DESY Summer Programm 2007
Work Report
(Period: July 25 - September 18)

Conformal Inversion and the Size of Instantons

by
Bryan Zaldívar Montero
Superior Institute for Applied Science and Technology (Havana, Cuba).
(HELEN Fellow at DESY)
Supervisor: Dr. Fridger Schrempp, Theory group

Contents

The first part of the study performed at DESY deals mainly with the topic of Instantons, and has been divided into steps, which are next described, following a chronological order:

1 Getting a feeling of what instantons are

1.1 Classical point of view

Starting from the form of the Euclidean action of a gauge theory [1],

$$S_E = \int d^4x \frac{1}{2g^2} F^{\mu\nu} F^{\mu\nu} \tag{1}$$

We look for non-trivial field-configurations that minimize this action. The lower boundary of the action is expressed with the help of the dual of the strength-energy tensor, i.e. $\tilde{F}_a^{\mu\nu}$, and it is straightforward to show that the minimum condition is fulfilled for those field configurations that solve the so called "self -(anti)dual" equations

$$F^{\mu\nu} = \pm \tilde{F}^{\mu\nu} \tag{2}$$

The requirement for the action to be finite, serves as our argument to express the boundary conditions of this equation, which in terms of the gauge field reads: $A_\mu(x) \to U^{-1}(x)\partial_\mu U(x)$ for $x \to \infty$ [2], where $U(x)$ is a gauge transformation.

From this asymptotic behaviour results and expression for the minimum action that is directly linked with the topological Pontryagin index (winding number) q, through the relation:

$$S_E^{min} = \frac{8\pi^2}{g^2}|q| \tag{3}$$

The winding number is related with the number of times the d-dimensional space-time boundary S_{d-1} is wrapped around the group space of $SU(N)$, being the latter the group to which the gauge transformation U belongs. It is easy to check that in the case we are dealing with, this winding number results equal to unity.

The search for a field solution that holds in the entire space-time begins with the proposal $A_\mu(x) = f(x)U^{-1}(x)\partial_\mu U(x)$ [3], where $f(x)$ is some function that behaves correctly at the boundary. Working with the $SU(2)$ group for simplicity, it is not hard to express the transformations U as linear combinations of the basis of this group, i.e. the 2x2 matrices $\tau_\mu = (-i\sigma_a, 1)$, where σ_a are the Pauli matrices. To obtain the $f(x)$ functions, we use the proposed form for A_μ to express the $F^{\mu\nu}$. With this, and imposing self-duality results in the final form of $f(x)$. The field solution can be then expressed [3]:

$$A_\mu(x) = \frac{(x_\alpha \tau_\alpha)^\dagger}{x^2 + \rho^2}\tau_\mu \tag{4}$$

The figure 1 shows a Mathematica-plot of the solution, for different values of ρ.

Being the term x^2 a space-time scalar, one sees that this solution is localised in space as well as in time. That's why this solution was called "instantons", accounting for the fact that

[1]The notation we followed here is the same as in [1]

[2]Having a finite action means that the strength-energy tensor $F^{\mu\nu}$ must vanish at infinity. Due to the definition of $F^{\mu\nu}$ for a non-abelian theory, the gauge fields A_μ that obey this condition are the vanishing field, plus the set of all the others obtained from that one by a gauge transformation, getting what is known as a "pure gauge".

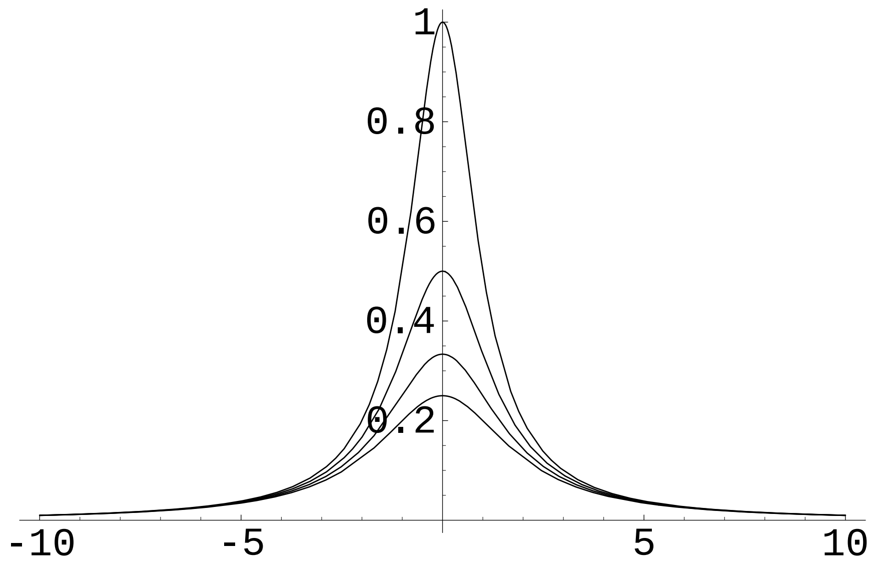

Figure 1: Plot of $A_\mu(x)$ vs. x for different values of ρ.

they live just an instant of time. The ρ parameter is called the *size* of the instanton, since it determines the "thickness" of this solution.

1.2 Quantum implications

1.2.1 Setting the stage

The instantons come into play in the quantum systems when studying the quantum tunneling process. Now one sees the reason of why the computation of the instanton solution was performed in the Euclidean world: It is easier to study a tunneling process occuring in, let's say, a double-well potential, if we switch from Minkowski metric to Euclidean metric, through a Wick rotation to imaginary times $t \to it$. The result of this change is that then we have an analogous system, but with the potential having opposite sign. So that the study of the transition via tunneling among two minima now becomes a simpler euclidean transition between two maxima without tunneling, and only now, the transition-amplitude path integral can be expanded around classical solutions of the equation of motion.

The study of the effect of instantons on transition amplitudes begins by analysing a system equipped with a periodic potential [4]. There, one sees that the infinitely degenerate vacuum energy is now a contiunous energy band, when tunneling is considered. The transition amplitude can be familiarly expanded in terms of intermediate states $|\phi_n>$, each with energy E_n. Then, looking at big times, the dominant part of this expansion comes from the lowest energies, and we can evaluate the transition-amplitude path integral using the known saddle-point approximation, leading to [2]:

$$G_E = \int \mathcal{D}_E[q(t')] e^{-\frac{1}{\hbar}S_E(q(t'))} = e^{-\frac{1}{\hbar}S_0} B_E(t) \left[Det \left(-\frac{d^2}{dt'^2} + V''(q_{cl}) \right) \right]^{-1/2} \tag{5}$$

where $B_E(t)$ is the measure of the gaussian integral that appears in the approximation, q_{cl} is the classical solution -the instanton in this case-, around which the expansion is made, and S_0 is the action evaluated on q_{cl}.

The fact that the action is invariant under the change of the instanton position t_0, causes the operator inside the determinant to have a zero eigenvalue (called "zero-mode"), and then, the whole expression appears to be divergent. However a trick can be made to get rid of this divergence. It consists in replacing the integration over the zero-mode coordinate (namely, t') by an integration over the collective-coordinate t_0, by means of an identity, first used by Gildener and Patrasciou 1977. The result is a determinant with the zero-mode been removed, which makes it finite. In general, this same procedure can be made with all the zero-modes that appears in the theory as a result of invariance of the action under any transformation (in addition to translation, Lorentz rotations, dilatation, special conformal transformation and gauge transformation can be considered).

The last thing to analyze from the 1-instanton contribution to the transition amplitude [3] is that we can divide the contribution of the determinant in two factors. The first one, let's called it K, accounting for the finite region in which the instanton solution lives, and the second one, accounting for the remaining infinite time region, where the system stands in the vacuum, and which can be succesfully recognized as the harmonic-oscillator path integral solution (together with the $B_E(t)$ factor), and then, it can be expressed in a rather simple way.

In order to correctly describe the energy band that appears as a result of tunneling, we need now to consider the multiinstanton contribution, in which intantons as well as antiinstantons are taken into account [4]. There are however two important restrictions for this multiinstanton configuration: 1) in order to obey the boundary condition (i.e. that the system at $t = -\infty$ to be in one vacuum and at $t = +\infty$ to be in the other), we need that the number of instantons (n_1) and the number of antiinstantons (n_2) obey the relation $n_1 - n_2 = 1$. 2) the (anti)instantons must be far apart from each other. The reason of this is that we must avoid overlapping of (anti)instantons, because otherwise, when evaluating the action on each overlapped solution, it would clearly not be minimized. A simpler way to say it is that if we alter the solutions that correctly minimize the action, then when evaluating the action on those, it would of course not get minimized. So, if condition 2) is obeyed, we have what has been called as the "dilute instanton-gas approximation". The contribution of n_1 instanton together with n_2 antiinstanton can be now computed easily from the 1-instanton result, because most of the terms are obtained by a $(n_1 + n_2)$-repetition of themselves, now including a correction factor that accounts for over-counting configurations where (anti)instantons merely exchange positions. The last step is to sum over the number of instantons and antiinstantons, taking into account all possible configurations that correctly give the total transition amplitude. The result, in the limit $t \to \infty$, is the following:

$$G_E = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \left(\frac{w}{\pi\hbar}\right)^{1/2} exp(e^{-S_0/\hbar} 2JKtcos\theta - \frac{1}{2}\omega t) \tag{6}$$

where J is the Jacobian of the change of variables (made to get rid of the zero-mode), and $\omega \equiv V''|_{(vac1,vac2)}$ [5]. The energy band levels are therefore [2]

[3] Up to now we have considered only the contribution of one instanton. This can be seen in another way: the transition from one vacuum to another, which we consider to happen in a big time $t \to \infty$, has been computed taking into account only one instanton living in a small region around t_0

[4] We need also antiinstantons because they account for transitions in the opposite direction

[5] The factor of the form $exp(e^{-S_0/\hbar})$ arises when considering the $(n_1 + n_2)$-repetition of the $e^{-S_0/\hbar}$ term under the double sum through n_1 and n_2, that eventually will has the form of a Taylor expansion of a common e^x function.

$$E_\theta = \frac{1}{2}\hbar\omega - \hbar(2JKe^{-S_0/\hbar}cos\theta) \tag{7}$$

which is the correct result, accounting for a correction to the free energy $E_0 = \frac{1}{2}\hbar\omega$ due to the quantum tunneling, as expected.

1.2.2 Analysing Yang-Mills theory

The obtained results allow us to study in a similar way the effect of instantons on the vacuum structure of the Yang-Mills theory, defined through the following Lagrangian:

$$\mathcal{L} = \frac{1}{2g^2}Tr(G_{\mu\nu}G_{\mu\nu}) \tag{8}$$

However some important constraint appears now because of the characteristics of this system: one of the Euler-Lagrange equations makes the system to be constrained to transform only on "small gauges" (i.e. $U(x) \to 1$ when $x \to \infty$). This implies that, as the field is not, in general, invariant under small gauges, then it can not be considered a physical one. In that sense, the correct physical field can be built as a linear combination of different non-physical ones. Each coefficient of the linear combination belongs to a different class (or "small" U), and it can be demonstrated [2] that this class classification can be related to an homotopy classification, in a way that each $U(x)$ correspond to a certain topological index N. The non-physical fields, that within the frame of the topological classification are called "topological vacua", tunnel each other. The physical field as the form

$$|\theta>= \sum_{N=-\infty}^{\infty} e^{iN\theta}|N> \tag{9}$$

which is the so called "θ-vacuum". The θ parameter comes into play because of the following: grouping all the space coordinates into one, we have, together with the time coordinate, a kind of 2-dimensional space-time, in whose perimeter leaves the boundary condition. These perimeter is topologically equivalent to a circle and can be parametrised by some angle θ. So, as the boundary conditions of the problem are θ-dependent, then the solution of that problem, i.e. the correct physical states that come out, must shown also this dependence. The form of the coefficients (i.e. the exponents) comes naturally from the theory, which is in agreement with the approach of the Bloch's theorem to the same problem. It is also worth mentioning that from this formulation comes the result that no topological states belonging to different θ-vacua can be related by any gauge-invariant operator, i.e. the transition between them is completely forbidden. So, taking this into account, the transition amplitude from a topological vacuum M to another N is expressed in the following way:

$$G_Q = \int(\mathcal{D}A_\mu)_Q e^{-S_E} \tag{10}$$

where $Q = N - M$. Here is explicit the fact that the transition depends of the difference between the topological vacua, being independent of the vacua themselves. So, in order to obtain the energy E_θ, a summation over all possible values of Q must be made, which as the form $\sum_Q e^{i\theta Q}G_Q$. Working with this expression, putting inside the G_Q-integral the exponential factor, we have a new way of defining the transition amplitude, which now is θ-dependent, and this dependence can be seen as been inside the definition of a new action S_θ, which therefore implies a redefinition of the lagrangian, that acquires a θ-dependent term of the form:

$$\Delta\mathcal{L}_\theta = \frac{\theta}{16\pi^2} Tr[G_{\mu\nu}G_{\mu\nu}] \tag{11}$$

1.3 Additional topics

In addition to all what have been discussed within this section, some other problems were studied. The reason of not discussing about them in the same way as before is because these are old-fashioned topics, that were insteresting in the time when the instantons studies were young, but nowadays the development of new approaches refutes them, having other new implications. Shortly, these topics are: 1) The infrared problem of large instantons. It was analysed that in the Yang-Mills theory, the instantons can have any size (a Yang-Mills-like theory was used to obtain the form of the exact instanton solution, see § 1.1, and notice that in principle ρ can be any finite quantity). Consequently, as discussed before, an integration over this degree of freedom must be taken into account, and what results is that this integral diverges. 2) Instantons and confinement. A description of the θ-vacuum-expectation-value of a quatity that accounts for the potential between two quarks, was made, for a 2-dimensional space-time, using the dilute instanton-gas approximation. The result was a linear dependence between this potential and the distance between the two quarks, which is a sign of confinement. However the analysis doesn't hold in a 4-dimensional space-time, the natural world of QCD, essentially because in this case the topology is different, but also the dilute instanton-gas approximation may be causing problems. 3) Suppresion of vacuum tunneling by massless fermions. Starting from the result that in a massless matter field theory the chirality is conserved, the effect of having an axial-vector-current anomaly is analyzed. When computing the transition amplitude for tunneling, the grassmanian integration of the matter field together with the presence of this anomaly leads to a null expression of the transition amplitude. This was a happy results since the computation didn't involve any dilute instanton approximation, but today it is know that the process doesn't work like that. 4) P,T violations in the Yang-Mills theory. The analysis of P and T symmetries in the $\Delta\mathcal{L}_\theta$ term was made, giving as a result that for this theory these symetries are violated, which we know from experiment that is not true. So in order to fix this problem, a very small θ-parameter must be consider, up to the order of $\theta = 10^{-5}$. The way in which the theory faces this, is supossing again the existence of massless quarks, and an anomaly in the axial current that causes the action to be rectified by a certain amount, which is similar to the S_θ term, up to a constant. So it appears to be that the condition of the fermions to be massless change the value of θ to some amount, thanks to a chiral rotation. Then it would means that the different θ-sectors are equivalent to each other, and we can always perform a rotation to obtain a $\theta = 0$-sector, restoring in that way the P and T symmetries.

2 Symmetries and Instantons

In order to study the symmetries that appears in the theory due to the instanton solution, some illuminating exercises were proposed by the supervisor, which are described next:

2.1 Local gauge-invariance of the Euclidean action

The considered action is the following:

$$S_E = \frac{1}{2} \int d^4x Tr(F_{\mu\nu} F_{\mu\nu}) \tag{12}$$

(where we have considered $g = 1$ for simplicity) and the idea was to show that it is invariant under a local gauge transformation of the $SU(2)$ group. This can be done in more than one way. Certainly, one way is to express the transformation law of the gauge field under the local transformation: $A_\mu(x) \to U^{-1}(x)A_\mu(x)U(x) + (i/g)U^{-1}(x)\partial_\mu U(x)$, and then, substitute this into the lagrangian of the theory, which eventually, after a bit of algebra, will manifest the gauge-invariant character. Another way to achieve this is appealing to the *modus operandi* followed when constructing a gauge theory. It is known that the construction of a gauge invariant lagrangian begins by replacing the ordinary derivative, by a covariant derivative, which introduces new fields in the original theory. For this reason we must add to the lagrangian a gauge-invariant expression of the gauge fields themselves. This new term must be a "kinetic-like" one only, because the inclusion of a "mass-like" term would again destroy our invariance. The idea is then to construct an invariant from covariant forms. The kinetic condition of this term suggests us to start considering the covariant derivative. It is straightforward to demonstrate that the 2nd covariant derivative transforms also in a covariant way. Now as the theory is constructed in such a way that the coefficients of the transformation belong to some group, e.g. $SU(2)$, then we want to have a Lie algebra that relates the coefficients of the transformation through the structure constants, and this is done by a commutation relation. So it is convenient to consider the commutator of the covariant derivatives, that eventually will provide the desired algebra. Of course this commutator transforms also covariantly. When computing the commutator, the result is some function that we can call $F_{\mu\nu}$; especifically [5]:

$$[D_\mu, D_\nu] = iF_{\mu\nu} \tag{13}$$

so in this case $F_{\mu\nu}$ transforms covariantly, and, when doing the contraction $F_{\mu\nu}F_{\mu\nu}$, the invariant character of it is checked directly. So it leads inmediatly to the invariance of the action in (12).

2.2 Evaluation of the theory in the instanton solution

The lengthy (algebraically speaking) obtention of the instanton solution (4) can be used to evaluate the lagrangian in a more direct way. Appealing to self-duality of the strength-energy tensor, the substitution of the gauge field -the instanton (4)- into it is easier, giving the following result:

$$F_{\mu\nu} = \frac{\rho^2}{(x^2 + \rho^2)^2} (\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu) \tag{14}$$

Then, working with the properties of the Pauli matrices, the product $F_{\mu\nu}F_{\mu\nu}$ can be computed, and the result is then [6]

$$\mathcal{L} = 48 \left(\frac{\rho}{\rho^2 + x^2} \right)^4 \tag{15}$$

To obtain the action, an integration of (15) must be made over the whole 4-dimensional space. This was performed by hand and checked using *Mathematica*. An interesting trick

[6] see Appendix A for some comments about the computation

was performed, considering the spherical symmetry of the problem, to translate the volume element d^4x into a correspondient one-variable integration, which makes things easier, by far [7]. The result was, as expected, that the action is just a constant, independent of the size of the instanton, and of course independent of any gauge transformation $U(x)$ that could appear. In order to check the behaviour of the action when changing the location of the instanton, from $x = 0$ to some other value $x = x_0$, which is expressed in the solution in the form:

$$\frac{(x_\alpha \tau_\alpha)^\dagger}{x^2 + \rho^2}\tau_\mu \rightarrow \frac{(x_\alpha \tau_\alpha)^\dagger}{(x - x_0)^2 + \rho^2}\tau_\mu, \tag{16}$$

some problems were faced when doing the integration, so an alternative method was used, and it was simply to look for the graphical solution of the action. The volume under the curve of the integrand (which gives the action) was plotted for different values of the instanton location. Then it was checked analytically that the percent of the maximum value of the integrand, obtained for a given displacement (away from the center of the distribution), is independent of the center itself, showing that the volume under the integrand, i.e. the action, is independent of the location of the instanton. A *Mathematica* plot of a 2-dimensional integrand was made (Figure 2), for different values of x_0. Even when the perspective of this 3D plot doesn't provide a good sight of the traslation-invariance of the volume under the curve, this can be directly checked looking at the expressions in the *Mathematica* worksheet (see Apendix A).

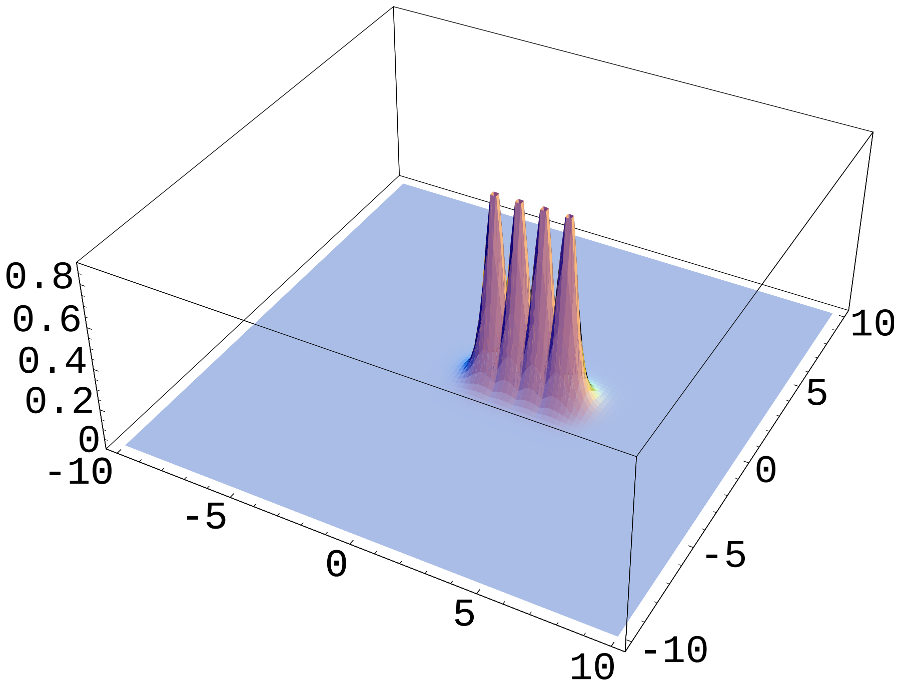

Figure 2: The volumes under these 2-dimensional curves represent the action for different locations of the instanton.

[7] see also Appendix A

2.3 Computation of the winding number q

The expression (3) is the result of a more specific equation that comes from the asymptotic behaviour of the strength-energy tensor. It comes naturally from the calculations that

$$\int_V d^4x Tr[F_{\mu\nu}\tilde{F}_{\mu\nu}] = -\frac{16\pi^2}{g^2}q \tag{17}$$

By definition, the winding number, expressed as a surface integral, has the form [6]

$$q = -\frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\int_{S^3} d^3Sn^\mu Tr[(U^{-1}\partial_\nu U)(U^{-1}\partial_\rho U)(U^{-1}\partial_\sigma U)] \tag{18}$$

So, expressing (17) as a surface integral by means of the Gauss's theorem, and comparing with (18), it is easy to check that, for this case, $q = 1$. This means that the space-time hypersphere S^3 is wrapped one time around the group space of $SU(2)$, or, said in fancy words, this theory as a topological charge equals to unity.

2.4 Relation between regular and singular gauges

When studying instantons, there are two important ways of expressing the instanton solution. One is called "the regular gauge", and the other, "the singular gauge". The former is just the one expressed in (4). It is not difficult to check that (4) can be expressed also in the following way:

$$A_\mu^{reg}(x) = \frac{2}{g}\frac{\sigma_{\mu\nu}x_\nu}{x^2+\rho^2} \tag{19}$$

On the other hand, the singular gauge as the form [7]

$$A_\mu^{sing}(x) = \frac{2}{g}\frac{\bar{\sigma}_{\mu\nu}x_\nu}{x^2+\rho^2}\frac{\rho^2}{x^2} \tag{20}$$

The idea is to relate these two forms by some gauge transformation, which would provide a desired equivalence between them. In order to achieve this, what one can do first is to think about one transformation $U(x)$ that eliminates the singularity present in the singular gauge. As before,

$$A'_\mu(x) = U^{-1}(x)A_\mu^{sing}(x)U(x) + (i/g)U^{-1}(x)\partial_\mu U(x) \tag{21}$$

were the $A'_\mu(x)$ must not have the singularity $(1/x^2)$. Making a strong use of the properties of the Pauli matrices, comparing what we want to have in the left hand side of (21) with what we have in its right hand side, then we get the following form of the transformation [8]:

$$U(x) = \frac{\sigma_\nu x_\nu}{\sqrt{x^2}} \tag{22}$$

Then, having a transformation U that eliminates the singularity, it is expected that if applying the inverse, i.e. U^{-1}, a singularity must be generated. In order to see this, we checked the following equality [8]:

$$A_\mu^{sing}(x) = U(x)A_\mu^{reg}(x)U^{-1}(x) + (i/g)U(x)\partial_\mu U^{-1}(x) \tag{23}$$

[8] see Appendix B for some details

that accounts for applying a transformation U^{-1} to the regular gauge. As expected, the last equality holds, so the demonstration that the singular and regular gauges are equivalent is complete.

3 Introducing conformal symmetry in the game

The last part of the study up to now, deals with the basic notions of conformal symmetry and its role played in the instantons description.

3.1 Basics of conformal invariance [9]

The study of conformal invariance will be an essential part of my work from now on, but up to now what has been done is to absorb the basic notions of it.

3.1.1 The conformal group

The conformal transformation of the coordinates ($x \longrightarrow x'$) leaves the metric invariant, up to a scale factor:

$$g_{\mu\nu}(\mathbf{x}) = \sigma(\mathbf{x}) g'_{\mu\nu}(\mathbf{x}') \tag{24}$$

Taking an infinitesimal transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(\mathbf{x})$, and the way the metric change under it; also imposing the transformation to be conformal, characterised by a function $f(\mathbf{x})$, then a constraint relation is obtained for this function, related with the dimension of space-time:

$$(d-1)\partial^2 f = 0 \tag{25}$$

For $d > 1$ it implies that f must be linear in the coordinates:

$$f(\mathbf{x}) = A + B_\mu x^\mu \tag{26}$$

From the relation of $f(\mathbf{x})$ with ϵ^μ follows a quadratic expression for the latter. All this yields to the following infinitesimal transformation law of x^μ:

$$x'^\mu = x^\mu + 2(\mathbf{x} \cdot \mathbf{b})x^\mu - b^\mu \mathbf{x}^2 \tag{27}$$

which is called Special Conformal Transformation (SCT).

The finite transformations corresponding to (27) are translations, dilations, rotations and SCT:

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu \\
x'^\mu &= \alpha x^\mu \\
x'^\mu &= M^\mu_\nu x^\nu \\
x'^\mu &= \tfrac{x^\mu - b^\mu \mathbf{x}^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2\mathbf{x}^2}
\end{aligned}$$

which is called the Conformal Group of transformations. The generators of the correspondient infinitesimal transformations can be expressed with the help of the generator's definition, and are:

$$
\begin{aligned}
&T_\mu = -i\partial_\mu \\
&D = -ix^\mu \partial_\mu \\
&L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \\
&K_\mu = -i(2x_\mu x^\nu \partial_\nu - \mathbf{x}^2 \partial_\mu)
\end{aligned}
$$

There are four generators accounting for translation, one for dilation, six for Lorentz rotation and four for SCT, which makes a total of 15 generators for the entire conformal group. They obey some commutation rules that actually define the conformal algebra.

It can be demonstrated that the inversion transformation (inversion on a circle [9]), which can be expressed covariantly in the form:

$$x'_\mu = \frac{b^2}{x^2} x_\mu \tag{28}$$

(being b the radius of inversion in this case), generates dilation and SCT in the following way [12]:

$$D_{a^2/b^2} = I_{a^2} I_{b^2} \tag{29}$$

$$K = I_{c^2} T_a I_{b^2} \tag{30}$$

respectively, where T_a represents a translation. However, the inversion itself can not be considered as part of the conformal group of transformations, since we can not express an infinitesimal generator for it (the change $x' \to x$ under inversion can never be infinitesimal). Anyway, due to this relations (28)-(29), the inversion plus the Poincare group of transformations can generate the whole conformal group.

Building conformal invariant functions can be achieved in the following way: Invariant under Poincare transformations requires that these functions can depend only of terms of the form $|\mathbf{x}_i - \mathbf{x}_j|$. Then, dilation invariance requires dependence of the form $\frac{|\mathbf{x}_i - \mathbf{x}_j|}{|\mathbf{x}_k - \mathbf{x}_l|}$. Finally, SCT-invariance requires a less trivial but straightforward dependence, of the form $\frac{|\mathbf{x}_i - \mathbf{x}_j||\mathbf{x}_k - \mathbf{x}_l|}{|\mathbf{x}_i - \mathbf{x}_k||\mathbf{x}_j - \mathbf{x}_l|}$. So for constructing a conformal invariant we need at least 4 points.

3.1.2 Conformal invariance in Classical Field Theory

Being a conformal transformation parametrized by w_g, a matrix representation T_g is the generator of that transformation if

$$\phi'(x') = (1 - i w_g T_g)\phi(x) \tag{31}$$

To find the form of the generators, we proceed in the following way: The subgroup (of the conformal group) that leaves the origin $x = 0$ invariant is generated by rotations, dilations, and SCT [10]. So defining the values of the generators of this subgroup for the origin, the next step is to translate this generators to a nonzero value of x [11]. This is done for all the operators of the subgroup, obtaining the action of them to the field $\phi(x)$

[9] In the plane, the inverse of a point P in respect to a circle of center O and radius R is a point P' such that P and P' are on the same ray going from O, and the following relation is obeyed: $OP \times OP' = R^2$

[10] analysing this for SCT, one sees that, starting from $x = 0$, a 1st inversion would give $x \to \infty$, then, after a translation by a small amount, a 2nd inversion leads to practically the same point $x = 0$

[11] this is done with the help of the Haussdorf formula for two operators A and B. See ref

Scale transformations are those were the field, in the presence of a coordinate transformation of the form $\mathbf{x}' = \lambda\mathbf{x}$, transform in the form

$$\phi'(\lambda\mathbf{x}) = \lambda^{-\Delta}\phi(\mathbf{x}) \tag{32}$$

where λ is the dilation factor and where Δ is the scaling dimension of the field. Taking into account that the Jacobian of the coordinate transformation is just λ^d, it can be demonstrated that in order to have an scale-invariant action $S = \int d^d x \partial_\mu \phi \partial^\mu \phi$, the scaling dimension must obey the relation $\Delta = d/2-1$. That's one of the reason of why the study of conformal invariance for 2 dimensions is taken with special care. However, another requirement for having, not only an scale-invariant, but a conformal-invariant action, involves the energy-momentum tensor, which must be symmetric and traceless. The way in which we obtain a valid tensor with this properties is the following: looking at the definition of the energy-momentum tensor by means of the Noether's theorem, we see that we have the freedom to add a divergence of an antisymmetric quantity (a 3rd rank tensor). With this modification the energy-momentum tensor is called the Belifante tensor, which is symmetric. Now, for having tracelessness, a 2nd quantity can be added, which has to do with the virial of the field, and is a double divergence of a 4th rank tensor. With this two modifications we can get a symmetric-traceless energy-momentum tensor, and our theory is succesfully conformal invariant.

3.1.3 Conformal invariance in Quantum Field Theory

Under a conformal transformation, a spinless field transform as

$$\phi(\mathbf{x}) = \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|^{-\Delta/d} \phi(\mathbf{x}) \tag{33}$$

If the field transforms like that it is called "quasi-primary field". The 2-point correlation function of 2 quasi-primary fields transforms, according to the conformal invariance of the action, in the form:

$$< \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2) >= \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_1}^{\Delta_1/d} \left|\frac{\partial \mathbf{x}'}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_2}^{\Delta_2/d} < \phi_1(\mathbf{x}_1')\phi_2(\mathbf{x}_2') > \tag{34}$$

The requirement of conformal invariance holds only if $\Delta_1 = \Delta_2$, which physically means that 2 quasi-primary fields are correlated only if they have the same scaling dimension.

Another aspects, like the form of the Ward identity implied by the conformal invariance, are been studying up to this point. So this topic is to be completed in the very-near future.

4 What is next?

The role of conformal symmetries will be appreciated next. When studying the *instanton-size distribution* $D(\rho)$ [12]:

$$D(\rho) \simeq \frac{dn^{(I)}}{d^4 z d\rho} \tag{35}$$

[12] It defines the probability $n^{(I)}$ to find an instanton with size within the range $[\rho; \rho + d\rho]$ in the volume element $d^4 z$

by means of the Instanton perturbation theory (I-pT), one encounters a divergence for this quantity in the infrared region (i.e. for large sizes of instantons). However, the computation of (35) using lattice simulations gives rise to a very different result. The comparison between the two computations is shown in Figure 3 [13].

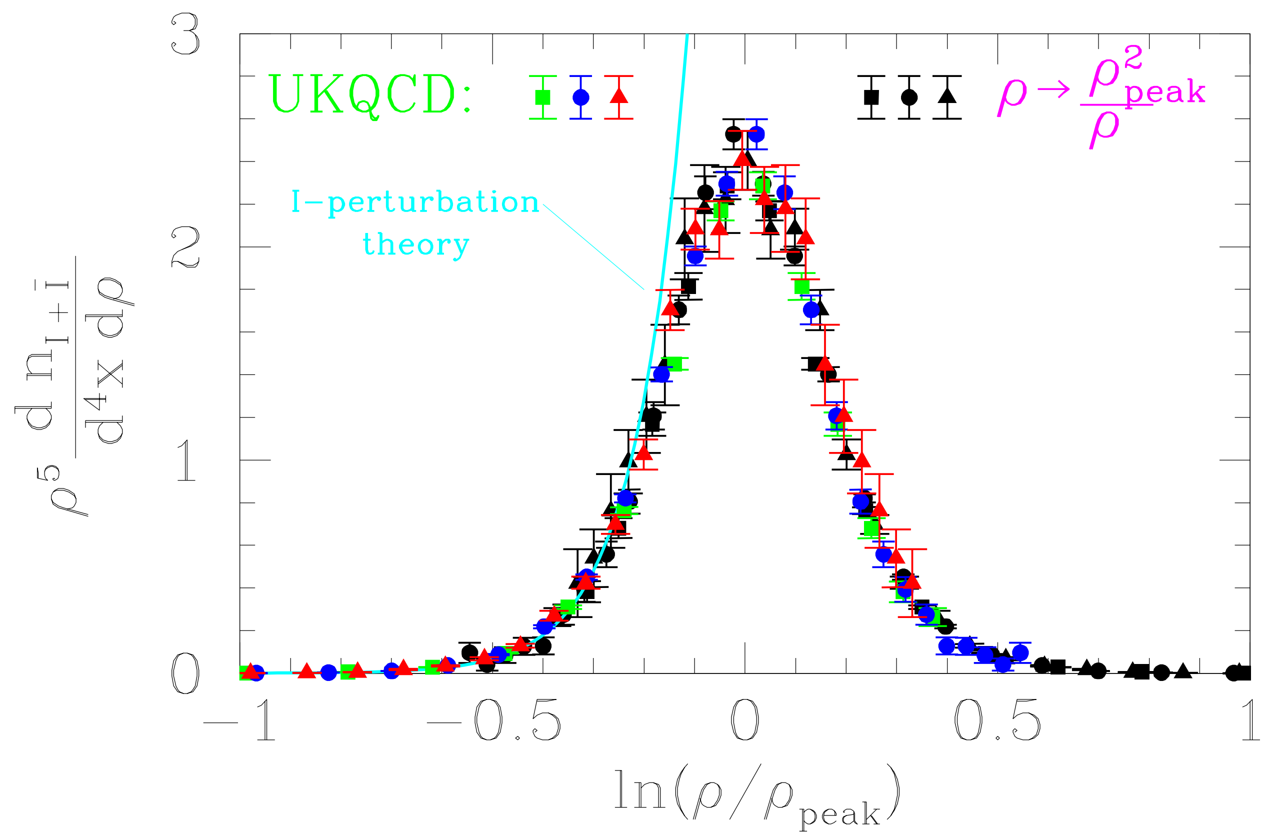

Figure 3: UKQCD lattice data [10] (solid symbols) for the (instanton+antiinstanton)-size distribution. It is displayed such as to suggest a virtually perfect *inversion symmetry* under $\rho \Rightarrow \rho_{peak}^2/\rho$ with $\rho_{peak} \approx 0.6fm$ (open and solid data symbols fit onto one universal, symmetric curve). The solid line refers to the Instanton-perturbation theory plot. Figure taken from Ref.[11].

The first important thing to note from this result is the concordance between the I-pT and lattice simulations until the size region of about 0.35 fm. Beyond that value, the former starts to diverge quickly, but the latter shows a very interesting behaviour, definning a "characteristic size" of instantons, around 0.5 fm; thus, the theory seems to protect instantons of being too large or too short. The strong symmetric appearance of this plot motivated Schrempp [11, 12, 13] to look further and a surprising aspect came out: the lattice plot appears to be invariant under an inversion of the I-size, with respect to the characteristic size ρ_{peak}, i.e.:

$$\rho \Leftrightarrow \rho' = \frac{\rho_{peak}^2}{\rho}. \tag{36}$$

From this point, considering the importance of the inversion operation for the conformal group of transformations, it is clear the necessity of studying the instanton-size distribution

[13] Within the frame of the UKQCD collaboration, a lattice data for the instanton-size distribution was obtained, and analysed by Smith and Teper, and Schrempp and Ringwald [10]

under the light of conformal symmetries. On the other hand, a beautiful issue comes into play, which is that, for classical instantons, the instantons of size ρ in regular gauge change under a conformal inversion of the coordinates (see (28), but with b replaced by ρ_{peak}) to anti-instantons of inverted size ρ'_{peak} in singular gauge:

$$A'^{(I)reg}_{\mu}(x,\rho) \Leftrightarrow A^{(\bar{I})sing}_{\mu}(x,\rho') \tag{37}$$

Note that the transformation leaves the instanton (or anti-instanton) in the same space-time point x. This is exactly the result obtained by [11, 12, 13]. So the rising question that follows from this, is if this same behaviour is still present for quantum instantons. A serious work [12, 13] has been devoted mainly to this matter.

This is the present state of my studies. Additional comments will be added to this report soon.

Thank you.

Appendix A

Computation of the Lagrangian using the Instanton solution

As shown in Eq.(12), in order to compute the Lagrangian we must take the trace of the product of strength-energy tensors, for which is useful to work with the expression (14). The square of it, i.e.

$$F_{\mu\nu}F_{\mu\nu} = \left(\frac{\rho}{\rho^2 + x^2}\right)^4 (\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu)^2 \tag{38}$$

contains a product of matrices which can be managed by the following tools:

$$\tau^\mu = (-i\sigma_a, \mathbf{1}_{2\times 2}) \tag{39}$$

where the σ_a are the Pauli matrices with $a = 1, 2, 3$ and the fourth component of τ^μ is just the unitary 2×2-matrix. When doing the product $\bar{\tau}^\mu \tau^\nu$ the following notation can be used:

$$\bar{\tau}^\mu \tau^\nu = \sigma_a \sigma_b + i\sigma_a \mathbf{1}_{2\times 2}\delta_b^0 - i\sigma_b \mathbf{1}_{2\times 2}\delta_a^0 \tag{40}$$

where these "delta-functions" account for the fourth value that μ or ν can take. So if $\nu = 4$ then $\delta_b^0 = 1$ and if $\mu = 4$ then $\delta_a^0 = 1$ and the remaining terms are zero. It is of most importance the known relation

$$\sigma_a \sigma_b = \delta_{ab} \cdot \mathbf{1}_{2\times 2} + i\epsilon_{abc}\sigma_c \tag{41}$$

where ϵ_{abc} is the totally antisymmetric tensor. Doing a bit of algebra comes the result shown in (15).

Computation of the Action using the Instanton solution

The integration of (12) can be implemented in a simple way if we consider the spherical symmetry of the integrand, which allows us to express the volume element as a function only of the radius. The hyper-dimensional volume of the space in which a surface S_{n-1} encloses the n-dimensional sphere is given by the expression:

$$V_n = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2} + 1)} \tag{42}$$

that in our 4-dimensional case is reduced to $V_4 = \frac{1}{2}\pi^2 R^4$. Then the variable of integration can be expressed as:

$$d^4x = dV_4 = 2\pi^2 R^3 dR \tag{43}$$

With this ansatz the problem is reduced to the computation of an integral of the type

$$I = \int_0^\infty \frac{R^3}{(\rho^2 + R^2)^4} dR \tag{44}$$

which must be integrated by parts to times, resulting in the expected result:

$$S = 8\pi^2 \tag{45}$$

where the same consideration as in (12) was made, i.e $g = 1$.

Appendix B

Equivalence between regular and singular gauges

Calling $\mathcal{A}$ to be the singular part of A_μ^{sing}, we can say that the transformation we are looking for acts in the following way:

$$\mathcal{A}' = 0 = U^{-1}(x)\mathcal{A}U(x) + (i/g)U^{-1}(x)\partial_\mu U(x) \tag{46}$$

in order for this expression to vanishes, we propose the following to be obeyed

$$\mathcal{A} = \frac{i}{g}U(x)(\partial_\mu U^{-1}(x)) \tag{47}$$

because when plugging it into (46), and considering the useful trick

$$(\partial_\mu U^{-1})U = \partial_\mu(U^{-1}U) - U^{-1}(\partial_\mu U) = -U^{-1}(\partial_\mu U) \tag{48}$$

the vanishing of it is straightforward. Now, considering our interest in relate the singular gauge with the regular one, an intuitive proposal can be made:

$$iU(\partial_\mu U^{-1}) = \frac{2\bar{\sigma}_{\mu\nu}x_\nu}{x^2} \tag{49}$$

Doing a bit of algebra one can re-express the $\bar{\sigma}_{\mu\nu}$ to the form:

$$\bar{\sigma}_{\mu\nu} \equiv \frac{1}{4i}(\sigma_\mu\bar{\sigma_\nu} - \sigma_\nu\bar{\sigma_\mu}) = -i\sigma_\mu\bar{\sigma_\nu} \tag{50}$$

and when applying this form to x_ν as required, it is not so hard to identify

$$U(x) = \frac{1}{\sqrt{x^2}}\sigma_\nu x_\nu \tag{51}$$

provided we check that this form of $U(x)$ obeys all the requirements from above.

References

[1] Ta-Pei Cheng and Ling-Fong Li. Gauge theory of elementary particle physics. Clarendon Press (Oxford), 1984.

[2] R. Rajaraman. Solitons and Instantons. North-Holland Physics Publishing, 1987.

[3] C. Nash and S. Sen. Topology and Geometry for Physicists. Academic Press, 1997.

[4] R. J. Rivers. Path Integral Methods in Quantum Field Theory. Cambridge University Press, 1988.

[5] M. E. Peskin, D. V. Schroeder. An Introduction to Quantum Field Theory. Addison-Wesley Publishing Company, 1996.

[6] D. Bailin, A. Love. Introduction to Gauge Field Theory, 1994.

[7] S. Moch, A. Ringwald and F. Schrempp. Instantons in Deep Inelastic Scattering -The simplest process. hep-ph/9609445, 1997.

[8] Personal notes from Dr. F. Schrempp.

[9] P. Di Francesco, P. Mathieu, D. Senechal. Conformal Field Theory. Springer. 1997.

[10] A. Ringwald and F. Schrempp. Confronting instanton perturbation theory with qcd lattice results. Phys. Lett., B459:249-258, 1999.

[11] F. Schrempp. J. Phys. G28, 915(2002); [hep-ph/0109032].

[12] Daniela Klammer. QCD-Instantons and Conformal Inversion Symmetry. Master thesis. DESY-THESIS-2006-018. Supervisor: Dr. F. Schrempp. 2006.

[13] D. Klammer and F. Schrempp; to be published.

[14] L. H. Ryder. Quantum Field Theory. Cambridge University Press, 1996.

[15] web-site. http://ww.wikipedia.org.