From $\mathbb{Z}_7$-Orbifolds to the MSSM

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We provide a general characterization of $\mathbb{Z}_7$ heterotic orbifold models. We classify all admissible gauge embeddings in the absence of Wilson lines. We focus then on models containing an E_6 or $SO(10)$ gauge sector and show that there exists a supersymmetric model with standard model gauge group, three matter generations, gauge coupling unification at higher energies and vector-like exotics. This is the only $\mathbb{Z}_7$-model with these properties. We compare this model to previous findings.

Contents

I. INTRODUCTION

To date, the standard model of particle physics [1–3] succeeds in describing the vast majority of high energy physics experiments. Nevertheless, some observed phenomena as well as theoretical aspects give rise to the belief that it is part of a more fundamental theory.

One of these aspects is the so called **hierarchy problem**. Assuming that the standard model correctly describes physics up to an energy scale Λ, the Higgs boson receives radiative corrections to its mass of the form $\delta m_H \sim g^2\Lambda^2$, where g is the electroweak coupling constant. If the scale Λ is very large, it is hard to see why the Higgs mass should be of the order of the electroweak scale as required by unitarity constraints [4, 5]. Within the framework of the standard model, this hierarchy between the electroweak scale and the scale of new physics cannot be explained without an unnatural fine tuning of parameters at every order of perturbation theory. It was found in the eighties that **supersymmetry** provides a solution to the hierarchy problem [6–8]. In supersymmetric models, every particle has a supersymmetric partner with the same quantum numbers but opposite statistics. Divergent loop diagrams are thereby cancelled by loops containing the corresponding supersymmetric partner. In this way, even in a theory with broken supersymmetry, it is possible for the radiative corrections to the Higgs mass to be reasonably small and thus for the hierarchy, once fixed at tree-level, to be preserved in all orders.

A second aspect indicating the existence of physics beyond the standard model is the experimental and theoretical evidence for the **unification of gauge coupling constants**. Evaluation of the renormalization group equations for the standard model shows that gauge coupling constants approximately achieve a common value at a high energy scale. Regarding this aspect within a supersymmetric theory, this unification at a scale of $\Lambda_{\mathrm{GUT}} \approx 10^{16}$ GeV becomes much more accurate. This behavior is not believed to be a coincidence but an evidence for the existence of supersymmetry, as well as an indication towards new physics at a large energy scale Λ_{GUT}, described by a **grand unified theory** (GUT). Several models with different unifying gauge groups have been proposed since the pioneering work of Pati and Salam in the seventies [11], the most popular being the Georgi-Glashow model $SU(5)$ [10], $SO(10)$ and E_6 [12, 13]. Especially $SO(10)$ provides the standard model matter in a remarkably elegant way since one $\underline{16}$-plet contains a whole generation of the standard model and a right-handed neutrino. All of these models are accompanied by problems such as couplings allowing for proton decay or baryon number violating interactions.

Although a consistent GUT would be a great step towards a better understanding of the strong and electroweak force, it still would not give answer to a more fundamental problem, namely how to incorporate gravity into the quantum field theoretical framework of the gauge forces. There is a widespread belief that at the Planck scale Λ_{P}, unification of all four forces occurs. A quantum field theoretical approach towards gravity however seems to be impossible, since the theory is non-renormalizable. Up to date, the most promising candidate for a "theory of everything" is **string theory**. The fundamental objects of string theory are one-dimensional open or closed strings of length of the order $l_{\mathrm{P}} \approx 10^{-33}$ cm, whose vibrational excitations appear as particles. The particle spectrum naturally features a spin-2 particle which can be identified as a graviton and thus, string theory provides a framework for a quantum theory of gravity [14, 15]. It was found in the eighties that five consistent supersymmetric string theories exist, whose space-time dimensionalities are required to be $D = 10$. Our universe however seems to be four dimensional and as a consequence, if string theory is to be true, there should be a mechanism that conceals the six extra-dimensions and reproduces the standard model as a low energy approximation to the UV theory.

In this paper, we describe a way to obtain a four dimensional MSSM-like theory from the heterotic $E_8 \times E_8$ string [16, 17] by compactifying six spatial dimensions on a $\mathbb{Z}_7$-orbifold [18, 19]. We classify all admissible gauge embeddings without Wilson lines and investigate those containing a GUT gauge sector. We find a successful $SU(5)$ GUT model. However the mechanism of breaking $SU(5)$ down to standard model is difficult to achieve. Furthermore, we show that there is a unique model in the $\mathbb{Z}_7$-landscape providing standard model gauge group, three matter generations, gauge unification and only vector-like exotic particles.

II. REVIEW ON STRING THEORY

This chapter shortly reviews basic principles of the bosonic, supersymmetric and heterotic string. A more comprehensive introduction can be found e.g. in [21–23].

A. Bosonic String Theory

In a theory of point particles, the position of a particle is described by coordinates $X^\mu(\tau)$, $\mu = 0, ..., D-1$ depending on one timelike coordinate τ. In string theory, particles are replaced by strings $X^\mu(\tau,\sigma)$, $\mu = 0, ..., D-1$ which are functions of a timelike coordinate τ and a spacelike coordinate σ, normalized to $\tau \in \mathbb{R}$ and $\sigma \in [0,\pi]$. Thus, for fixed τ, a curve is traced out as σ varies. The space-time coordinates $X^\mu(\tau,\sigma)$, $\mu = 0, ..., D-1$ can formally be seen as fields of a two-dimensional field theory living in a two-dimensional manifold called **worldsheet**, which is parameterised by the worldsheet coordinates (τ,σ).

1. Bosonic String Action and Equations of Motion

The action of the bosonic string in D dimensions is given by

$$S = -\frac{T}{2}\int d\sigma^2 (-\det\, h)^{-\frac{1}{2}} h^{\alpha\beta}\eta^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu \tag{1}$$

where $h^{\alpha\beta}(\tau,\sigma)$, $\alpha,\beta = 0,1$ is the **worldsheet metric**, η is the usual space-time metric in D dimensions and T is the tension of the string. The action is invariant under the following symmetries:

Worldsheet reparametrization

$$\delta X^\mu = \xi^\alpha \partial_\alpha X^\mu; \quad \delta h^{\alpha\beta} = \xi^\gamma \partial_\gamma h^{\alpha\beta} + \partial_\gamma \xi^\alpha h^{\gamma\beta} + \partial_\gamma \xi^\beta h^{\gamma\alpha}\ , \tag{2}$$

Space-time Poincaré-transformations

$$\delta X^\mu = l^\mu_\nu X^\nu + a^\mu; \quad \delta h^{\alpha\beta} = 0; \quad \text{with} \quad l_{\mu\nu} = \eta_{\mu\rho} l^\rho_\nu\ , \tag{3}$$

Conformal invariance

$$\delta h^{\alpha\beta} = \Lambda(\tau,\sigma) h^{\alpha\beta}; \quad \delta X^\mu = 0\ . \tag{4}$$

The equations of motion obtained from this action by varying with respect to X^μ and $h^{\alpha\beta}$ are

$$\partial_\alpha (-\det\, h)^{\frac{1}{2}} h^{\alpha\beta} \partial_\beta X^\mu = 0\ , \tag{5}$$

$$T_{\alpha\beta} = -\partial_\alpha X^\mu \partial_\beta X_\mu + \frac{1}{2} h_{\alpha\beta} h_{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu = 0\ . \tag{6}$$

By exploiting reparameterization invariance and conformal invariance, the worldsheet metric $h^{\alpha\beta}$ can be simplified to the form

$$h_{\alpha\beta} = \eta_{\alpha\beta}\ . \qquad \textbf{(covariant gauge)} \tag{7}$$

For equations (5) and (6) to be valid, surface terms of the variation are required to vanish, which is the case if (covariant gauge)

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)\ , \qquad \textbf{(closed string)} \tag{8}$$

$$\frac{\partial X^\mu}{\partial \sigma} = 0\ . \qquad \textbf{(open string)} \tag{9}$$

In the following, only the closed string (needed for heterotic string theory) will be further investigated.

2. Mode expansion and Canonical Quantization

The general solution to the equation of motion (5) in covariant gauge is

$$X^\mu = x^\mu + l^2 p^\mu \tau + \frac{i}{2} l \sum_{n\neq 0} \left(\frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} + \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \right) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma) \ , \quad (10)$$

with

$$\begin{aligned} X_R^\mu(\tau-\sigma) &= \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau-\sigma) + \frac{i}{2} l \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau-\sigma)} \ , \\ X_L^\mu(\tau+\sigma) &= \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau+\sigma) + \frac{i}{2} l \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \ , \end{aligned} \quad (11)$$

referred to as **right-** and **left-movers**. Calculating the canonical momentum $P^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu}$ and imposing the canonical equal-τ commutation relations

$$\begin{aligned} [X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] &= [\dot{X}^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = 0 \ , \\ [P^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] &= i\delta(\sigma-\sigma')\eta^{\mu\nu}, \end{aligned} \quad (12)$$

yields the commutation relations for the centre-of-mass variables x^μ and p^μ and the oscillators α_n^μ and $\tilde{\alpha}_n^\mu$:

$$\begin{aligned} [x^\mu, p^\nu] &= -i\eta^{\mu\nu}, \\ [\alpha_m^\mu, \alpha_n^\nu] &= -m\delta_{m+n,0}\eta^{\mu\nu}, \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= -m\delta_{m+n,0}\eta^{\mu\nu}, \\ [\tilde{\alpha}_m^\mu, \alpha_n^\nu] &= 0 \ . \end{aligned} \quad (13)$$

3. Virasoro Algebra and Mass Operator

To implement the second equation of motion (6), which is equivalent to $T_{++} = T_{00} + T_{01} = 0$ and $T_{--} = T_{00} - T_{01} = 0$, in the quantized theory, **Virasoro operators** are introduced:

$$\begin{aligned} L_m &= \frac{T}{2} \int_0^\pi d\sigma\, e^{2im(\tau-\sigma)} T_{--} \\ &= \int_0^\pi d\sigma\, e^{2im(\tau-\sigma)} \left(\frac{\partial X_R^\mu}{\partial \tau} \right)^2 \\ &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^\mu \alpha_{\mu n} \ , \qquad (m \neq 0) \end{aligned} \quad (14)$$

$$\begin{aligned} \tilde{L}_m &= \frac{T}{2} \int_0^\pi d\sigma\, e^{2im(\tau+\sigma)} T_{++} \\ &= \int_0^\pi d\sigma\, e^{2im(\tau+\sigma)} \left(\frac{\partial X_L^\mu}{\partial \tau} \right)^2 \\ &= -\frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}^\mu \tilde{\alpha}_{\mu n} \ , \qquad (m \neq 0) \end{aligned} \quad (15)$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{l}{2} p^\mu$. Let us define L_0' and $\tilde{L}_0'$ as

$$L_0' = -\frac{1}{2} \alpha_0^\mu \alpha_{0\mu} - \sum_{n=1}^{\infty} \alpha_{-n}^\mu \alpha_{\mu n} - a \ , \quad (16)$$

$$\tilde{L}_0' = -\frac{1}{2} \tilde{\alpha}_0^\mu \tilde{\alpha}_{0\mu} - \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{\mu n} - a \ , \quad (17)$$

with a arising from the commutators of α_n^μ and $\tilde{\alpha}_m^\mu$. L_0 and $\tilde{L}_0$ are then defined to be the normal-ordered products

$$L_0 = -\frac{1}{2} : \sum_{n=-\infty}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu n} := L_0' + a \ , \tag{18}$$

$$\tilde{L}_0 = -\frac{1}{2} : \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_{\mu n} := \tilde{L}_0' + a \ . \tag{19}$$

The constraints (6) can now be imposed on physical states:

$$L_m|\phi\rangle = 0, \qquad (m > 0) \tag{20}$$
$$L_0|\phi\rangle = a|\phi\rangle \ . \tag{21}$$

From the commutators (13), the commutation relations for L_m and $\tilde{L}_m$ can be calculated, which furnish the so called **Virasoro algebra**:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0} \tag{22}$$
$$[\tilde{L}_m, \tilde{L}_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0} \tag{23}$$
$$[\tilde{L}_m, L_n] = 0 \ . \tag{24}$$

From the condition (20) and the mode expansion, the squared-mass operator M^2 can be derived as

$$\begin{aligned}\frac{1}{8}M^2 &= \frac{1}{8}p^\mu p_\mu \\ &= -\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_{\mu n} - \frac{1}{2}\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^{\mu}\tilde{\alpha}_{\mu n} - a \\ &= \frac{1}{8}(M_R^2 + M_L^2) \ .\end{aligned} \tag{25}$$

The masses of states have equal contributions from left- and right-movers, i.e. $M_R^2 = M_L^2$.

4. Lightcone Gauge

Even though some free parameters of the theory have been fixed by implementing the covariant gauge, there is still some residual gauge freedom left, that can be used to reduce the number of physical degrees of freedom. It can be shown, that the so called **lightcone coordinates** $X^+ = X^0 + X^{D-1}$ and $X^- = X^0 - X^{D-1}$ can be fixed by setting $X^+ = x^+ l^2 p^+ \tau$ and calculating X^- from the equations of motion. By imposing **lightcone gauge**, the physical degrees of freedom are reduced from D to $(D-2)$. The mass-squared operator in lightcone gauge has the form

$$\begin{aligned}\frac{1}{4}M_R^2 &= \sum_{n=1}^{\infty}\alpha_{-n}^{i}\alpha_{in} - a \ , \\ \frac{1}{4}M_L^2 &= \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^{i}\tilde{\alpha}_{in} - a \ , \\ M_R^2 &= M_L^2 \ ,\end{aligned} \tag{26}$$

where the sum over i goes from 1 to $D-2$. So far, we have not specified the number of space-time dimensions D and the normal-ordering constant a. It turns out that for a bosonic string theory to be free of quantum anomalies, the only choice is $D = 26$ and $a = 1$. With these values, it can be seen from equation (26), that the ground state $|0\rangle_L \otimes |0\rangle_R$ is tachyonic, i.e. has negative mass-square $M_L^2 + M_R^2 = -8$ which is an undesirable feature of the bosonic string and will be overcome in superstring theory and the heterotic string. The massless states of the bosonic string are $\alpha^i_{-1}|0\rangle_R \otimes \tilde{\alpha}^j_{-1}|0\rangle_L$, which, when decomposed into traceless symmetric part, antisymmetric part and trace can be interpreted as a **graviton** $g^{\mu\nu}$, an **antisymmetric tensor** $B^{\mu\nu}$ and a scalar Φ called **dilaton**. The tachyonic ground state and the fact that there are no fermionic states make bosonic string theory unsuitable as a realistic theory, both of these problems are not present in super string theory.

B. Superstring Theory

1. Action of the Superstring and Equations of Motion

In superstring theory, for each bosonic degree of freedom X^μ, a fermionic degree of freedom Ψ^μ in form of a two-component Majorana spinor is introduced to build up a **worldsheet supersymmetric** field theory coupled to two-dimensional supergravity. Beside worldsheet supersymmetry, the action of the superstring possesses **conformal invariance** as well as a **superconformal symmetry** involving the gravitino χ_α. The latter two can be used to impose the covariant gauge

$$h_{\alpha\beta} = \eta_{\alpha\beta}; \qquad \chi_\alpha = 0 \qquad \textbf{(covariant gauge)} \tag{27}$$

The equations of motion follow from varying the action with respect to X^μ, Ψ^μ, $h^{\alpha\beta}$ and χ^α:

$$\begin{aligned}
\partial_\alpha \partial^\alpha X^\mu &= 0 \ , &(28)\\
i\rho^\alpha \partial_\alpha \Psi^\mu &= 0 \ , &(29)\\
T_{\alpha\beta} &= -\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{4}\bar{\Psi}^\mu(\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha)\Psi_\mu \\
&\quad + \frac{\eta_{\alpha\beta}}{2}(\partial^\gamma X^\mu \partial_\gamma X_\mu + \frac{i}{2}\bar{\Psi}^\mu \rho^\gamma \partial_\gamma \Psi_\mu) = 0 \ , &(30)\\
J^\alpha &= \frac{1}{2}\rho^\beta \rho^\alpha \Psi^\mu \partial_\beta X_\mu = 0 \ . &(31)
\end{aligned}$$

For the surface terms to vanish, boundary conditions for the fermionic degrees of freedom can be periodic or antiperiodic

$$\begin{aligned}
\Psi^\mu_R(\tau, \sigma + \pi) &= \Psi^\mu_R(\tau, \sigma) \ , \qquad \textbf{(Ramond)} &(32)\\
\Psi^\mu_R(\tau, \sigma + \pi) &= -\Psi^\mu_R(\tau, \sigma) \ , \qquad \textbf{(Neveu-Schwarz)} &(33)
\end{aligned}$$

and can be chosen for right- and leftmovers independently.

2. Mode expansion and Canonical Quantization

The mode expansion and quantization of the bosonic degrees of freedom are as in the previous section. For the fermionic degrees of freedom, we find from the equations of motion:

$$\begin{aligned}
\Psi^\mu_R &= \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)}; \qquad \Psi^\mu_L = \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)} \ , \qquad \text{(R)} &(34)\\
\Psi^\mu_R &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} b^\mu_n e^{-2in(\tau-\sigma)}; \qquad \Psi^\mu_L = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_n e^{-2in(\tau+\sigma)} \ . \qquad \text{(NS)} &(35)
\end{aligned}$$

For quantization, we impose canonical anti-commutation relations

$$\begin{aligned}
\{\Psi^\mu_R(\tau,\sigma), \Psi^\nu_R(\tau,\sigma'\} &= \{\Psi^\mu_L(\tau,\sigma), \Psi^\nu_L(\tau,\sigma'\} = -2\pi\delta(\sigma-\sigma')\eta^{\mu\nu}, \\
\{\Psi^\mu_R(\tau,\sigma), \Psi^\nu_L(\tau,\sigma'\} &= 0 \ , &(36)
\end{aligned}$$

to find for the oscillators

$$\begin{aligned}
\{d^\mu_m, d^\nu_n\} &= \{\tilde{d}^\mu_m, \tilde{d}^\nu_n\} = -\delta_{m+n,0}\eta^{\mu\nu}, \qquad \text{(R)} \\
\{b^\mu_r, b^\nu_s\} &= \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = -\delta_{r+s,0}\eta^{\mu\nu}, \qquad \text{(NS)} &(37)
\end{aligned}$$

with left- and right-mover oscillators anticommuting.

3. Super-Virasoro Algebra

The implementation of the classical equations of motion (30) and (31) on physical states in the Hilbert space is achieved in terms of **Super-Virasoro Operators**. It is convenient to write equations (30) and (31) in terms of T_{++}, T_{--}, as well as

$$\begin{aligned} J_+ &= \Psi^\mu_L \partial_+ X_{\mu L} \ , & (38)\\ J_- &= \Psi^\mu_R \partial_- X_{\mu R} \ , & (39) \end{aligned}$$

with $\partial_\pm = \frac{1}{2}\partial_\tau \pm \partial_\sigma$. Beside L_m and $\tilde{L}_m$, which are defined as in bosonic string theory, we introduce G_r, $\tilde{G}_r$, $r \in \mathbb{Z} + \frac{1}{2}$ as the Fourier modes of J_- and J_+ for Neveu-Schwarz boundary conditions and F_m, $\tilde{F}_m$, $m \in \mathbb{Z}$ for Ramond boundary conditions. As we did explicitly in the previous section, the algebra of these operators can now be worked out and constraints on physical states be imposed (for further details, see [21]). Depending on whether we have (NS) or (R) boundary conditions, normal-ordering constants a_{NS} and a_R arise. It can be shown, that if commutators of Lorentz generators are to be free of quantum anomalies, the only choice is

$$\begin{aligned} a_{NS} &= \frac{1}{2} \ , \\ a_R &= 0 \ , \\ D &= 10 \ . \qquad (40) \end{aligned}$$

This choice also ensures, that there are no ghost states (states with negative norm) in the Hilbert space. So far, the degrees of freedom of the theory are not independent. To work only with physical degrees of freedom, we can fix the values of $X^+ = X^0 + X^{D-1}$ and $\Psi^+ = \Psi^0 + \Psi^{D-1}$

$$\begin{aligned} X^+(\tau,\sigma) &= x^+ + p^+\tau \ , \\ \Psi^+ &= 0 \ , \qquad (41) \end{aligned}$$

and express X^- and Ψ^- in terms of the transverse degrees of freedom X^i and Ψ^i, $i = 1, ..., 9$ (**light cone gauge**).

4. Superstring Ground State and Massless States

The mass-squared operator in light cone gauge has the form

$$\begin{aligned} M^2 &= M_R^2 + M_L^2 \ , \\ \frac{1}{4}M_R^2 &= \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n d^i_{-n} d^i_n \ , \\ \frac{1}{4}M_L^2 &= \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{n=1}^{\infty} n \tilde{d}^i_{-n} \tilde{d}^i_n \ , \qquad (42) \end{aligned}$$

for Ramond boundary conditions, while for Neveu-Schwarz boundary conditions, we find

$$\begin{aligned} \frac{1}{4}M_R^2 &= \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r - \frac{1}{2} \ , \\ \frac{1}{4}M_L^2 &= \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{r=\frac{1}{2}}^{\infty} r \tilde{b}^i_{-r} \tilde{b}^i_r - \frac{1}{2} \ . \qquad (43) \end{aligned}$$

The ground state is a product right- and left-mover states. The right-mover ground state of the Neveu-Schwarz sector has to fulfill

$$\alpha^\mu_m|0\rangle_R = b^\mu_r|0\rangle_R = 0 \qquad m, r > 0 \qquad (44)$$

and is therefore tachyonic with mass-square $M_R^2 = -2$ as can be seen from the mass squared operator. This problem will be addressed in the next section. For the right-mover ground state of the Ramond sector we require

$$\alpha_m^\mu |0\rangle_R = d_m^\mu |0\rangle_R = 0 \qquad\qquad m > 0 \; . \tag{45}$$

Note that in contrast to the NS sector, there is a zero-mode d_0^μ from the fermionic oscillators. Equation (43) implies that the application of d_0^μ to a state leaves the mass of the state unchanged. Defining $\gamma^\mu = i\sqrt{2}d_0^\mu$, we see from the anticommutation relations that these γ-matrices form a representation of the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \; . \tag{46}$$

The ground state of the Ramond sector therefore transforms as an irreducible spinor representation of $SO(1,9)$.

5. *GSO-Projection*

As we have seen in the previous section, the right- and leftmover ground state of the NS sector are tachyonic. This problem may be overcome by applying a projection to the states of the theory (Gliozzi, Scherk and Olive), the **GSO-projection**. For a detailed treatment, consult i.e. [21–23], here we will just mention, that it deletes the tachyonic ground-state and, depending on which mechanism is used in the Ramond sector, leads to so called **type-IIA** or **type-IIB theories**. Furthermore, the GSO-projection ensures **space-time supersymmetry** (not to be confused with worldsheet supersymmetry).

C. Heterotic String Theory

For a string theory to be suitable to describe nature, we should expect it to contain gauge fields for the known forces. In this aspect, heterotic string theory is an outstanding candidate, since, as we will see, gauge fields arise in a natural way. The **heterotic string** is a heterosis of left-movers of the 26-dimensional bosonic string with right-movers of the ten-dimensional superstring. Compactifying 16 of the bosonic dimensions on a torus yields a ten-dimensional space-time supersymmetric theory with gauge group $E_8 \times E_8$ or $SO(32)$. We will especially consider the former possibility, as E_8 contains naturally not only the standard model gauge group, but also theories such as E_6 and $SO(10)$, which might be an answer to the longstanding question of gauge unification. One E_8 factor might provide the gauge group for a grand unified theory (GUT) while the other E_8 factor might work as a hidden-sector (e.g. for SUSY-breaking).

1. *Quantization of the Heterotic String*

The mode expansions of the right-movers read

$$\begin{aligned}
X_R^\mu(\tau-\sigma) &= \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-2in(\tau-\sigma)} \; , \\
\Psi_R^\mu(\tau-\sigma) &= \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \; , \qquad\qquad \text{(R)} \\
\Psi_R^\mu(\tau-\sigma) &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2in(\tau-\sigma)} \; . \qquad\qquad \text{(NS)}
\end{aligned} \tag{47}$$

For the left-movers, we will label from now on the first ten degrees of freedom $\mu = 0, ..., 9$ and the 16 "internal" degrees of freedom $I = 10, ..., 25$.

$$\begin{aligned} X_R^\mu(\tau+\sigma) &= \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^\mu}{n}e^{-2in(\tau+\sigma)} \ , \qquad (\mu = 0, ..., 9) \\ X_R^I(\tau+\sigma) &= x_L^I + p_L^I(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^I}{n}e^{-2in(\tau+\sigma)} \ . \qquad (I = 10, ..., 25) \end{aligned} \tag{48}$$

For $\mu = 0, ..., 9$, the quantization is just as in the preceding section and the same commutation relations hold.

$$\begin{aligned} [x^\mu, p^\nu] &= -i\eta^{\mu\nu}, \\ [\alpha_m^\mu, \alpha_n^\nu] &= [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = -m\delta_{m+n,0}\eta^{\mu\nu}, \\ \{d_m^\mu, d_n^\nu\} &= -\delta_{m+n,0}\eta^{\mu\nu}, \\ \{b_r^\mu, b_s^\nu\} &= -\delta_{r+s,0}\eta^{\mu\nu}. \end{aligned} \tag{49}$$

To quantize the internal degrees of freedom, we have to account for the absence of corresponding right-movers. This can be worked out in detail (cf. [21]) and yields (note the factor of $\frac{1}{2}$)

$$[x_L^I, p_L^J] = -\frac{i}{2}\delta^{IJ} \ , \tag{50}$$

while the commutator for the oscillators is unchanged,

$$[\tilde{\alpha}_m^I, \tilde{\alpha}_n^J] = m\delta_{m+n,0}\delta^{IJ} \ . \tag{51}$$

2. Toroidal Compactification

To obtain a ten-dimensional theory with gauge fields, we will now compactify the 16 internal degrees of freedom on a torus. Let the torus be defined by some lattice $\Lambda = \{\sum_{j=1}^{16} R_j n_j e_j \ : \ n_j \in \mathbb{Z}\}$, which is spanned by arbitrary vectors $e_j \in \mathbb{R}^{16}$, $(j = 1, ..., 16)$, normalized to $|e_j| = \sqrt{2}$. For X^I, including right-movers (which will be eliminated later on), we identify equivalent points on the lattice:

$$x^I = x^I + \sqrt{2}\pi \sum_j R_j n_j e_j^I \ . \tag{52}$$

Making this identification, we see that there are now additional ways to satisfy the boundary condition, corresponding to winding the string round the torus:

$$X^I(\tau, \sigma+\pi) = X^I(\tau, \sigma) + 2\pi L^I \ , \tag{53}$$

with $L^I = \frac{1}{\sqrt{2}}\sum_{j=1}^{16} R_j n_j e_j^I$ referred to as the **winding number**. The mode expansions for left- and right-movers are

$$\begin{aligned} X_R^I(\tau-\sigma) &= x_R^I + p_R^I(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^I e^{-2in(\tau-\sigma)} \ , \\ X_L^I(\tau+\sigma) &= x_L^I + p_L^I(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^I e^{-2in(\tau+\sigma)} \ , \end{aligned} \tag{54}$$

with

$$\begin{aligned} p_R^I &= \frac{1}{2}(p^I - 2L^I) \ , \\ p_L^I &= \frac{1}{2}(p^I + 2L^I) \ , \\ x^I &= x_R^I + x_L^I \ . \end{aligned} \tag{55}$$

We eliminate the right-movers by demanding

$$\begin{aligned} p_R^I &= 0 \ , \\ x_R^I &= 0 \ , \end{aligned} \tag{56}$$

which implies

$$\begin{aligned} p_L^I &= 2L^I \ , \\ x_L^I &= x_L^I + 2\pi L^I \ . \end{aligned} \tag{57}$$

Furthermore, we require the translation operator $\exp(2i\sum_{I=1}^{16} p_L^I x_L^I)$ to be single-valued, when x^I is replaced by an equivalent point (the unusual factor of two in the exponent arises because of the commutator (50)). If Λ^* denotes the dual lattice defined by $\sum_{I=1}^{16} e_j^I e_i^{*I} = \delta_{ij}$, then p_L^I must be of the form

$$p_L^I = \frac{1}{\sqrt{2}} \sum_{j=1}^{16} \frac{m_j}{R_j} e_j^{*I} \ . \tag{58}$$

3. Massless States

The mass-squared operator in light cone gauge can be written as

$$\frac{1}{4} M_R^2 = N \ , \tag{59}$$

$$\frac{1}{4} M_L^2 = \frac{1}{2} \sum_{I=1}^{16} (p_L^I)^2 + \tilde{N} - 1 \ , \tag{60}$$

with

$$\begin{aligned} N &= \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_{in} + n d_{-n}^i d_{in}) \ , && \text{(R)} \\ N &= \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_{in} + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_{ir} \ , && \text{(NS)} \\ \tilde{N} &= \sum_{n=1}^{\infty} (\tilde{\alpha}_{-n}^i \tilde{\alpha}_{in} + \tilde{\alpha}_{-n}^I \tilde{\alpha}_{In}) \ . \end{aligned} \tag{61}$$

Noticing, that

$$\begin{aligned} N\alpha_{-m}^j |0\rangle_R &= m\alpha_{-m}^j |0\rangle_R \ , \\ N d_{-m}^j |0\rangle_R &= m d_{-m}^j |0\rangle_R \ , && \text{(R)} \\ N b_{-r}^j |0\rangle_R &= r b_{-r}^j |0\rangle_R \ , && \text{(NS)} \\ \tilde{N}\tilde{\alpha}_{-m}^j |0\rangle_L &= m\tilde{\alpha}_{-m}^j |0\rangle_L \ , \\ \tilde{N}\tilde{\alpha}_{-m}^J |0\rangle_L &= m\tilde{\alpha}_{-m}^J |0\rangle_L \ , \end{aligned} \tag{62}$$

where $j = 1, ..., 8$ and $J = 1, ..., 16$, we can now work out the massless states of the theory (note that these have to satisfy $M_R^2 = M_L^2 = 0$). First of all, we find 16 massless vector bosons of Kaluza-Klein type (for an introduction to Kaluza-Klein theory, see e.g. [24])

$$\tilde{V}_I^i = b_{-\frac{1}{2}}^i |0\rangle_R \otimes \tilde{\alpha}_{-1}^I |0\rangle_L \ , \qquad I = 1, ..., 16 \tag{63}$$

providing a gauge group $U(1)^{16}$. This gauge group can be enlarged for special choices of the lattice Λ and the radii R_j. We will denote a state with momentum p_L^I in the compactified dimensions by $|p^I\rangle$. According to equations (59) and (60), further massless states of the form

$$W^i(p_L^I) = b_{-\frac{1}{2}}^i |0\rangle_R \otimes |p_L^I\rangle \tag{64}$$

are present, provided that

$$\sum_I (p_L^I)^2 = 2 \ . \tag{65}$$

The choice of lattice is restricted by a fundamental constraint on string theories called **modular invariance**, ensuring the absence of gravitational and quantum anomalies and finiteness of string loop contributions to scattering amplitudes. As a consequence of modular invariance, the lengths of the radii are fixed to

$$R_j = \frac{1}{\sqrt{2}} \ , \tag{66}$$

the lattice has to be **self-dual**

$$\Lambda^* = \Lambda \ , \tag{67}$$

even

$$\sum_{I=1}^{16} e_j^I e_j^I \in 2\mathbb{Z} \text{ for all } j \ , \tag{68}$$

and **integer**

$$\sum_{I=1}^{16} e_j^I e_k^I \in \mathbb{Z} \text{ for all } j,k \ . \tag{69}$$

In 16 dimensions, there are only two lattices that satisfy these constraints, namely Γ_{16}, which is a sublattice of the root lattice of $SO(32)$ and $\Gamma_{E_8\times E_8}$, the direct product of two E_8 root lattices. We will investigate the latter possibility here. Taken together, the Kaluza-Klein fields (63) and the fields with internal momenta $(p_L^I)^2 = 2$ are the gauge fields of an $E_8 \times E_8$ gauge group. The remaining massless states

$$b^i_{-\frac{1}{2}}|0\rangle_R \otimes \tilde{\alpha}^j_{-1}|0\rangle_L \qquad i,j = 1,...,8 \qquad \text{(NS)} \tag{70}$$

$$|0\rangle_R \otimes \tilde{\alpha}^j_{-1}|0\rangle_L \qquad j = 1,...,8 \qquad \text{(R)} \tag{71}$$

generate the complete multiplet of ten-dimensional $\mathcal{N} = 1$ supergravity. Note that the theory is free of tachyons, since physical states have to obey $M_R^2 = M_L^2$ and the (NS) ground state has $M_R^2 = -2$, while the only bosonic state with negative mass squared has $M_L^2 = -4$.

III. HETEROTIC ORBIFOLDS

To obtain a four dimensional theory from the heterotic string, further six dimensions need to be compactified. A toroidal compactification would lead to a theory with $\mathcal{N} = 4$ SUSY, which is undesirable, since theories with $\mathcal{N} > 2$ SUSY are non-chiral. One possible way to achieve $\mathcal{N} = 1$ SUSY, is compactification on Calabi-Yau-manifolds, another, which we will discuss here, is orbifold compactifications.

A. Geometry

Let Γ be the lattice of a semisimple lie algebra of rank six with basis vectors $\{e_\alpha\}$ and $\mathcal{P}$ be a cyclic abelian group (the **point group**, i.e. $\mathbb{Z}_N$), acting crystallographically on the lattice. We define the **space group** $\mathcal{S}$, acting on $\mathbb{R}^6$, as the semisimple product of the point group with the group of lattice translations

$$S = P \ltimes \Gamma = \{g = (\Theta, n_\alpha e_\alpha) \mid \Theta \in P, n_\alpha \in \mathbb{N}\} \ , \tag{72}$$

implying that for $g_1 = (\Theta_1, n^\alpha e_\alpha),\ g_2 = (\Theta_2, m_\alpha e_\alpha) \in \mathcal{S}$, the product is

$$g_1 g_2 = (\Theta_1\Theta_2, n_\alpha e_\alpha + m_\alpha \Theta_1 e_\alpha) \ . \tag{73}$$

The action of $\mathcal{S}$ on the six spatial dimension is given by

$$X^i \longmapsto (\Theta X)^i + n_\alpha e_\alpha \qquad\qquad (i = 3, ..., 8) \ . \tag{74}$$

The spacegroup can be embedded into the 16 compactified degrees of freedom $S \hookrightarrow G$ in terms of a **shift vector** V and **Wilson lines** A_α according to

$$(\theta^k, n_\alpha e_\alpha) \longmapsto (kV, n_\alpha A_\alpha) \ , \tag{75}$$

acting on the 16 internal degrees of freedom as

$$X^I \longmapsto X^I + kV^I + n_\alpha A^I_\alpha \qquad\qquad (I = 1, ..., 16) \ . \tag{76}$$

The choices of V and A_α, which are restricted by several constraints to be discussed, will determine the breaking of the gauge group. The space

$$\mathcal{O} = \mathbb{R}^6/S \otimes T^{16}/G \tag{77}$$

with the above structure is called a **heterotic orbifold**.

B. Algebraic and Physical Constraints on the Structure of the Orbifold

Demanding a theory with $\mathcal{N} = 1$ SUSY as well as modular invariance imposes constraints on the different constituents of the orbifold. Furthermore, the embedding of the space-group into the gauge degrees of freedom is restricted by algebraic requirements.
Before discussing these, let us introduce some further notation. We will use a complex notation for the space-time coordinates

$$\begin{aligned}
Z^0 &= \frac{1}{\sqrt{2}}(X^1 + iX^2) \ , \\
Z^1 &= \frac{1}{\sqrt{2}}(X^3 + iX^4) \ , \\
Z^2 &= \frac{1}{\sqrt{2}}(X^5 + iX^6) \ , \\
Z^3 &= \frac{1}{\sqrt{2}}(X^7 + iX^8) \ ,
\end{aligned} \tag{78}$$

where we define $Z^{\bar a} = \bar Z^a$ $(a = 0, ..., 3)$. In the following, we will label the 16 gauge degrees of freedom by $I = 1, ..., 16$, the six compactified spatial dimensions by $a = 1, 2, 3,\ \bar a = \bar 1, \bar 2, \bar 3$ and the four uncompactified dimensions by $\mu = 0, 1, 2, 3$. We assume that the six dimensional torus $T^6 = \mathbb{R}/\Gamma$ is factorizable $T^6 = T^2 \times T^2 \times T^2$, so that the action of the point group P can be represented by a diagonal matrix

$$\theta = \mathrm{diag}\ (e^{2\pi i v_1}, e^{2\pi i v_2}, e^{2\pi i v_3}) \ , \tag{79}$$

acting on the complex coordinates according to

$$Z^a \longmapsto e^{2\pi i v_a} Z^a \qquad\qquad (a = 1, 2, 3). \tag{80}$$

The vector $v = (v_1, v_2, v_3)$ is referred to as the **twist vector**.
Demanding $\mathcal{N} = 1$ SUSY, the choice for the twist vector gets restricted by the condition [18, 19]

$$v_1 + v_2 + v_3 = 0 \ . \tag{81}$$

Since the embedding $S \hookrightarrow G$ is a group homomorphism, the cyclic nature of P restricts the form of V in the following way:

$$\theta^N = 1 \quad \Rightarrow \quad NV \in \Gamma_{E_8 \times E_8} . \tag{82}$$

A canonical form of V is the so called **standard embedding** $V = (v^1, v^2, v^3, 0^{13})$. Wilson lines are restricted by the action of the pointgroup on the lattice generators e_α. If for example $\theta e_\alpha = e_\beta$, then $A_\alpha \approx A_\beta$, which means that the two Wilson lines are equivalent up to lattice translations. In this way also the order of the Wilson lines N_α is fixed.
Finally, modular invariance imposes constraints on the orbifold parameters which are for $\mathbb{Z}_N$-orbifolds without Wilson lines expressed by

$$N(V^2 - v^2) = 0 \mod 2 \ . \tag{83}$$

Including Wilson lines, this has to be extended to [28]

$$\begin{aligned} N(V^2 - v^2) &= 0 \mod 2 \ , \\ N_\alpha(A_\alpha \cdot V) &= 0 \mod 2 \ , \\ N_\alpha(A_\alpha^2) &= 0 \mod 2 \ , \\ Q_{\alpha\beta}(A_\alpha \cdot A_\beta) &= 0 \mod 2 \ , \end{aligned} \tag{84}$$

where N_α denotes the order of the Wilson line A_α and $Q_{\alpha\beta}$ is the greatest common divisor of N_α and N_β [1].

C. Matter Spectrum of the Orbifold

Let us investigate the matter spectrum of $\mathbb{Z}_N$-orbifolds and how it is influenced by the structure of the orbifold. In heterotic string theory, matter is described by closed strings. Due to the orbifold compactification, there are now additional ways to satisfy the closed string boundary conditions. We will distinguish between the **untwisted sector**

$$Z(\tau, \sigma + \pi) = Z(\tau, \sigma) + n_\alpha e_\alpha \ , \tag{85}$$

and the **twisted sector**

$$Z(\tau, \sigma + \pi) = \Theta Z(\tau, \sigma) + n_\alpha e_\alpha = g Z(\tau, \sigma) \ , \tag{86}$$

where g is called **constructing element**. The latter boundary condition tells us in particular that the centre-of-mass coordinate z^a appearing in the mode expansion of Z^a has to coincide with a fixed point. It is therefore appropriate to say that strings in the twisted sector are twisted around fixed points. We will denote the untwisted sector by U and the twisted sectors, belonging to a constructing element $g = (\theta^k, n_\alpha e_\alpha)$ by T_k
Since massive string states have masses of the order of the Planck mass and therefore cannot be observed, we are only interested in the massless spectrum of the theory. In general, the massless states are given by

$$|q\rangle_R \otimes \tilde{\alpha}^x_{-1} |p\rangle_L \ , \tag{87}$$

for the untwisted sector, where q is a weight of $SO(8)$, $\tilde{\alpha}^x_{-1}$ is an oscillator acting in either direction $x = \mu$, $x = a/\bar{a}$ or $x = I$ and $p \in \Gamma_{E_8 \times E_8}$. The massless states in the twisted sector, belonging to a constructing element $g = (\theta^k, n_\alpha A_\alpha)$, are of the form

$$|q_{sh}\rangle_R \otimes \tilde{\alpha} |p_{sh}\rangle_L = |q + v_g\rangle_R \otimes \tilde{\alpha} |p + V_g\rangle_L \ , \tag{88}$$

with $\tilde{\alpha}$ denoting a product of oscillator excitations in the complex directions a / $\bar{a}$ and $v_g = kv$ and $V_g = kV + n_\alpha A_\alpha$, referred to as the **local twist** and the **local shift vector** respectively. For these states to be massless, they have to fulfill the mass equation to be discussed in a moment.

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[1] In principle, modular invariance merely demands $N_\alpha(A_\alpha \cdot V) = 0 \mod 1$ and $Q_{\alpha\beta}(A_\alpha \cdot A_\beta) = 0 \mod 1$. It was found in [28] that it is difficult to determine the spectrum of these models. This is not a problem, since a shift and a Wilson line satisfying the less stringent condition can always be transformed to a form that fulfills equation (84) by lattice translations or Weyl reflexions.

Since the orbifold group $O \subset S \otimes G$ is a symmetry of the theory, we request physical states to be invariant under its action. To be more precise, states belonging to a sector with constructing element $g = (\theta^k, n_\alpha A_\alpha)$ must be invariant under the action of every element of the centralizer $\mathcal{Z}_g = \{h \in S \mid [h,g] = 0\}$:

$$|q_{sh}\rangle_R \otimes \tilde{\alpha}|p_{sh}\rangle_L \stackrel{h}{\longmapsto} e^{2\pi i\Phi}|q_{sh}\rangle_R \otimes \tilde{\alpha}|p_{sh}\rangle_L \stackrel{!}{=} |q_{sh}\rangle_R \otimes \tilde{\alpha}|p_{sh}\rangle_L \ . \tag{89}$$

1. *Untwisted Sector*

Let us first investigate the matter spectrum of the untwisted sector. The mass equation reads

$$\frac{m_L^2}{4} = \frac{1}{2}p^2 + \tilde{N} - 1 = 0 = \frac{1}{2}q^2 - \frac{1}{2} = \frac{m_R^2}{4} \ . \tag{90}$$

As in the uncompactified theory, there are two ways to satisfy the equation for the left movers, namely $p^2 = 0$, $\tilde{N} = 1$ and $p^2 = 2$, $\tilde{N} = 0$. Nevertheless, we will find a different spectrum, since some states are projected out by the orbifold, i.e. the gauge group is broken. The phases obtained by the orbifold projections read

$$|q\rangle_R \otimes |p\rangle_L : \qquad e^{2\pi i\Phi} = e^{2\pi i[p\cdot V_h - q\cdot v_h]} \ , \tag{91}$$

$$|q\rangle_R \otimes \tilde{\alpha}_{-1}^x|0\rangle_L : \qquad e^{2\pi i\Phi} = e^{2\pi i[-q\cdot v_h + (\delta_{x,a} - \delta_{x,\bar{a}})\cdot v_h^a]} \ . \tag{92}$$

Only states that get no overall phase survive in the four dimensional theory. These can be categorized as follows:

The four dimensional graviton $g^{\mu\nu}$, antisymmetric tensor $B^{\mu\nu}$ and dilaton ϕ are given by the traceless symmetric part, the antisymmetric part and the trace of

$$|q\rangle_R \otimes \tilde{\alpha}_{-1}^\mu|0\rangle_L \ . \tag{93}$$

Since $\tilde{\alpha}_{-1}^\mu$ and $|0\rangle_L$ do not transform under the action of the orbifold, $|q\rangle_R$ has to transform trivially, which implies

$$q = \begin{cases} \pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \\ \pm(1, 0, 0, 0) \end{cases} . \tag{94}$$

Geometrical moduli are states of the type

$$|q\rangle_R \otimes \tilde{\alpha}_{-1}^{a/\bar{a}}|0\rangle_L \ , \tag{95}$$

satisfying $q \cdot v_h + (\delta_{x,a} - \delta_{x,\bar{a}})v_h^a = 0 \mod 1$.

The uncharged gauge bosons correspond to the 16 Cartan generators of the gauge group of the four dimensional theory $\mathcal{G}_{4D}$. From the 16 ten dimensional uncharged gauge bosons, only their four dimensional components associated with the uncompactified space are present in the four dimensional theory:

$$|q\rangle_R \otimes \tilde{\alpha}_{-1}^I|0\rangle_L \ , \tag{96}$$

with q given by equation (94). Note that q implicitly carries a minkowskian index since it is a weight of $SO(8)$. The fact that there are 16 Cartan generators in $\mathcal{G}_{4D}$ implies that the rank of the gauge group is unchanged.

The charged gauge bosons of the four dimensional gauge group $\mathcal{G}_{4D}$ are states of the form $|q\rangle_R \otimes |p\rangle_L$ satisfying

$$q \cdot v_h = 0 \mod 1 \tag{97}$$

$$p \cdot V_h = 0 \mod 1 \ . \tag{98}$$

They transform as the adjoint representation of the four dimensional gauge group. In case not all 480 left moving momenta satisfy the above equation, the gauge group will be broken.

The untwisted charged matter is composed of states $|q\rangle_R \otimes |p\rangle_L$ with non-trivially transforming left and right moving parts that meet the invariance condition

$$p \cdot V_h - q \cdot v_h = 0 \mod 1 \ . \tag{99}$$

Their gauge transformation properties depend on their momenta p.

2. Twisted Sector

Writing the mass equations for the twisted sector, we have to account for the presence of twisted oscillators $\tilde{\alpha}^a_{-\eta^a}$ and $\tilde{\alpha}^{\bar{a}}_{-\eta^{\bar{a}}}$, causing a shift of the ground state energy by

$$\delta c = \frac{1}{2}\sum_a \eta^a(1-\eta^a) \ . \tag{100}$$

The fractional oscillator frequencies η^a and $\eta^{\bar{a}}$ obey

$$\begin{aligned} \eta^a &= v_g^a \mod 1 \ , & (101)\\ \eta^{\bar{a}} &= -v_g^a \mod 1 \ , & (102) \end{aligned}$$

so that $0 < \eta^a \ , \eta^{\bar{a}} \leq 1$. The oscillator number $\tilde{N}$ is defined as

$$\tilde{N} = \sum_{a=1}^{3} \eta^a \tilde{N}_g^a + \eta^{\bar{a}} \tilde{N}_g^{\bar{a}} \ , \tag{103}$$

where $\tilde{N}_g^a$ and $\tilde{N}_g^{\bar{a}}$ are integer oscillator numbers counting the number of oscillator excitations in direction a and $\bar{a}$ respectively. Including the transformation phase of oscillators, the overall phase in the twisted sector is

$$e^{2\pi i \Phi} = e^{2\pi i [p_{sh}\cdot V_h - q_{sh}\cdot v_h + (\tilde{N}_g - \tilde{N}_g^*)\cdot v_h - \frac{1}{2}(V_g \cdot V_h - v_g \cdot v_h)]} \ . \tag{104}$$

IV. PHENOMENOLOGY OF THE $\mathbb{Z}_7$-ORBIFOLD

In recent years, $\mathbb{Z}_N$-orbifolds have undergone extensive study. It is well known that $\mathbb{Z}_3$- and $\mathbb{Z}_4$-orbifolds fail to yield realistic [2] models [27], while it is possible to obtain several MSSM-like models from $\mathbb{Z}_6$- as well as $\mathbb{Z}_{12}$-orbifolds [31], [32]. The next step is now to investigate $\mathbb{Z}_7$-orbifolds with the intention of finding a model with standard model gauge group and three generations of quarks and leptons.

A. Classification of $\mathbb{Z}_7$-Orbifold models

The underlying lattice of a $\mathbb{Z}_7$-orbifold is $\Gamma_{SU(7)}$, the root-lattice of $SU(7)$ on which the $\mathbb{Z}_7$-symmetry is realized by the twist $v = (\frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$. To get a valid embedding of the space group $\mathcal{S} = \mathbb{Z} \ltimes \Gamma_{SU(7)}$ into the internal degrees of freedom, the shift vector V has to obey the constraints discussed in chapter 3:

$$\begin{aligned} 7V &\in \Gamma_{E_8\times E_8} \ , \\ 7(V^2 - v^2) &= 0 \mod 2 \ , \\ 7(A\cdot V) &= 0 \mod 2 \ , \\ 7A^2 &= 0 \mod 2 \ . \end{aligned} \tag{105}$$

[2] By realistic model, we mean a model with standard model gauge group, three matter generations and gauge unification where the exotic particles can be decoupled from the low energy theory.

Writing these conditions, we have used the fact that there is only one independent Wilson line A in $\mathbb{Z}_7$-orbifolds and that it is of order seven (cf. Appendix, Wilson lines). The breaking of the $E_8 \times E_8$ gauge group of the heterotic string by orbifold compactification depends on the embedding of the space group into the gauge degrees of freedom and thus on the choice of the shift vector V and the Wilson line A. For $\mathbb{Z}_7$-orbifolds, we found 110 shifts consistent with the constraints of equation (105) of which 40 yield different gauge groups. These are listed in table I. The result is in accordance with [26].

Table of gauge groups			
E8	E8'	7V	7V'
$SU(8)$	$SU(5) \times SU(4)$	$\frac{1}{2}(13,1,1,1,1,1,1,1)$	$(6,1,1,1,1,0,0,0)$
$SU(8)$	$SU(2) \times E6$	$\frac{1}{2}(13,1,1,1,1,1,1,1)$	$(4,3,1,0,0,0,0,0)$
$SU(8)$	$SU(2) \times SU(6)$	$\frac{1}{2}(13,1,1,1,1,1,1,1)$	$\frac{1}{2}(9,3,3,1,1,1,1,1)$
$SU(8)$	$SO(12)$	$\frac{1}{2}(13,1,1,1,1,1,1,1)$	$(5,1,0,0,0,0,0,0)$
$SU(8)$	$SO(10) \times SU(3)$	$\frac{1}{2}(13,1,1,1,1,1,1,1)$	$(3,1,1,1,0,0,0,0)$
$SU(2) \times SU(7)$	$SU(2) \times SU(7)$	$(6,1,1,1,1,1,1,0)$	$(6,1,1,1,1,1,1,0)$
$SU(2) \times SU(7)$	$SO(8) \times SU(3)$	$(6,1,1,1,1,1,1,0)$	$(5,1,1,1,0,0,0,0)$
$SU(2) \times SU(7)$	$E6$	$(6,1,1,1,1,1,1,0)$	$(3,2,1,0,0,0,0,0)$
$SU(2) \times SU(7)$	$E8$	$(6,1,1,1,1,1,1,0)$	$(0,0,0,0,0,0,0,0)$
$SU(5) \times SU(4)$	$SU(2) \times SO(10)$	$(6,1,1,1,1,0,0,0)$	$(5,2,1,0,0,0,0,0)$
$SU(5) \times SU(4)$	$SU(2) \times SU(5) \times SU(3)$	$(6,1,1,1,1,0,0,0)$	$(5,1,1,1,1,1,0,0)$
$SU(5) \times SU(4)$	$SU(7)$	$(6,1,1,1,1,0,0,0)$	$\frac{1}{2}(7,3,1,1,1,1,1,1)$
$SU(5) \times SU(4)$	$SO(14)$	$(6,1,1,1,1,0,0,0)$	$(4,0,0,0,0,0,0,0)$
$SU(5) \times SU(4)$	$E7$	$(6,1,1,1,1,0,0,0)$	$(1,1,0,0,0,0,0,0)$
$SU(3) \times SO(10)$	$SU(2) \times SU(5) \times SU(3)$	$(6,1,1,0,0,0,0,0)$	$(5,2,1,1,1,0,0,0)$
$SU(3) \times SO(10)$	$SU(7)$	$(6,1,1,0,0,0,0,0)$	$\frac{1}{2}(11,1,1,1,1,1,1,-1)$
$SU(3) \times SO(10)$	$E7$	$(6,1,1,0,0,0,0,0)$	$(3,3,0,0,0,0,0,0)$
$SU(3) \times SO(10)$	$SU(2) \times SO(10)$	$(6,1,1,0,0,0,0,0)$	$(4,1,1,0,0,0,0,0)$
$SU(3) \times SO(10)$	$SO(14)$	$(6,1,1,0,0,0,0,0)$	$(2,0,0,0,0,0,0,0)$
$SU(2) \times SU(3) \times SU(5)$	$SU(2) \times SU(6)$	$\frac{1}{2}(11,3,3,1,1,1,1,-1)$	$\frac{1}{2}(11,3,1,1,1,1,1,1)$
$SU(2) \times SU(3) \times SU(5)$	$SO(12)$	$\frac{1}{2}(11,3,3,1,1,1,1,-1)$	$(4,2,0,0,0,0,0,0)$
$SU(2) \times SU(3) \times SU(5)$	$SU(2) \times E6$	$\frac{1}{2}(11,3,3,1,1,1,1,-1)$	$(2,1,1,0,0,0,0,0)$
$SU(2) \times SU(6)$	$SO(14)$	$\frac{1}{2}(11,3,1,1,1,1,1,1)$	$(6,0,0,0,0,0,0,0)$
$SU(2) \times SU(6)$	$SU(2) \times SO(10)$	$\frac{1}{2}(11,3,1,1,1,1,1,1)$	$(4,2,1,1,0,0,0,0)$
$SU(2) \times SU(6)$	$SU(7)$	$\frac{1}{2}(11,3,1,1,1,1,1,1)$	$\frac{1}{2}(9,1,1,1,1,1,1,1)$
$SU(2) \times SU(6)$	$E7$	$\frac{1}{2}(11,3,1,1,1,1,1,1)$	$(2,2,0,0,0,0,0,0)$
$SU(2) \times SO(10)$	$SU(2) \times E6$	$(5,2,1,0,0,0,0,0)$	$(4,3,1,0,0,0,0,0)$
$SU(2) \times SO(10)$	$SO(12)$	$(5,2,1,0,0,0,0,0)$	$(5,1,0,0,0,0,0,0)$
$SU(2) \times E6$	$SU(7)$	$(4,3,1,0,0,0,0,0)$	$\frac{1}{2}(7,3,1,1,1,1,1,1)$
$SU(2) \times E6$	$SO(14)$	$(4,3,1,0,0,0,0,0)$	$(4,0,0,0,0,0,0,0)$
$SU(2) \times E6$	$E7$	$(4,3,1,0,0,0,0,0)$	$(1,1,0,0,0,0,0,0)$
$SO(14)$	$SO(12)$	$(6,0,0,0,0,0,0,0)$	$(4,2,0,0,0,0,0,0)$
$SU(7)$	$SO(12)$	$\frac{1}{2}(11,1,1,1,1,1,1,-1)$	$(3,1,0,0,0,0,0,0)$
$SO(8) \times SU(3)$	$SO(8) \times SU(3)$	$(5,1,1,1,0,0,0,0)$	$(5,1,1,1,0,0,0,0)$
$SO(8) \times SU(3)$	$E6$	$(5,1,1,1,0,0,0,0)$	$(3,2,1,0,0,0,0,0)$
$SO(8) \times SU(3)$	$E8$	$(5,1,1,1,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$
$SO(12)$	$E7$	$(5,1,0,0,0,0,0,0)$	$(1,1,0,0,0,0,0,0)$
$E6$	$E6$	$(3,2,1,0,0,0,0,0)$	$(3,2,1,0,0,0,0,0)$
$E6$	$E8$	$(3,2,1,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$
$E8$	$E8$	$(0,0,0,0,0,0,0,0)$	$(0,0,0,0,0,0,0,0)$

TABLE I: Z_7-models without Wilson lines

In order to obtain the MSSM at low energies, the most interesting models without Wilson lines are those with a GUT gauge group which can then be broken to $SU(3) \times SU(2) \times U(1)$ by including Wilson lines. Table II shows the numbers of models among the above containing a certain GUT group.

Gauge Group	Number of Models
$SU(4) \times SU(2) \times SU(2)$	0
$SU(5)$	10
$SO(10)$	10
E_6	10
E_7	5
E_8	3

TABLE II: Z_7-models with GUT gauge sectors

According to the classification of $\mathbb{Z}_7$-orbifolds in [27], there exist approximately 630000 inequivalent [3] models. For the $\mathbb{Z}_6$-II-orbifold, it was found to be a successful strategy to search for models containing an E_6 or $SO(10)$ gauge sector. We adhere to this strategy here and scan the "landscape" of $\mathbb{Z}_7$ models for those containing E_6 or $SO(10)$ in absence of Wilson lines. In order for these models to be able to reproduce the spectrum of the MSSM by adding a Wilson line, they have to contain $\underline{27}$-plets or $\underline{16}$-plets respectively. Out of ten models with E_6- and ten models with $SO(10)$-factors in the gauge group, we found in each case three models containing matter states transforming under the respective family representations. These are listed in table III.

Gauge Group	7V
$SU(8) \times SU(10)' \times SU(3)'$	$\frac{1}{2}(13,1,1,1,1,1,1,1)\ (3,1,1,1,0,0,0,0)$
$SU(3) \times SO(10) \times E_7'$	$(6,1,1,0,0,0,0,0)\ (3,3,0,0,0,0,0,0)$
$SU(2) \times SO(10) \times SU(2)' \times E_6'$	$(5,2,1,0,0,0,0,0)\ (4,3,1,0,0,0,0,0)$
$SU(2) \times E_6 \times SO(14)'$	$(4,3,1,0,0,0,0,0)\ (4,0,0,0,0,0,0,0)$
$SU(2) \times E_6 \times E_7'$	$(4,3,1,0,0,0,0,0)\ (1,1,0,0,0,0,0,0)$
$E_6 \times E_8'$	$(3,2,1,0,0,0,0,0)\ (0,0,0,0,0,0,0,0)$

TABLE III: Z_7-models containing E_6 or $SO(10)$ factors with $\underline{27}$-plets or $\underline{16}$-plets

Note that table III contains the standard embedding $7V_{SE} = (3,2,1,0^5)(0^8) \approx (1,2,-3,0^5)(0^8)$. This particular shift represents the simplest non-trivial orbifold construction and, as we will see later, also yields the *only* possible MSSM-like $\mathbb{Z}_7$-orbifold model.

B. Standard Embedding without Wilson lines

We will explicitly determine the matter spectrum of the model belonging to the standard embedding $7V = (1,2,-3,0^5)(0^8)$ without Wilson lines. For later convenience, we will use a different but equivalent form of the shift vector $7V' = (2^2,1^6)(0^8)$ related to the standard form by a Weyl reflexion.

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[3] In this context, two models are considered to be equivalent if they contain the same representations with respect to the non-abelian gauge groups and the same number of gauge singlets. This criterion might underestimate the number of truly inequivalent numbers.

In the untwisted sector, we find aside from the graviton, the antisymmetric tensor and the dilaton the uncharged and charged gauge bosons as well as a variety of untwisted charged matter states. Let us first concentrate on the gauge bosons. As mentioned in the previous section, all of the 16 uncharged gauge bosons are present in the four dimensional theory. The charged gauge bosons are the states $|q\rangle_R \otimes |p\rangle_L$ that meet the invariance condition $p \cdot V = 0$ mod 1. We find 72 momenta in the first E_8-factor and 240 in the second one which implies that the second E_8-factor is unbroken. Including the eight Cartan generators, the $72 + 8$ momenta in the first E_8 factor furnish a $\underline{78} \oplus \underline{1} \oplus \underline{1}$ representation of $E_6 \times U(1) \times U(1)$. The gauge group is therefore broken to $E_6 \times U(1)^2 \times E_8'$.
Next, we consider the untwisted charged matter spectrum. Table IV shows the possible phases acquired by momenta p and q under the action of the constructing element g.

Phase	q's	No. of p's
$-\frac{1}{7}$	$(0,-1,0,0), (-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$	28
$-\frac{2}{7}$	$(0,0,-1,0), (-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	27
$-\frac{3}{7}$	$(0,0,0,1), (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$	27
$-\frac{4}{7}$	-	1
$-\frac{5}{7}$	-	1

TABLE IV: Untwisted charged matter

The chirality of a state is defined positive if the first entry of its 8_s-weight is positive. Since equivalent states of opposite chirality enter into one physical state, it suffices to count states of negative chirality. Tensoring together the states represented by the momenta in table IV and identifying the representations, we can summarize the untwisted charged matter spectrum as

$$3(\overline{\underline{27}}, \underline{1}) \oplus 3(\underline{1}, \underline{1}), \tag{106}$$

where we have omitted the $U(1)$ charges.

In the twisted sector, we will determine the matter spectrum of one sector explicitly and state the results for the other ones without calculation. Let us investigate the twisted matter spectrum of $g = (\theta^3, 0)$. The mass equation for $\mathbb{Z}_7$-orbifolds is given by

$$p_{sh}^2 = (p + V_g)^2 = 2(1 - \delta c - \tilde{N}) = \frac{10}{7} - 2\tilde{N} \tag{107}$$

$$q_{sh}^2 = (q + v_g)^2 = \frac{3}{7} \,. \tag{108}$$

For the above fixed point, equation (107) has solutions for $\tilde{N} = 0, \frac{1}{7}, \frac{2}{7}, \frac{4}{7}$. Once all the weights p and q satisfying equation (107) have been identified, invariance under centralizer elements has to be imposed, i.e. $\Phi = 0 \mod 1$ (cf. equation 104). It can be shown that in $\mathbb{Z}_7$-orbifolds, equation (104) is fulfilled by all massless states (see Appendix, Orbifold Projections in $\mathbb{Z}_7$) and thus, all of these states will appear in the matter spectrum. For the contemplated oscillator numbers $\tilde{N}$, the number of solutions to equation (107) are listed in table V.

The oscillator numbers can be assembled by different combinations of $\tilde{N}_g^a$, $\tilde{N}_g^{\bar{a}}$ (cf. equation (103)), to be precise, we find one possibility to get $\tilde{N} = \frac{1}{7}$, two possibilities for $\tilde{N} = \frac{2}{7}$ and

$\tilde{N}$	q_{sh}'s	No. of p's
0	$(-\frac{1}{2}, -\frac{1}{14}, \frac{5}{14}, \frac{3}{14}), (0, -\frac{4}{7} - \frac{1}{7} - \frac{3}{7})$	27
$\frac{1}{7}$	$(-\frac{1}{2}, -\frac{1}{14}, \frac{5}{14}, \frac{3}{14}), (0, -\frac{4}{7} - \frac{1}{7} - \frac{3}{7})$	1
$\frac{2}{7}$	$(-\frac{1}{2}, -\frac{1}{14}, \frac{5}{14}, \frac{3}{14}), (0, -\frac{4}{7} - \frac{1}{7} - \frac{3}{7})$	1
$\frac{4}{7}$	$(-\frac{1}{2}, -\frac{1}{14}, \frac{5}{14}, \frac{3}{14}), (0, -\frac{4}{7} - \frac{1}{7} - \frac{3}{7})$	1

TABLE V: Momenta q and p surviving the orbifold projection for different oscillator numbers $\tilde{N}$ in sector T_3 with $X^f = 0$

five possibilities for $\tilde{N} = \frac{4}{7}$. Therefore, the matter spectrum for sector T_3 at the fixed point $X^f = 0$ is

$$1(\overline{\underline{27}}, \underline{1}) \oplus 8(\underline{1}, \underline{1}). \tag{109}$$

Proceeding similarly for the other sectors, we can determine the whole matter spectrum of the model, which is summarized in table VI, including $U(1)$ charges. Note that in the absence of Wilson lines all fixed points are equivalent. As a consequence, we only have to investigate the matter content of the sectors T_k at the origin and multiply the numbers of irreducible representations by seven, since for every value k, there are seven fixed points (cf. Appendix, Fixed Points).

Sector	Matter content
U	$1(\underline{1}, \underline{1})_{-7,0}$
	$1(\overline{\underline{27}}, \underline{1})_{\frac{7}{2},\frac{7}{2}}$
	$1(\overline{\underline{27}}, \underline{1})_{-\frac{7}{2},\frac{7}{2}}$
	$1(\underline{1}, \underline{1})_{\frac{7}{2},-\frac{21}{2}}$
	$1(\overline{\underline{27}}, \underline{1})_{0,-7}$
	$1(\underline{1}, \underline{1})_{\frac{7}{2},\frac{21}{2}}$

Sector	Matter content
T_3	$7(\overline{\underline{27}}, \underline{1})_{-1,2}$
	$35(\underline{1}, \underline{1})_{\frac{5}{2},-\frac{3}{2}}$
	$7(\underline{1}, \underline{1})_{-1,9}$
	$14(\underline{1}, \underline{1})_{-\frac{9}{2},-\frac{3}{2}}$
T_5	$7(\overline{\underline{27}}, \underline{1})_{-\frac{1}{2},-\frac{5}{2}}$
	$35(\underline{1}, \underline{1})_{-\frac{1}{2},\frac{9}{2}}$
	$7(\underline{1}, \underline{1})_{-4,-6}$
	$14(\underline{1}, \underline{1})_{3,-6}$

Sector	Matter content
T_6	$7(\overline{\underline{27}}, \underline{1})_{\frac{3}{2},\frac{1}{2}}$
	$35(\underline{1}, \underline{1})_{-2,-3}$
	$7(\underline{1}, \underline{1})_{5,-3}$
	$14(\underline{1}, \underline{1})_{\frac{3}{2},\frac{15}{2}}$

TABLE VI: Matter content of all sectors. The quantum numbers are shown with respect to $E_6 \times E_8$, the two U(1) charges are given by subscripts.

The $U(1)$-directions were chosen to be $t_1 = (\frac{7}{2}, \frac{7}{2}, 0^{14})$ and $t_2 = (0^2, \frac{7}{2}^6, 0^8)$ with the corresponding charges given by $(p \cdot t_1, p \cdot t_2)$.

Geometrical moduli appear as states $|q\rangle_R \otimes \tilde{\alpha}^{a/\bar{a}}|0\rangle_L$ satisfying $q \cdot v_h + (\delta_{x,a} - \delta_{x,\bar{a}})v_h^a = 0 \mod 1$. The momenta q satisfying this constraint for different values of x are listed in table VII.

Counting states of negative helicity and recalling that due to supersymmetry a physical state must contain a Ramond weight along with an Neveu-Schwarz weight, we find one invariant state for each complex direction and thus three geometrical moduli for each fixed point (in accordance with the calculation in Appendix, Geometrical Moduli).

x	q's
1	$(0,1,0,0),(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
2	$(0,0,1,0),(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
3	$(0,0,0,1),(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$\bar{1}$	$(0,-1,0,0),(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2})$
$\bar{2}$	$(0,0,-1,0),(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2})$
$\bar{3}$	$(0,0,0,-1),(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$

TABLE VII: Momenta q of the geometrical moduli

C. Standard Embedding including Wilson lines: $SU(5)$ GUT

As we have mentioned in chapter 3, the gauge group can be further broken by applying Wilson lines. Let the Wilson line (recall that there is only one in $\mathbb{Z}_7$-orbifolds) be given by the vector

$$A = \frac{1}{7}(0^2, -1, -4, -3^3, 0, 3, 1^3, 0^4). \tag{110}$$

While Wilson lines do not affect geometrical moduli, the graviton, the antisymmetric tensor nor the dilaton, since these are characterized by the momenta q and the twist v only, it does influence the gauge bosons, as well as the matter content in the untwisted and twisted sectors.

Gauge Group. Weights corresponding to gauge bosons in a theory with Wilson lines satisfy

$$p \cdot A \;=\; 0 \quad \text{mod } 1 \;, \tag{111}$$

$$p \cdot V \;=\; 0 \quad \text{mod } 1 \;, \tag{112}$$

as can be seen in equation (98). Conditions (111) and (112) are fulfilled by 20 momenta belonging to the first E_8-factor, which together with the eight Cartan generators $28 = 24+4$ form the adjoint representation of $SU(5) \times U(1)^4$. In the second factor, we find 46 momenta that together with the Cartan generators $54 = 45+8+1$ characterise the gauge fields of an $SO(10) \times SU(3) \times U(1)$. The Wilson line therefore breaks the gauge group according to

$$\mathcal{G}_{4D} = E_6 \times U(1)^2 \times E_8' \quad \xrightarrow{A} \quad \mathcal{G}_{4D}^{\mathrm{WL}} = SU(5) \times U(1)^4 \times SO(10)' \times SU(3)' \times U(1)' \;.$$

Untwisted charged matter. Similar to the case without Wilson lines but with the additional condition (111) the representations of the untwisted matter states with respect to the new gauge group can be identified:

$$3(\underline{5},\underline{1},\underline{1}) \oplus 3(\underline{1},\underline{1},\underline{1}). \tag{113}$$

Twisted sectors. More changes are reflected in the twisted sector, since a Wilson line lifts the degeneracy of the different fixed points. To see this, let us determine the spectra of two different fixed points belonging to the sector T_3. Without Wilson lines, the matter contents for $g_1 = (\theta^3, 0)$ and $g_2 = (\theta^3, 3e_1)$ are identical (eq. (109)). Including a Wilson line, the local shift of the second fixed point is now $V_{g_2} = 3V + 3A$ and thus the surviving weights p differ from those of g_1. Explicitly, we find:

$$g_1: \qquad 1(\overline{\underline{10}},\underline{1},\underline{1}) \oplus 2(\underline{5},\underline{1},\underline{1}) \oplus 1(\overline{\underline{5}},\underline{1},\underline{1}) \oplus 10(\underline{1},\underline{1},\underline{1}), \tag{114}$$

$$g_2: \qquad 1(\underline{1},\underline{10},\underline{1}) \oplus 1(\underline{1},\underline{1},\underline{3}) \;. \tag{115}$$

Altogether, the matter content of all sectors (with $U(1)$-charges omitted) is summarized in table VIII.

Optimistically, we might interpret this as the matter content of a $SU(5)$ GUT, provided that we can find a way to decouple the $\underline{3}$- and $\overline{\underline{3}}$-plets of $SU(3)$, 8 pairs of $\underline{5}$- and $\overline{\underline{5}}$-plets of $SU(5)$ and three $\underline{10}$-plets of $SU(10)$ from the low-energy theory by giving them large masses. In this case, the model would be phenomenologically acceptable as long as an external mechanism breaking $SU(5)$ to standard model can be applied.

Sector	Matter content
U	$3(\underline{5},\underline{1},\underline{1})$
	$(\underline{1},\underline{1},\underline{1})$
T_3, T_5, T_6	$1(\overline{\underline{10}},\underline{1},\underline{1})$
	$3(\underline{5},\underline{1},\underline{1})$
	$3(\overline{\underline{5}},\underline{1},\underline{1})$
	$3(\underline{1},\underline{1},\underline{3})$
	$3(\underline{1},\underline{1},\overline{\underline{3}})$
	$1(\underline{1},\underline{10},\underline{1})$
	$23(\underline{1},\underline{1},\underline{1})$

TABLE VIII: Matter content of all sectors. The spectra for T_3, T_5 and T_6 are identical up to $U(1)$ charges and only one copy is listed. The quantum numbers are given with respect to $SU(5) \times SO(10) \times SO(3)$

D. MSSM Candidate

We have now the tools at hand to determine the matter spectrum of orbifold models with Wilson lines. So far, we have shown that it is possible to obtain a model with $SU(5)$ gauge group and three generations of matter. The next step is now to see, whether we can find a Wilson line for one of the models of table III that directly breaks the gauge group to $SU(3) \times SU(2) \times U(1)$. Out of the candidates in table III, it is indeed possible for one model to provide the gauge group of the standard model and three matter generations after applying a Wilson line, namely standard embedding with the following Wilson line:

$$A = \frac{1}{7}(-1, 5, -5, 0^2, 3^2, 5)(0^2, 2^4, 3, 7).$$

The gauge group turns out to be

$$\begin{aligned} \mathcal{G}_{\text{4D}} &= SU(3) \times SU(2) \times U(1)^5 \times SU(8)' \times U(1)' \\ &= SU(3) \times SU(2) \times U(1)_Y \times [SU(8) \times U(1)^5] \ , \end{aligned} \tag{116}$$

where we have separated what we will call the **observable sector** from the **hidden sector** in the second line. The generator of the hypercharge was set to

$$t_Y = \frac{1}{6}(2, -2, 4, 0^2, -1, -1, -2)(0^8). \tag{117}$$

The matter spectrum of this model is summarized in tables IX (standard model matter), X (gauge singlets) and XI (exotic particles).

Let us summarize a few properties of this model here. The gauge group of the ten dimensional heterotic string is broken according to

$$E_8 \times E_8 \quad \xrightarrow{V} \quad E_6 \times E_8 \quad \xrightarrow{A} \quad SU(3) \times SU(2) \times U(1)_Y \times [SU(8) \times U(1)^5] \ . \tag{118}$$

Generically, orbifold models contain one anomalous U(1) factor. A generalized Green-Schwarz mechanism can be applied in order to cancel the anomalies [29] resulting in a supersymmetry

No.	Representation	Label	No.	Anti-representation	Label
3	$(\underline{3},\underline{2};\underline{1})_{\frac{1}{6}}$	Q			
7	$(\underline{\overline{3}},\underline{1};\underline{1})_{\frac{1}{3}}$	$\overline{d}$	4	$(\underline{3},\underline{1};\underline{1})_{-\frac{1}{3}}$	d
3	$(\underline{\overline{3}},\underline{1};\underline{1})_{-\frac{2}{3}}$	$\overline{u}$			
7	$(\underline{1},\underline{2};\underline{1})_{-\frac{1}{2}}$	L	4	$(\underline{1},\underline{2};\underline{1})_{\frac{1}{2}}$	$\overline{L}$
3	$(\underline{1},\underline{1};\underline{1})_{1}$	$\overline{E}$			

TABLE IX: Matter content of the observable sector. Quantum numbers are shown with respect to $SU(3)\times SU(2)\times SU(8)'$ and hypercharge is given by the subscript.

No.	Representation	Label	No.	Anti-representation	Label
32	$(\underline{1},\underline{1};\underline{1})_{0}$	N			
6	$(\underline{1},\underline{1};\underline{1})_{\frac{1}{7}}$	s_1	6	$(\underline{1},\underline{1};\underline{1})_{-\frac{1}{7}}$	$\overline{s_1}$
2	$(\underline{1},\underline{1};\underline{8})_{\frac{2}{7}}$	s_2	2	$(\underline{1},\underline{1};\underline{\overline{8}})_{-\frac{2}{7}}$	$\overline{s_2}$
11	$(\underline{1},\underline{1};\underline{1})_{\frac{3}{7}}$	s_3	11	$(\underline{1},\underline{1};\underline{1})_{-\frac{3}{7}}$	$\overline{s_3}$
9	$(\underline{1},\underline{1};\underline{1})_{\frac{4}{7}}$	s_4	9	$(\underline{1},\underline{1};\underline{1})_{-\frac{4}{7}}$	$\overline{s_4}$
2	$(\underline{1},\underline{1},\underline{8})_{\frac{1}{7}}$	s_5	2	$(\underline{1},\underline{1};\underline{\overline{8}})_{-\frac{1}{7}}$	$\overline{s_5}$

TABLE X: Gauge Singlets with respect to non-abelian standard model quantum numbers. Quantum numbers are shown with respect to $SU(3)\times SU(2)\times SU(8)'$ and hypercharge is given by the subscript.

breaking **Fayet-Iliopoulos term** in the superpotential. The removal of this term necessary for preservation of supersymmetry in the four dimensional theory requires some gauge singlets N to take large VEVs. As a consequence, many of the gauge group factors might be broken and at best the low energy theory is equipped with a gauge group $\mathcal{G}_{\mathrm{SM}}\times\mathcal{G}_{\mathrm{hidden}}$, where the standard model matter is uncharged with respect to the hidden sector. In this way, also the vector-like exotic particles s_i and x_j of tables X and XI can be given large masses, provided that the relevant Yukawa couplings are allowed. The $U(1)_Y$ factor belonging to the hypercharge is defined as a linear combination of the six occurring $U(1)$-groups. In our model, the hypercharge is orthogonal to the anomalous U(1) which ensures that it is not broken by the Green-Schwarz mechanism. Furthermore, we checked that $U(1)_Y$ arises as a subgroup of $SU(5)$ and the proper normalization to $t_Y^2=\frac{5}{6}$ ensures unification of coupling constants at high energies.

No.	Representation	Label	No.	Anti-representation	Label
1	$(\underline{1},\underline{2};\underline{1})_{\frac{5}{14}}$	x_1	1	$(\underline{1},\underline{2};\underline{1})_{-\frac{5}{14}}$	$\overline{x_1}$
4	$(\underline{\overline{3}},\underline{1};\underline{1})_{\frac{5}{21}}$	x_2	4	$(\underline{3},\underline{1};\underline{1})_{-\frac{5}{21}}$	$\overline{x_2}$
5	$(\underline{1},\underline{2};\underline{1})_{\frac{1}{14}}$	x_3	5	$(\underline{1},\underline{2};\underline{1})_{-\frac{1}{14}}$	$\overline{x_3}$
3	$(\underline{3},\underline{1};\underline{1})_{-\frac{2}{21}}$	x_4	3	$(\underline{3},\underline{1};\underline{1})_{\frac{2}{21}}$	$\overline{x_4}$

TABLE XI: Exotic particles. Quantum numbers are shown with respect to $SU(3)\times SU(2)\times SU(8)'$ and hypercharge is given by the subscript.

A similar model with the same gauge group has been proposed in reference [30], p. 54, example 2. The gauge group however arises from a different breaking mechanism:

$$E_8 \times E_8 \quad \xrightarrow{V} \quad SU(7) \times SU(2) \times U(1) \times E_8 \quad \xrightarrow{A} \quad SU(3) \times SU(2) \times [SU(8) \times U(1)^6] \ . \quad (119)$$

While the intermediate theory without Wilson lines has gauge group $SU(7) \times SU(2) \times U(1)$, our model has the desirable feature to provide the more attractive GUT-group E_6. Furthermore, no hypercharge was defined for the model in [30]. Actually, we have shown that no combination of the six available $U(1)$ renders the correct hypercharge for quarks and leptons.
It is worth mentioning that the model we presented is unique among all possible $\mathbb{Z}_7$-models in the sense that it provides the MSSM matter content, vector-like exotics and a non-anomalous, properly normalized hypercharge allowing for gauge unification [27].

V. CONCLUSIONS

We have investigated the heterotic $E_8 \times E_8$ string compactified on a $\mathbb{Z}_7$-orbifold. Out of 40 different models without Wilson lines, we focused on those containing a GUT gauge sector with the intention of finding realistic models through non-trivial Wilson lines. In this respect, standard embedding yields rich phenomenology. We presented an $SU(5)$ GUT model in four dimensions. Even more interestingly, standard embedding also yields a realistic model with standard model gauge group, three matter generations, gauge coupling unification and vector-like exotic particles. Among approximately 630000 $\mathbb{Z}_7$-models, this is the only one with these features. Further phenomenological aspects such as Yukawa couplings, supersymmetry breaking scale and proton decay remain still to be investigated.

APPENDIX A: GEOMETRICAL MODULI OF $SU(7)/\mathbb{Z}_7$

Initially, a six-dimensional lattice has 21 parameters (six lengths and 15 angles), which can be reduced by requiring the metric

$$g_{ij} = \alpha_i \cdot \alpha_j = |\alpha_i|\,|\alpha_j| \cos\theta_{ij} \qquad \text{(A1)}$$

to be invariant under the point group $\mathcal{P}$:

$$\Theta g \Theta^T = g \ . \qquad \text{(A2)}$$

Here, Θ denotes the Coxeter element, i.e. the matrix realizing the point group on the orbifold and θ_{ij} is the angle between the vectors α_i and α_j. Parameters that are not constrained by equation (A2) are called **geometrical moduli**. The geometrical moduli of a $\mathbb{Z}_N$-orbifold can be obtained by identifying the free parameters of equation (A2).
In case of $SU(7)/\mathbb{Z}_7$, equation (A2) is equivalent to (with $r_i = |\alpha_i|$):

$$\begin{aligned} r_1 = r_2 = r_3 = r_4 = r_5 &= r_6 \\ \cos\theta_{12} = \cos\theta_{23} = \cos\theta_{34} = \cos\theta_{45} = \cos\theta_{56} &= -\frac{1}{2} - \cos\theta_{25} - \cos\theta_{35} \\ \cos\theta_{13} = \cos\theta_{24} = \cos\theta_{46} &= \cos\theta_{35} \\ \cos\theta_{14} = \cos\theta_{15} = \cos\theta_{26} = \cos\theta_{36} &= \cos\theta_{25} \end{aligned}$$

The geometrical moduli are therefore r_6, θ_{25} and θ_{35} in agreement with [25].

APPENDIX B: FIXED POINTS

Due to the equivalence of points in the orbifold that differ only by a lattice translation, nontrivial fixed points with respect to the action of the point group can appear. In general, these have to fulfill

$$Z^a = \theta^k Z^a + n_\alpha e^a_\alpha \ . \qquad \text{(B1)}$$

Solving for Z^a yields

$$Z^a = (1-\theta^k)^{-1} n_\alpha e_\alpha^a \ , \tag{B2}$$

where the vector $n_\alpha e_\alpha$ is referred to as the **reverse vector**, since it reverses the action of the point group and translates the rotated vector back to the initial one. Using formula (B2) to calculate the fixed points, we find seven independent fixed points, characterized by the reverse vectors

$$\begin{array}{l} 0 \\ e_1 \\ e_1+e_2 \\ e_1+e_2+e_3 \\ e_1+e_2+e_3+e_4 \\ e_1+e_2+e_3+e_4+e_5 \\ e_1+e_2+e_3+e_4+e_5+e_6 \ . \end{array}$$

All in all, we find seven fixed points for each sector $k = 0, ..., 6$.

APPENDIX C: WILSON LINES

We can see from the Coxeter Element of $\mathbb{Z}_7$ on the $SU(7)$-lattice

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \ . \tag{C1}$$

that the action of the point group on the lattice basis vectors e_α is

$$e_1 \mapsto e_2 \mapsto e_3 \mapsto e_4 \mapsto e_5 \mapsto e_6 \mapsto -e_1-e_2-e_3-e_4-e_5-e_6 \ , \tag{C2}$$

which yields the following equivalences for the Wilson lines:

$$A_1 \approx A_2 \approx A_3 \approx A_4 \approx A_5 \approx A_6, \text{ and } 7A_1 \approx 0 \ . \tag{C3}$$

We therefore have only one independent Wilson line of order 7 in the $\mathbb{Z}_7$-orbifold.

APPENDIX D: ORBIFOLD PROJECTIONS IN $\mathbb{Z}_7$

In the following calculation, we will explicitly show that in $\mathbb{Z}_7$-orbifolds, no massless states are projected out by the action of the orbifold group. For a state to be invariant under the action of an element h of the centralizer $\mathcal{Z}_g$, the phase

$$\Phi = \Phi' - \frac{1}{2}(V_g \cdot V_h - v_g \cdot v_h) = p_{sh} \cdot V_h - q_{sh} \cdot v_h + (\tilde{N}_g - \tilde{N}_g^*) \cdot v_h - \frac{1}{2}(V_g \cdot V_h - v_g \cdot v_h) \tag{D1}$$

must be integer, where $p_{sh} = p + V_g$ and $q_{sh} = q + v_g$ denote the shifted momenta. First of all we notice that for $h \in \mathcal{Z}_g$,

$$[g,h] = 0 \quad \Rightarrow \quad h = g^k \qquad\qquad \text{for } k \in \mathbb{N} \ , \tag{D2}$$

which implies $V_h = kV_g$ and $v_h = kv_g$ for the shift vectors and the twists. In addition, we will use the mass equations

$$\begin{aligned} (p+V_g)^2 \ &= \ p^2 + 2p\cdot V_g + V_g^2 = \frac{10}{7} - 2\tilde{N} \\ &\Rightarrow \ p \cdot V_g = \frac{1}{2}\left(\frac{10}{7} - 2\tilde{N} - p^2 - V_g^2\right) \ , \end{aligned} \tag{D3}$$

as well as

$$
\begin{aligned}
(q+v_g)^2 &= q^2+2q\cdot v_g+v_g^2=\frac{3}{7} \\
&\Rightarrow\ q\cdot v_g=\frac{1}{2}\left(\frac{3}{7}-q^2-v_g^2\right)\ .
\end{aligned}
\tag{D4}
$$

Using these relations, we can determine Φ'

$$
\begin{aligned}
\Phi' &= p_{sh}\cdot V_h-q_{sh}\cdot v_h+(\tilde{N}_g-\tilde{N}_g^*)\cdot v_h \\
&= p\cdot V_h-q\cdot v_h+V_g\cdot V_h-v_g\cdot v_h+(\tilde{N}_g-\tilde{N}_g^*)\cdot v_h \\
&\overset{h=g^k}{=} k(p\cdot V_g-q\cdot v_h)+k(V_g^2-v_g^2)+k(\tilde{N}_g-\tilde{N}_g^*)\cdot v_g \\
&\overset{(D3),(D4)}{=} \frac{k}{2}(1-2\tilde{N}-(p^2-q^2)-(V_g^2-v_g^2)+2(V_g^2-v_g^2))+k(\tilde{N}_g-\tilde{N}_g^*)\cdot v_g \\
&= \frac{k}{2}(1-(p^2-q^2)-(V_g^2-v_g^2)+2(V_g^2-v_g^2))\ ,
\end{aligned}
\tag{D5}
$$

where we have used the fact that $(\tilde{N}_g-\tilde{N}_g^*)\cdot v_h=\tilde{N}$. Since $p^2\in 2\mathbb{Z}$ and $q^2\in 2\mathbb{Z}+1$, we can leave away $\frac{1}{2}(1-(p^2-q^2))\in\mathbb{Z}$, since we are calculating modulo 1. We are then left with

$$
\Phi' = \frac{k}{2}(V_g^2-v_g^2) \quad \mathrm{mod}\ 1\ . \tag{D6}
$$

The overall phase $\Phi=\Phi'-\frac{1}{2}(V_g\cdot V_h-v_g\cdot v_h)=\Phi'-\frac{k}{2}(V_g^2-v_g^2)$ therefore vanishes modulo 1, which means that no states are projected out by the action of the orbifold group.

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