Spinning Strings in AdS-spaces.

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Abstract

The aim of work is to investigate classical dynamics of strings in a curved AdS_3 background and to find the relation between spin and energy of rotating strings in AdS_5. The reformulation of the problem in terms of matrix valued gauge fields and connection between these two approaches are considered.

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1 Introduction

It is known that large N limits of certain conformal field theories in d dimensions can be described in terms of supergravity and string theory on the product of $d+1$-dimensional Anti-deSitter spacetime with a compact manifold [1, 2, 3]. For instance, there is duality between a supersymmetric $\mathcal{N} = 4$ Yang-Mills gauge theory (which is a conformal field theory) on the 4-dimensional boundary of AdS_5 (with gauge group $SU(N)$ and coupling constant g_{YM}) and Type IIB string theory on $AdS_5 \times S^5$ space (with string coupling constant g_{st} proportional to g^2_{YM}, N units of five-form flux on S^5, and radius of curvature $(g^2_{YM}N)^{1/4}$). In the large N limit with $\lambda = g^2_{YM}N$ fixed but large, the string theory is weakly coupled and supergravity is a good approximation to it. One can hope that for large N and large λ, the $\mathcal{N} = 4$ theory in four dimensions is controlled by the tree approximation to supergravity.

One problem that evolved from this correspondence consists in finding relation between energy and spin of rotating strings in AdS-spaces[4]. In this work we investigate specific case of rotation in AdS_3. In the first section we introduce the problem that is considered with strings and in the second section the reformulated one.

2 Classical String Behavior

In this section we would like to introduce objects we will work with and to describe their dynamics in a classical case[1]. The common way of investigation for system dynamics starts from definition of action (or Lagrangian) and includes working with quantities come from symmetry properties of this action. Let us begin by reviewing how it can be used to describe a massive point particle.

2.1 World-Line Description of a Point Particle

A point particle sweeps out a trajectory (or world line) in spacetime. This can be described by functions $x^\mu(\tau)$ that describe how the world line, parameterized by τ, is embedded in the spacetime, whose coordinates are denoted x^μ. For simplicity, let us assume that the spacetime is flat Minkowski space with a Lorentz metric

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Then, the Lorentz invariant line element is given by

$$ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu. \tag{2}$$

In units $\hbar = c = 1$, the action for a particle of mass m is given by

$$S = -m \int ds. \tag{3}$$

This could be generalized to a curved spacetime by replacing $\eta_{\mu\nu}$ by a metric $g_{\mu\nu}(x)$, but we will not do so here. In terms of the embedding functions, $x^\mu(t)$, the action can be rewritten in the form

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{4}$$

where dots represent τ derivatives. An important property of this action is invariance under local reparametrizations. This is a kind of gauge invariance, whose meaning is that the form of S is unchanged under an arbitrary reparametrization of the world line $\tau \to \tau(\tilde{\tau})$. Actually, one should require that the function $\tau(\tilde{\tau})$ is smooth and monotonic $\left(\frac{d\tau}{d\tilde{\tau}} > 0\right)$. The reparametrization invariance is a one-dimensional analog of the four-dimensional general coordinate invariance of general relativity.

The reparametrization invariance of S allows us to choose a gauge. A nice choice is the "static gauge"

$$x^0 = \tau. \tag{5}$$

In this gauge (renaming the parameter t) the action becomes

$$S = -m \int \sqrt{1 - v^2} dt, \tag{6}$$

where

$$\vec{v} = \frac{d\vec{x}}{dt}. \tag{7}$$

Requiring this action to be stationary under an arbitrary variation of $\vec{x}(t)$ gives the Euler–Lagrange equations

$$\frac{d\vec{p}}{dt} = 0, \text{ where } \vec{p} = \frac{\delta S}{\delta \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - v^2}}, \tag{8}$$

which is the usual result. So we see that usual relativistic kinematics follows from the action $S = -m \int ds$.

[1]Extended description of string dynamics one can find in the classical manual [5]

2.2 World-Volume Actions

We can now generalize the analysis of the massive point particle to an extended object of various dimensionality (p-brane) of "tension" T_p. The action in this case involves the invariant $(p+1)$-dimensional volume and is given by

$$S_p = -T_p \int d\mu_{p+1}, \tag{9}$$

where the invariant volume element is

$$d\mu_{p+1} = \sqrt{-\det(-\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu)}\, d^{p+1}\sigma. \tag{10}$$

Here the embedding of the p-brane into d-dimensional spacetime is given by functions $x^\mu(\sigma^\alpha)$. The index $\alpha = 0, \ldots, p$ labels the $p+1$ coordinates σ^α of the p-brane world-volume and the index $\mu = 0, \ldots, d-1$ labels the d coordinates x^μ of the d-dimensional spacetime. We have defined

$$\partial_\alpha x^\mu = \frac{\partial x^\mu}{\partial \sigma^\alpha}. \tag{11}$$

The determinant operation acts on the $(p+1)\times(p+1)$ matrix whose rows and columns are labeled by α and β. The tension T_p is interpreted as the mass per unit volume of the p-brane. For a 0-brane, it is just the mass.

Let us now specialize to the string, $p = 1$. Evaluating the determinant gives

$$S[x] = -T \int d\sigma d\tau \sqrt{\dot{x}^2 x'^2 - (\dot{x}\cdot x')^2}, \tag{12}$$

where we have defined $\sigma^0 = \tau$, $\sigma^1 = \sigma$, and

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}, \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma}. \tag{13}$$

This action, called the Nambu–Goto action [6, 7], is equivalent to the action

$$S[x,h] = -\frac{T}{2}\int d^2\sigma \sqrt{-h}\, h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu, \tag{14}$$

where $h_{\alpha\beta}(\sigma,\tau)$ is the world-sheet metric, $h = \det h_{\alpha\beta}$, and $h^{\alpha\beta}$ is the inverse of $h_{\alpha\beta}$. The Euler–Lagrange equation obtained by varying $h^{\alpha\beta}$ are

$$T_{\alpha\beta} = \partial_\alpha x \cdot \partial_\beta x - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta}\partial_\gamma x \cdot \partial_\delta x = 0. \tag{15}$$

In addition to reparametrization invariance, the action $S[x,h]$ has another local symmetry, called conformal invariance (or Weyl invariance). Specifically, it is invariant under the replacement

$$\begin{aligned} h_{\alpha\beta} &\to \Lambda(\sigma,\tau) h_{\alpha\beta} \\ x^\mu &\to x^\mu. \end{aligned} \tag{16}$$

This local symmetry is special to the $p = 1$ case (strings).

The two reparametrization invariance symmetries of $S[x,h]$ allow us to choose a gauge in which the three functions $h_{\alpha\beta}$ (this is a symmetric 2×2 matrix) are expressed in terms of just one function. A convenient choice is the "conformally flat gauge"

$$h_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi(\sigma,\tau)}. \tag{17}$$

Here, $\eta_{\alpha\beta}$ denoted the two-dimensional Minkowski metric of a flat world sheet. However, because of the factor $e^{\phi}, h_{\alpha\beta}$ is only "conformally flat." Classically, substitution of this gauge choice into $S[x,h]$ leaves the gauge-fixed action

$$S = \frac{T}{2} \int d^2\sigma \eta^{\alpha\beta} \partial_\alpha x \cdot \partial_\beta x. \tag{18}$$

The gauge-fixed action is quadratic in the x's. Mathematically, it is the same as a theory of d free scalar fields in two dimensions. The equations of motion obtained by varying x^μ are simply free two-dimensional wave equations:

$$\ddot{x}^\mu - x''^\mu = 0. \tag{19}$$

This is not the whole story, however, because we must also take account of the constraints $T_{\alpha\beta} = 0$. Evaluated in the conformally flat gauge, these constraints are

$$\begin{aligned} T_{01} &= T_{10} = \dot{x} \cdot x' = 0 & (20) \\ T_{00} &= T_{11} = \frac{1}{2}(\dot{x}^2 + x'^2) = 0. & (21) \end{aligned}$$

For the next purposes we can define variables

$$\sigma^- = \tau - \sigma \quad \text{and} \quad \sigma^+ = \tau + \sigma.$$

The derivatives conjugate to $\sigma^\pm$ are defined by

$$\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{22}$$

If one writes the world-sheet energy-momentum tensor $T_{\alpha\beta}$ in the $\sigma^\pm$ coordinate system according to the standard rules of tensor analysis, one finds using (20) and (21) that

$$T_{++} = \frac{1}{2}(T_{00} + T_{01}) = \partial_+ x^\mu \cdot \partial_+ x_\mu \tag{23}$$

and

$$T_{--} = \frac{1}{2}(T_{00} - T_{01}) = \partial_- x^\mu \cdot \partial_- x_\mu. \tag{24}$$

We will use these definitions below.

2.3 Boundary Conditions

To go further, one needs to choose boundary conditions. We are interested in the case of closed-string. The general solution of the 2$\mathbb{D}$ wave equation (19) is given by a sum of "right-movers" and "left-movers":

$$x^\mu(\sigma, \tau) = x^\mu_R(\sigma^-) + x^\mu_L(\sigma^+). \tag{25}$$

These should be subject to the following additional conditions:

- $x^\mu(\sigma, \tau)$ is real
- $x^\mu(\sigma + \pi, \tau) = x^\mu(\sigma, \tau)$
- $(x'_L)^2 = (x'_R)^2 = 0$ (These are the $T_{\alpha\beta} = 0$ constraints)

The first two of these conditions can be solved explicitly in terms of Fourier series:

$$\begin{aligned} x^\mu_R &= \frac{1}{2}x^\mu + \ell_s^2 p^\mu(\tau - \sigma) + \frac{i}{\sqrt{2}}\ell_s \sum_{n \neq 0} \frac{1}{n} \alpha^\mu_n e^{-2in(\tau - \sigma)} & (26) \\ x^\mu_L &= \frac{1}{2}x^\mu + \ell_s^2 p^\mu(\tau + \sigma) + \frac{i}{\sqrt{2}}\ell_s \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-2in(\tau + \sigma)}, & \end{aligned}$$

where the expansion parameters α_n^μ, $\tilde{\alpha}_n^\mu$ satisfy

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger, \quad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^\dagger. \tag{27}$$

The center-of-mass coordinate x^μ and momentum p^μ are also real. The fundamental string length scale ℓ_s is related to the tension T by

$$T = \frac{1}{2\pi\alpha'}, \quad \alpha' = \ell_s^2. \tag{28}$$

The parameter α' is called the universal Regge slope, since the string modes lie on linear parallel Regge trajectories with this slope.

2.4 Solution for string spinning in AdS_5

In this section we consider a spinning closed string in AdS_5 to understand the relation between dimension Δ and spin S for leading Regge trajectory closed strings[2]. To carry out this calculation it is convenient to use the global AdS_5 metric

$$ds^2 = R^2(-dt^2 cosh^2\rho + d\rho^2 + sinh^2\rho d\Omega_3^2), \tag{29}$$

so that the energy is identified with the conformal dimension in the dual CFT. The string is at the equator of S^3 and the azimuthal angle depends on time[3]:

$$\phi = \omega t$$

In the Nambu action for the string we pick a gauge where $\tau = t$ and ρ is a function of σ. The Lagrangian becomes

$$L = -4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\, \sqrt{\cosh^2\rho - (\dot{\phi})^2 \sinh^2\rho} \tag{30}$$

The maximum radial coordinate is ρ_0, and the factor of 4 comes since there are four segments of the string stretching from 0 to ρ_0, which is determined by

$$\coth^2\rho_0 = \omega^2 \ . \tag{31}$$

The energy and the spin of the string are

$$E = \frac{\partial L}{\partial\dot{\phi}}\dot{\phi} - L = 4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\, \frac{\cosh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} \tag{32}$$

$$S = \frac{\partial L}{\partial\dot{\phi}} = 4\frac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\, \frac{\omega\sinh^2\rho}{\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} \tag{33}$$

The same expressions can also be derived in the conformal gauge where the world sheet action is

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\, G_{ij}\partial_\alpha X^i \partial^\alpha X^j \ . \tag{34}$$

Now we need to impose the conditions

$$T_{++} = \partial_+ X^i \partial_+ X^j G_{ij} = 0$$

$$T_{--} = \partial_- X^i \partial_- X^j G_{ij} = 0,$$

[2]this section is a similar with section 3 from [4], but includes the extra calculations.
[3]actually we use here the AdS_3 metric and we can extend the result to the case of AdS_3

where G_{ij} is the metric tensor and, according to (29):

$$G_{ij} = R^2 \begin{pmatrix} -cosh^2\rho & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & sinh^2\rho \end{pmatrix} .$$

Inserting $t = e\tau$, $\phi = e\omega\tau$, $\rho = \rho(\sigma)$ into these equations, we find

$$(\rho')^2 = e^2(\cosh^2\rho - \omega^2 \sinh^2\rho). \tag{35}$$

Thus,

$$d\sigma = \frac{d\rho}{e\sqrt{\cosh^2\rho - \omega^2\sinh^2\rho}} . \tag{36}$$

We may now adjust e so that the period of σ is 2π.

The space-time energy is given by

$$E = \frac{R^2}{2\pi\alpha'} e \int_0^{2\pi} d\sigma\, \cosh^2\rho \,,$$

and the spin by

$$S = \frac{R^2}{2\pi\alpha'} e\omega \int_0^{2\pi} d\sigma\, \sinh^2\rho \,.$$

Changing the integration variable from σ to ρ we find the previously derived expressions.

Since $R^4 = \lambda\alpha'^2$, where λ is the 't Hooft coupling, these expressions specify $E/\sqrt{\lambda}$ and $S/\sqrt{\lambda}$ as functions of ω. Therefore, the dependence of $E/\sqrt{\lambda}$ on $S/\sqrt{\lambda}$ is known in parametric form. Actually, the integrals in (32) and (33) can be expressed in terms of elliptic or hypergeometric functions. Here it will suffice to give approximate expressions in the limits where the string is much shorter or much longer than the radius of curvature of AdS_5.

Short strings. For large ω, $\rho_0 \approx \tanh\rho_0 \approx 1/\omega$. Here the string is not stretched much compared to the radius of curvature of AdS_5, so we can approximate AdS_5 by flat metric near the center. The calculation reduces to the standard spinning string in flat space, and we get

$$\begin{aligned} E &\approx 4\tfrac{R^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \,\tfrac{1}{\sqrt{1-\omega^2\rho^2}} = \tfrac{2R^2}{\pi\alpha'\omega} \arcsin(\omega\rho)|_0^{\rho_0} = \tfrac{R^2}{\alpha'\omega} \;, \\ S &\approx 4\tfrac{R^2}{2\pi\alpha'} \int_0^{\rho_0} d\rho \,\tfrac{\omega\rho^2}{\sqrt{1-\omega^2\rho^2}} = \langle\langle x \equiv \omega\rho, \xi \equiv \tfrac{2R^2}{\pi\alpha'\omega^2}\rangle\rangle = \\ &= \xi \int_0^1 x^2(1-x^2)^{-1/2}dx = \langle\langle t \equiv x^2\rangle\rangle = \tfrac{\xi}{2}\int_0^1 t^{1/2}(1-t)^{-1/2} = \\ &= \tfrac{\xi}{2}B(\tfrac{1}{2},\tfrac{3}{2}) = \tfrac{\xi}{2}\tfrac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)} = \tfrac{\frac{\xi}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{1!} = \tfrac{R^2}{2\alpha'\omega^2} \;, \end{aligned} \tag{37}$$

so that

$$E^2 = R^2\frac{2S}{\alpha'} \;. \tag{38}$$

Using the AdS/CFT correspondence, we have $\Delta = E$. For large ω, $S \ll \sqrt{\lambda}$. In this regime we find agreement with the AdS/CFT result

$$\Delta^2 \approx m^2R^2 \;. \tag{39}$$

Indeed, for the leading closed string Regge trajectory, $m^2 = \frac{2(S-2)}{\alpha'}$, so the precise factor in (38) agrees with the AdS/CFT formula.

Long strings. The situation where $S \gg \sqrt{\lambda}$ corresponds to ω approaching 1 from above:

$$\omega = 1 + 2\eta\,, \tag{40}$$

where $\eta \ll 1$. Then ρ_0 becomes very large, so that the string feels the metric near the boundary of AdS:

$$\omega \overset{\vee}{=} \coth\rho_0 = \frac{1+\exp(-2\rho_0)}{1-\exp(-2\rho_0)} \approx (1+\exp(-2\rho_0))^2 \approx 1 + 2\,\underline{\exp(-2\rho_0)} \overset{\vee}{=} 1 + 2\,\underline{\eta}$$

i.e.

$$\rho_0 \to \frac{1}{2}\ln(1/\eta)\ . \tag{41}$$

Expanding the integrals for E and S, one finds

$$\begin{aligned}
E &= 4\tfrac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\,\tfrac{\cosh^2\rho}{\sqrt{\cosh^2\rho-(1+2\eta)^2\sinh^2\rho}} \approx 4\tfrac{R^2}{2\pi\alpha'}\int_0^{\rho_0} d\rho\,\tfrac{\cosh^2\rho}{\sqrt{1-4\eta\sinh^2\rho}} = \\
&= \xi\int_0^{\rho_0}\{\cosh^2\rho + 2\eta\sinh^2\rho + \ldots\}d\rho = \\
&= \xi\int_0^{\rho_0}\{\tfrac{1}{2}(1+\cosh 2\rho) + \eta(-1+\cosh 2\rho) + \ldots\}d\rho = \\
&= \xi\int_0^{\rho_0}\{\tfrac{1}{2}(1+\cosh 2\rho) + \ldots\}d\rho = \tfrac{\xi}{2}(\rho_0 + \tfrac{1}{2}\sinh(2\rho_0) + \ldots) = \\
&= \tfrac{\xi}{2}\{\tfrac{1}{2}\ln(1/\eta) + \tfrac{1}{2}\sinh 2(\tfrac{1}{2}\ln(1/\eta)) + \ldots\} = \tfrac{R^2}{2\pi\alpha'}(\tfrac{1}{\eta} + \ln(1/\eta) + \ldots)\ , \\
S &= \ldots = \tfrac{R^2}{2\pi\alpha'}(\tfrac{1}{\eta} - \ln(1/\eta) + \ldots)\ .
\end{aligned} \tag{42}$$

It follows that $E - S$ does not approach a constant in the limit $S \gg \sqrt{\lambda}$, but instead behaves as follows:

$$E - S = \frac{\sqrt{\lambda}}{\pi}\ln(S/\sqrt{\lambda}) + \ldots \tag{43}$$

The logarithmic asymptotics in (43) are qualitatively the same as in perturbative gauge theories.

3 The dual problem

In this section we will reformulate spinning string problem in other terms and prove equivalence between this two formulations. The one concept represents a strings dynamics in AdS-spaces with the law of motion (25,26) and the other one represents a gauge theory on $2\mathbb{D}$-spacetime with a matrix-valued gauge fields in case of zero curvature. The formal problem definition in both of these concepts and connection between these problems are followed.

3.1 The reformulation basis

Let us introduce variables:

$$\begin{aligned} x^0 + ix^3 &= \cosh\rho e^{i\tau} \\ x^1 + ix^2 &= \sinh\rho e^{i\phi} \end{aligned} \tag{44}$$

$$g \equiv \begin{pmatrix} x^0 + x^1 & -x^2 + x^3 \\ -x^2 - x^3 & x^0 - x^1 \end{pmatrix} \in \mathbb{SL}(2,\mathbb{R}) \tag{45}$$

$$R_i \equiv -g^{-1}\partial_i g,$$

where $\partial_0 \equiv \frac{d}{d\tau}, \partial_1 \equiv \frac{d}{d\sigma}$ and $x^\mu = x^\mu(\sigma, \tau), \mu = \overline{0,3}$.

Proposition. There is equivalence between two systems[4]:

$$\begin{cases} \partial_0 R_1 - \partial_1 R_0 + [R_1 R_0] = 0 & (I.1) \\ \partial_0 R_0 - \partial_1 R_1 = 0 & (I.2) \\ Tr(R_0 R_1) = 0 & (I.3) \\ Tr(R_0^2 + R_1^2) = 0 & (I.4) \end{cases} \qquad \begin{cases} T_{--} = 0 & (II.1) \\ T_{++} = 0 & (II.2) \\ \partial_+ \partial_- x^\mu = 0, \mu = \overline{0,3} & (II.3) \end{cases}$$

□ One can check by direct calculation that (I.1) is an identity with x^μ as in definition (44). By the simple computation we can note that

$$(dx^0)^2 + (dx^3)^2 - (dx^1)^2 - (dx^2)^2 = 1 \tag{46}$$

Therefore the $\mathbb{SL}(2, \mathbb{R})$ group manifold is a 3-dimensional hyperboloid. The metrics on this manifold,

$$ds^2 = -(dx^0)^2 - (dx^3)^2 + (dx^1)^2 + (dx^2)^2 = |d(x^1 + ix^2)|^2 - |d(x^0 + ix^3)|^2, \tag{47}$$

is expressed in coordinates (τ, ρ, ϕ) as

$$ds^2 = -d\tau^2 cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\phi^2 \tag{48}$$

We will work on the universal cover of the hyperboloid (46), and τ is non-compact. Let us prove equivalence between (I.3)&(I.4) and (II.1)&(II.2). The eq.(II.1) means

$$\partial_- x^\mu \partial_- x^{\overline{\mu}} G_{\mu\overline{\mu}} = 0$$

where $G_{\mu\overline{\mu}}$ is a metric tensor and, according to (48),

$$G_{\mu\overline{\mu}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since $G_{\mu\overline{\mu}}$ has diagonal form we can write (II.1)&(II.2) in the form

$$((\partial_0 x^\mu)^2 + (\partial_1 x^\mu)^2 \pm 2\partial_0 x^\mu \partial_1 x^\mu) G_{\mu\mu} = 0$$

and in terms of metrics:

$$\frac{(ds)^2}{(d\tau)^2} + \frac{(ds)^2}{(d\sigma)^2} \pm 2\frac{(ds)^2}{d\sigma d\tau} = 0. \tag{49}$$

Therefore, because of plus-minus sign in (49)

$$\frac{(ds)^2}{d\sigma d\tau} = 0 \quad \text{and} \quad \frac{(ds)^2}{(d\tau)^2} + \frac{(ds)^2}{(d\sigma)^2} = 0 \tag{50}$$

If we transform conjunction (50) to the corresponding alternation then we will get an useful constant-curvature condition:

$$\frac{(ds)^2}{(d\tau)^2} = 0 \text{ and } \frac{(ds)^2}{(d\sigma)^2} = 0 \tag{51}$$

If we express (51,50,I.3,I.4) in terms of coordinates (τ, ρ, ϕ) then by direct calculation we can establish the equivalence (51)⇔(I.4),(50)⇔(I.3).

[4] a "gauge" system (I) and a "string" system (II)

Now let us prove the equivalence between (I.2) and (II.3). We can use the following representation for the matrix g:

$$g = \begin{pmatrix} \cos\tau\cosh\rho + \cos\phi\sinh\rho & \sin\tau\cosh\rho - \sin\phi\sinh\rho \\ -\sin\tau\cosh\rho - \sin\phi\sinh\rho & \cos\tau\cosh\rho - \cos\phi\sinh\rho \end{pmatrix} = e^{i\frac{\tau+\phi}{2}\sigma_2} e^{\rho\sigma_3} e^{i\frac{\tau-\phi}{2}\sigma_2}$$

where $\sigma^\mu = (I, \overrightarrow{\sigma}), \mu = \overline{0,3}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix and

$$\overrightarrow{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

is the set of Pauli matrices.

The exponential representation is useful to find the inverse matrix

$$g^{-1} = e^{-i\frac{\tau-\phi}{2}\sigma_2} e^{-\rho\sigma_3} e^{-i\frac{\tau+\phi}{2}\sigma_2} = \begin{pmatrix} \cos\tau\cosh\rho - \cos\phi\sinh\rho & -\sin\tau\cosh\rho + \sin\phi\sinh\rho \\ \sin\tau\cosh\rho + \sin\phi\sinh\rho & \cos\tau\cosh\rho + \cos\phi\sinh\rho \end{pmatrix}$$

It is also useful to expand g in sum by the σ^μ basis and to write matrices in the equivalent form as vectors:

$$A = \sum_{k=0}^{3} a^k \sigma_k \sim (a^0, a^1, a^2, a^3). \tag{52}$$

The algebra of these vectors is obtained from the following multiplication rule:

$$\begin{array}{c|cccc} \sigma & 0 & 1 & 2 & 3 \\ \hline 0 & \sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & \sigma_1 & \sigma_0 & i\sigma_3 & -i\sigma_2 \\ 2 & \sigma_2 & -i\sigma_3 & \sigma_0 & i\sigma_1 \\ 3 & \sigma_2 & i\sigma_2 & -i\sigma_1 & \sigma_0 \end{array} \tag{53}$$

In this representation we can write g and g^{-1} in the simple form:

$$g \sim (x^0, -x^2, ix^3, x^1) \text{ and } g^{-1} \sim (x^0, x^2, -ix^3, -x^1) \tag{54}$$

and

$$\begin{array}{rccccc} -R_i = g^{-1}\partial_i g = & (x^0\partial_i x^0 & -x^2\partial_i x^2 & +x^3\partial_i x^3 & -x^1\partial_i x^1, \\ & -x^0\partial_i x^2 & +x^2\partial_i x^0 & -x^3\partial_i x^1 & +x^1\partial_i x^3, \\ & ix^0\partial_i x^3 & -ix^3\partial_i x^0 & +ix^2\partial_i x^1 & -ix^1\partial_i x^2, \\ & x^0\partial_i x^1 & -x^1\partial_i x^0 & +x^2\partial_i x^3 & -x^3\partial_i x^2) \end{array}$$

The left part of eq.(II.2) is

$$\partial_0 R_0 - \partial_1 R_1 = -\partial_0 g^{-1}\partial_0 g + \partial_1 g^{-1}\partial_1 g - g^{-1}\partial_0^2 g + g^{-1}\partial_1^2 g.$$

We find that the sum of first and second terms vanishes if we take into account eq.(51).

$$\begin{array}{rccccc} \partial_i g^{-1}\partial_i g = & (\partial_i x^0\partial_i x^0 & -\partial_i x^2\partial_i x^2 & +\partial_i x^3\partial_i x^3 & -\partial_i x^1\partial_i x^1, \\ & -\partial_i x^0\partial_i x^2 & +\partial_i x^2\partial_i x^0 & -\partial_i x^3\partial_i x^1 & +\partial_i x^1\partial_i x^3, \\ & i\partial_i x^0\partial_i x^3 & -i\partial_i x^3\partial_i x^0 & +i\partial_i x^2\partial_i x^1 & -i\partial_i x^1\partial_i x^2, \\ & \partial_i x^0\partial_i x^1 & -\partial_i x^1\partial_i x^0 & +\partial_i x^2\partial_i x^3 & -\partial_i x^3\partial_i x^2) \\ \\ = & (\partial_i x^0\partial_i x^0 & -\partial_i x^2\partial_i x^2 & +\partial_i x^3\partial_i x^3 & -\partial_i x^1\partial_i x^1, \\ & & & 0 & , \\ & & & 0 & , \\ & & & 0 &) \end{array}$$

Therefore the $\sigma_{1,2,3}$-components for $\partial_0 g^{-1}\partial_0 g - \partial_1 g^{-1}\partial_1 g$ are equal to zero and the σ_0-component is equal to

$$(\partial_0 x^0)^2 + (\partial_0 x^3)^2 - (\partial_0 x^2)^2 - (\partial_0 x^1)^2 - (\partial_1 x^0)^2 - (\partial_1 x^3)^2 + (\partial_0 x^2)^2 + (\partial_0 x^1)^2 = (\frac{ds}{d\tau})^2 - (\frac{ds}{d\sigma})^2 = 0 \tag{55}$$

The last equality comes from eq.(51). So we have a nice result

$$\partial_0 R_0 - \partial_1 R_1 = -g^{-1}\partial_0^2 g + g^{-1}\partial_1^2 g = (0,0,0,0).$$

According to eq.(55) and (54) we have

$$g^{-1}(\partial_0^2 g - \partial_1^2 g) = 0 \Rightarrow \partial_+\partial_- g \sim (\partial_+\partial_- x^0, -\partial_+\partial_- x^2, i\partial_+\partial_- x^3, \partial_+\partial_- x^1) = 0, \tag{56}$$

where we use the definition (22) of derivatives. Obviously, from eq.(56), we have $\partial_+\partial_- x^\mu = 0, \mu = \overline{0,3}$.

So, we have proved the implication (I.2)&(I.4)⇒(II.3). The opposite implication one can check by putting $\partial_-\partial_+ \equiv 0$ into left part of eq.(I.2) and using eqs.(II.1,2).

Finally, we have established the equivalence between systems (I) and (II). ■

3.2 Solution of the reformulated problem

In this section we will find some particular solutions of the system (I) or investigate properties of a possible solution if we can't find it directly.

`Ansatz 1. Case of spinning string.`
Inserting $\phi = \omega\tau, \quad \rho = \rho(\sigma)$ into system (I), we find

$$\cos(\tau + \omega\tau)(\sinh 2\rho(\omega^2 - 1) + 2\rho'') = 0 \tag{57}$$
$$\sin(\tau + \omega\tau)(\sinh 2\rho(\omega^2 - 1) + 2\rho'') = 0 \tag{58}$$
$$\cosh^2\rho = \rho'^2 + \omega^2\sinh^2\rho = 0 \tag{59}$$

After integrating eqs.(57,58) over ρ and choosing a value of the arbitrary constant the system of this two equations becomes identical with eq.(59). So, actually we have only one equation (59) that corresponds with eq.(35), which was deduced with the string approach. The solution is, obviously, the same as in section 2.4.

`Ansatz 2. Sigma dependent angle.`
Inserting $\phi = \phi(\sigma,\tau), \quad \rho = \rho(\sigma)$ into eq.(I.3), and with taking into account the periodical condition from the strings image we find that if a nontrivial solution exists, then we should find it in form $\phi = \phi(\tau), \quad \rho = \rho(\sigma)$. Inserting this ansatz into eq.(I.2) one can find that $\ddot{\phi} = 0$ and, therefore, $\phi = \omega\tau + C$, where C is an arbitrary constant. So, we have returned to the ansatz 1.

`Ansatz 3. General case.`
Inserting $\phi = \phi(\sigma,\tau), \quad \rho = \rho(\sigma,\tau)$ into system (I) and simplifying, we find

$$\begin{cases} \sinh 2\rho\left(\phi^{(1,0)^2} - \phi^{(0,1)^2} + 1\right) + 2\left(\rho^{(0,2)} - \rho^{(2,0)}\right) = 0 \\ 4\rho^{(0,1)}\sinh^2\rho + 4\cosh^2\rho\left(\rho^{(1,0)}\phi^{(1,0)} - \rho^{(0,1)}\phi^{(0,1)}\right) + \sinh 2\rho\left(\phi^{(2,0)} - \phi^{(0,2)}\right) = 0 \\ \sinh\rho\left(2\cosh\rho\left(\rho^{(0,1)}\left(-1 + \phi^{(0,1)}\right) - \rho^{(1,0)}\phi^{(1,0)}\right) + \sinh\rho\left(\phi^{(0,2)} - \phi^{(2,0)}\right)\right) = 0 \\ \rho^{(0,1)}\rho^{(1,0)} + \sinh^2\rho\phi^{(0,1)}\phi^{(1,0)} = 0 \\ -\cosh^2\rho + \rho^{(0,1)^2} + \rho^{(1,0)^2} + \sinh^2\rho\left(\phi^{(0,1)^2} + \phi^{(1,0)^2}\right) = 0 \end{cases} . \tag{60}$$

This system may be useful in trying other substitutions or numerical computations.

4 Conclusion

The dependence between spin and energy of rotating strings in AdS_5 was found in approximation of short and long strings with using of two approaches: with direct calculation and with solving of the reformulated problem. The equivalence between these two approaches has proved. We hope that the reformulated problem will be useful in solving a problems of the string theory.

References

[1] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `hep-th/9711200`.

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, `hep-th/9802150`.

[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B428** (1998) 105–114,`hep-th/9802109`.

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, `hep-th/0204051`.

[5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, in 2 vols., Cambridge Univ. Press, 1987.

[6] Y. Nambu, Notes prepared for the Copenhagen High Energy Symposium (1970).

[7] T. Goto, *Prog. Theor. Phys.* **46** (1971) 1560.