log(-1)$ for fermionic complexes and $h_A = 0$ for other complexes. For the physical theory, the bulk free energy, defined as in (21) with Z in that equation replaced by the Witten index, is given by the minimal value of the following functional of the densities:

$$\mathcal{F} = \sum_A \int_{-\infty}^{\infty} du \left((R\epsilon_A \rho_A + h_A) - \left[\rho_A \log\left(1 + \frac{\bar{\rho}_A}{\rho_A}\right) + \bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right)\right]\right) \tag{41}$$

Here, R is the inverse temperature. Minimization of this functional with constraints (38) gives the following equations:

$$\log \mathcal{Y}_A(u) = K_{AB}(u,v) * \log[1 + 1/\mathcal{Y}_B(v)] + R\epsilon_A(u), \tag{42}$$

where $\mathcal{Y}_A = \frac{\bar{\rho}_A}{\rho_A}$. The minimal value of the functional (41) is

$$\mathcal{F} = \sum_A \int \frac{du}{2\pi} \frac{dp_A}{du} \log\left(1 + 1/\mathcal{Y}_A(u)\right), \tag{43}$$

and this quantity times R is the ground state energy of the mirror theory in volume R.

4 Derivation of the Y system for AdS_4/CFT_3.

In this section the main results of this work are presented.

Applying the fusion procedure to the Bethe equations (30), it can be found that the kernels K_{AB} are given by the entries of the following table (see the appendix for description of the notation used here):

$K_{AB} =$

$A \backslash B$	$\bigcirc_m$	$\oplus$	$\otimes$	$\triangle_m$	$\bullet_m$	$\odot_m$
$\bigcirc_n$	$+K_{n-1,m-1}$	$-K_{n-1}$	$+K_{n-1}$	0	0	0
$\oplus$	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{B}^{(01)}_{1m}$	$-\mathcal{B}^{(01)}_{1m}$
$\otimes$	$-K_{m-1}$	0	0	$+K_{m-1}$	$-\mathcal{R}^{(01)}_{1m}$	$-\mathcal{R}^{(01)}_{1m}$
$\triangle_n$	0	$-K_{n-1}$	$+K_{n-1}$	$+K_{n-1,m-1}$	$-\mathcal{R}^{(01)}_{n,m}$ $-\mathcal{B}^{(01)}_{n-2,m}$	$-\mathcal{R}^{(01)}_{n,m}$ $-\mathcal{B}^{(01)}_{n-2,m}$
$\bullet_n$	0	$+\mathcal{B}^{(10)}_{n1}$	$-\mathcal{R}^{(10)}_{n1}$	$-\mathcal{B}^{(10)}_{n,m-2} - \mathcal{R}^{(10)}_{n,m}$	$+\mathcal{S}_{nm}$ $+K_{nm}$	$+\mathcal{S}_{nm}$
$\odot_n$	0	$+\mathcal{B}^{(10)}_{n1}$	$-\mathcal{R}^{(10)}_{n1}$	$-\mathcal{B}^{(10)}_{n,m-2} - \mathcal{R}^{(10)}_{n,m}$	$+\mathcal{S}_{nm}$	$+\mathcal{S}_{nm}$ $+K_{nm}$

Introducing the functions Y_A

$$\{\mathcal{Y}_{\bigcirc_n}, \mathcal{Y}_{\oplus}, \mathcal{Y}_{\otimes}, \mathcal{Y}_{\triangle_n}, \mathcal{Y}_{\bullet_n}, \mathcal{Y}_{\odot_n}\} = \left\{Y_{\bigcirc_n}, Y_{\oplus}, \frac{1}{Y_{\otimes}}, \frac{1}{Y_{\triangle_n}}, \frac{1}{Y_{\bullet_n}}, \frac{1}{Y_{\odot_n}}\right\}, \tag{44}$$

we can write the equations (42) as:

$$\log Y_{\otimes} = +K_{m-1} * \log\frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} + \mathcal{R}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) + \mathcal{R}^{(01)}_{1m} * \log(1+Y_{\odot_m}) + i\pi \tag{45}$$

$$\log Y_{\oplus} = -K_{m-1} * \log\frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} - \mathcal{B}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) - \mathcal{B}^{(01)}_{1m} * \log(1+Y_{\odot_m}) - i\pi \tag{46}$$

$$\begin{aligned}\log Y_{\triangle_n} &= -K_{n-1,m-1} * \log(1+Y_{\triangle_m}) - K_{n-1} * \log\frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} \\ &+ \left(\mathcal{R}^{(01)}_{nm} + \mathcal{B}^{(01)}_{n-2,m}\right) * \log(1+Y_{\bullet_m}) + \left(\mathcal{R}^{(01)}_{nm} + \mathcal{B}^{(01)}_{n-2,m}\right) * \log(1+Y_{\odot_m})\end{aligned} \tag{47}$$

$$\log Y_{\bigcirc_n} = K_{n-1,m-1} * \log(1+1/Y_{\bigcirc_m}) + K_{n-1} * \log\frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} \tag{48}$$

$$\begin{aligned}\log Y_{\bullet_n} &= -R\epsilon_n + (-\mathcal{S}_{nm} - K_{nm}) * \log(1+Y_{\bullet_m}) - \mathcal{S}_{nm} * \log(1+Y_{\odot_m}) \\ &- \mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus}) + \mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes}) + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m})\end{aligned} \tag{49}$$

$$\begin{aligned}\log Y_{\odot_n} &= -R\epsilon_n + (-\mathcal{S}_{nm} - K_{nm}) * \log(1+Y_{\odot_m}) - \mathcal{S}_{nm} * \log(1+Y_{\bullet_m}) \\ &- \mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus}) + \mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes}) + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m}).\end{aligned} \tag{50}$$

Introduce the discrete Laplacian operator

$$\Delta K_n(u) \equiv K_n(u+i/2-i0) + K_n(u-i/2+i0) - K_{n+1}(u) - K_{n-1}(u).$$

Following [4], we apply this operator to the l.h.s. of the last set of equations, and obtain another set of equations for the Y functions, which is presented below.

u3 single roots:

$$\log \frac{Y^+_{\otimes} Y^-_{\otimes}}{Y_{\triangle_2} Y_{\bigcirc_2}} = \log \frac{(1+1/Y_{\bigcirc_2})(1+Y_{\bullet_1})(1+Y_{\odot_1})}{1+Y_{\triangle_2}} \tag{51}$$

u2 strings:

$$\log \frac{Y^+_{\bigcirc_n} Y^-_{\bigcirc_n}}{Y_{\bigcirc_{n+1}} Y_{\bigcirc_{n-1}}} = \log(1+1/Y_{\bigcirc_{n+1}})(1+1/Y_{\bigcirc_{n-1}}) \ , \ n>2 \tag{52}$$

$$\log \frac{Y^+_{\bigcirc_2} Y^-_{\bigcirc_2}}{Y_{\bigcirc_3}} = \log \frac{(1+Y_{\otimes_+})(1+1/Y_{\bigcirc_3})}{1+1/Y_{\oplus_+}}. \tag{53}$$

u4 strings:

$$\log \frac{Y^+_{\bullet_n} Y^-_{\bullet_n}}{Y_{\bullet_{n+1}} Y_{\bullet_{n-1}}} = \log \frac{1+Y_{\triangle_n}}{(1+Y_{\bullet_{n+1}})(1+Y_{\bullet_{n-1}})} \ , \ n>1 \tag{54}$$

$$\log \frac{Y^+_{\bullet_1} Y^-_{\bullet_1}}{Y_{\bullet_2}} = \log \frac{1+Y_{\otimes}}{1+Y_{\bullet_2}} . \tag{55}$$

u$\bar{4}$ strings:

$$\log \frac{Y^+_{\odot_n} Y^-_{\odot_n}}{Y_{\odot_{n+1}} Y_{\odot_{n-1}}} = \log \frac{1+Y_{\triangle_n}}{(1+Y_{\odot_{n+1}})(1+Y_{\odot_{n-1}})} \ , \ n>1 \tag{56}$$

$$\log \frac{Y^+_{\odot_1} Y^-_{\odot_1}}{Y_{\odot_2}} = \log \frac{1+Y_{\otimes}}{1+Y_{\odot_2}} . \tag{57}$$

Trapezias:

$$\log \frac{Y^+_{\triangle_n} Y^-_{\triangle_n}}{Y_{\triangle_{n+1}} Y_{\triangle_{n-1}}} = \log \frac{(1+Y_{\bullet_n})(1+Y_{\odot_n})}{(1+Y_{\triangle_{n+1}})(1+Y_{\triangle_{n-1}})} \ , \ n>2 \tag{58}$$

$$\begin{aligned} \log \frac{Y^+_{\triangle_2} Y^-_{\triangle_2}}{Y_{\triangle_3}} &= \log \frac{(1+Y_{\oplus})(1+Y_{\bullet_2})(1+Y_{\odot_2})Y_{\otimes}}{(1+Y_{\triangle_3})(1+Y_{\otimes})} - \log Y_{\otimes} Y_{\oplus} \\ &+ \sum_n (\mathcal{R}^{(01)}_{n1} - \mathcal{B}^{(01)}_{n1}) * \log(1+Y_{\bullet_n}) + \sum_n (\mathcal{R}^{(01)}_{n1} - \mathcal{B}^{(01)}_{n1}) * \log(1+Y_{\odot_n}). \end{aligned} \tag{59}$$

Adding up Eqs. (45), (46) we find that

$$\log Y_{\otimes} Y_{\oplus} = \sum_n (\mathcal{R}^{(01)}_{n1} - \mathcal{B}^{(01)}_{n1}) * \log(1+Y_{\bullet_n}) + \sum_n (\mathcal{R}^{(01)}_{n1} - \mathcal{B}^{(01)}_{n1}) * \log(1+Y_{\odot_n}). \tag{60}$$

Therefore, in Eq. (59) all the summands except the first one cancel, and Eq. (59) takes the form:

$$\log \frac{Y^+_{\triangle_2} Y^-_{\triangle_2}}{Y_{\triangle_3}} = \log \frac{(1+Y_{\oplus})(1+Y_{\bullet_2})(1+Y_{\odot_2})Y_{\otimes}}{(1+Y_{\triangle_3})(1+Y_{\otimes})}. \tag{61}$$

Thus, all the final equations for Y functions, except (60), are local in the sense that each includes no more than a fixed number of the Y functions. The final equations are indeed the Y-system equations conjectured in [5]. A graphical representation of this Y-system is given in Fig. 1. Each value of the index A corresponds to a node of this diagram. For each node A, the Y-system equation has the form

$$Y_A^+ Y_A^- = \frac{\prod_B (1+Y_B)}{\prod_C (1+1/Y_C)}, \tag{62}$$

where the index B (resp. C) labels the nodes connected to the A node by horizontal (resp. vertical) lines.

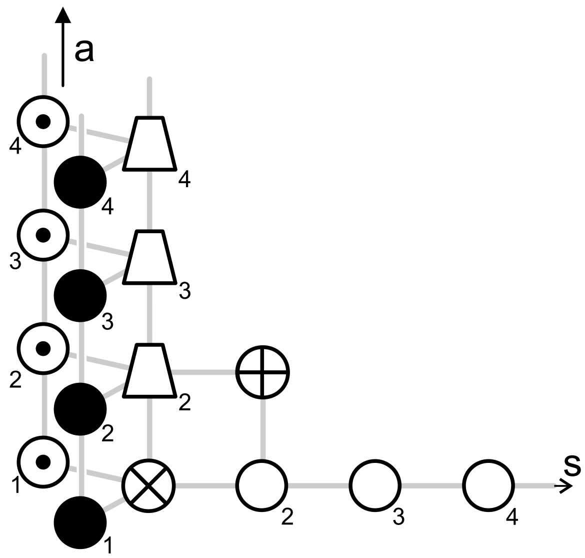

Figure 1: Graphical representation of the Y-system.

Appendix A: notation.

For expressions inolving the dressing kernel σ we use the following notation:

$$\theta(a_j, b_j, a_k, b_k) \equiv \chi(a_j, a_k) + \chi(b_j, b_k) - \chi(a_j, b_k) - \chi(b_j, a_k) - (k \leftrightarrow j), \tag{63}$$

$$\theta(x_j, x_k) \equiv \theta(x_j^+, x_j^-, x_k^+, x_k^-), \tag{64}$$

$$\sigma_{\mathrm{BES}}(u, v) \equiv e^{i\theta(x(u), x(v))}, \tag{65}$$

$$\sigma(u, v) \equiv \sigma_{\mathrm{BES}}(u, v), \tag{66}$$

$$\sigma(a_j, b_j, a_k, b_k) \equiv e^{i\theta(a_j, b_j, a_k, b_k)}, \tag{67}$$

where $\chi(a,b)$ in (63) is given by Eq. (1.12) in [3].

For entries of the table of kernels K_{AB}, the notation is as follows:

$$K_n(u,v) \equiv \frac{1}{2\pi i}\frac{\partial}{\partial v}\ln\frac{u-v+in/2}{u-v-in/2}, \tag{68}$$

$$K_{nm} \equiv \sum_{j=-\frac{m-1}{2}}^{\frac{m-1}{2}}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} K_{2j+2k+2}, \tag{69}$$

$$\mathcal{S}_{nm}(u,v) \equiv \frac{1}{2\pi i}\frac{\partial}{\partial v}\log\sigma(x^{[+n]}(u),x^{[-n]}(u),x^{[+n]}(v),x^{[-n]}(v)) \tag{70}$$

$$\mathcal{B}_{nm}^{(ab)}(u,v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i}\frac{\partial}{\partial v}\ln\frac{f(u+ia/2+ij,v-ib/2+ik)}{f(u-ia/2+ij,v+ib/2+ik)} \tag{71}$$

$$\mathcal{R}_{nm}^{(ab)}(u,v) \equiv \sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{1}{2\pi i}\frac{\partial}{\partial v}\ln\frac{g(u+ia/2+ij,v-ib/2+ik)}{g(u-ia/2+ij,v+ib/2+ik)}, \tag{72}$$

where

$$f(u,v) = 1-\frac{1}{x(u)x(v)}, \quad g(u,v) = x(u)-x(v). \tag{73}$$

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