DESY Summer Student Programm 2007

Work Report

Symmetries and Instantons

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1 Introduction

Nowadays we use gauge theories to descibe the physics around us. We all know the nice formulation of quantum electrodynamics, which is an abelian gauge theory. But there are also non-abelian theories, e.g. quantum chromodynamics (QCD). One of the most important aspects of a gauge theory is the related gauge invariance or gauge symmetry. By observing nature one can clearly detect symmetries, e.g. the physics stays the same all around the world and we expect them even to be the same on the other end of the universe. This is a translational symmetry, which I would like to illustrate in one further example. If one is able to do an experiment in one corner of a room one could as well move the whole experiment to the opposite corner and it will still work. Well of course this is a very general symmetry, actually it is a so called global symmetry. In the case of physics we are going even a step further as we claim to have local symmetries. This gives a first impression of the importance of symmetries in physics. Another very interesting aspect of the standard model is, that even if we are able to calculate values that are in exellent agreement with experiment, we still lack a total understanding of the theory. This becomes visible e.g. in terms of the QCD vacuum. Today we try to obtain more information about the QCD vacuum by using different techniques, that include simulating and approximating. One of these simulation methods is lattice gauge theory. So, when 'observing' the QCD vacuum via lattice gauge theory, we get something, that can be compared to a 'boiling soup'. Hence there exist objects in the QCD vacuum and they are actually described in terms of topology. What we want to study now is one of those objects, namly instantons. Instantons appear as solution to a classical field equation, but they are localised in space as well as in time, hence the name instanton. Instantons can also be related to quantum tunneling and to a tunneling between degenerated groundstates of a theory. But in this text we don't want to go into to much detail. We will treat instantons in a 'simple' context and will especially study an unexpected and therefore very interesting symmetry, that appears.
The following text first gives again a more detailed introduction to instantons and what they are. Then follow some calculations concerning basic properties of instantons before we take a look at conformal transformations and what kind of special symmetry appears when applying them to instantons. Finally a conlusion ends the report.

2 What are Instantons?

There are different ways to introduce instantons. In the following we want to consider two approaches to the subject.

2.1 Instantons in Yang-Mills Theories

In general Yang-Mills theories are non abelian gauge theories. The starting point is the Yang-Mills action

$$S = \frac{1}{2} \int d^4x Tr \left\{ F^{\mu\nu} F_{\mu\nu} \right\} \tag{1}$$

From this action a classical field equation for the gauge fields A_μ can be derived, the so called Yang-Mills equation. This equation can be written in terms of the field strength tensor $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$. Here the space is eucidean and four-dimensional. Using a simple mathmatical relation one gets a lower bound for the action

$$S = \frac{1}{2} \int d^4x Tr \left\{ F^{\mu\nu} F_{\mu\nu} \right\} = \frac{1}{4} \int d^4x \left(F^{\mu\nu} \pm \tilde{F}^{\mu\nu} \right)^2 \mp \frac{1}{2} \int d^4x F^{\mu\nu} \tilde{F}^{\mu\nu} \geq \frac{1}{2} \int d^4x Tr \left\{ F^{\mu\nu} \tilde{F}_{\mu\nu} \right\} \tag{2}$$

and therefore in order to get the minimal action the equation, that must hold, is

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \tag{3}$$

the classical Yang-Mills field equation. The simplest solution of this equation A_μ, that was discussed in 1975 [1], is then what we refer to as an instanton [2], or also one instanton solution. In this context instantons are also linked to topology. The so called winding number (topological charge) $q \in \mathbb{Z}$ is appointed to them. A positive winding number is interpreted as an instanton, a negative one as an anti-instanton. The winding number q is defined by:

$$|q| = \frac{g^2}{4\pi^2} \int d^4x Tr \left\{ F^{\mu\nu} \tilde{F}_{\mu\nu} \right\} \tag{4}$$

This yields

$$S = \frac{1}{2} \int d^4x Tr \left\{ F^{\mu\nu} F_{\mu\nu} \right\} = \frac{8\pi^2}{g^2} |q| \, . \tag{5}$$

The winding number is counting how often the euclidean sphere S_4 is wrapped around the SU(2) group. For the simplest solution A_μ $q = 1$, that is why it is called one instanton solution. In further calculations $q = 1$ or $q = -1$.

2.2 Instantons and Tunneling

Instantons also appear when we try to find a classical solution to a quantum mechanical problem, the tunnneling. We consider a one dimensional doublewell potential $V(x) = g\left(x^2 - x_0^2\right)^2$, Fig. 1.

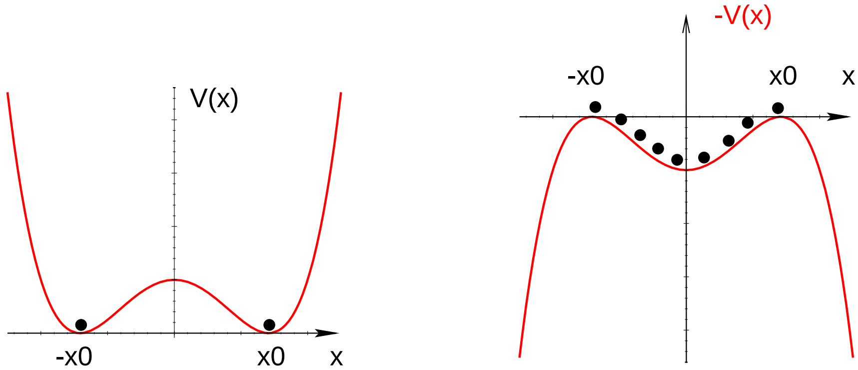

Figure 1:
On the left: the doublewell potential
On the right: The potential after going to imaginary time. [Figure taken from F.Schrempp, general colloquium 1999]

Then the classically allowed solutions must fulfill the condition $E - V > 0$, where E is the energy. So we obtain two degenerated groundstates with $E = 0$ at $x = \pm x_0$. According to this there is no way for $E = 0$ to go from $-x_0$ to $+x_0$. But as we know there is whatsoever a quantum mechanical probability of tunneling between the two groundstates. To calculate it classically we now go to imaginary time $t \longrightarrow i\tau$ (by means of a Wick rotation). Then the shape of the potential changes as is depicted in Fig. 1. So with $\left(\frac{dx}{dt}\right)^2 = -\left(\frac{dx}{d\tau}\right)^2$ we get

$$-E = \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)$$

and the differential equation to solve for $E = 0$ then reads

$$\frac{dx}{\sqrt{2V(x)}} = d\tau\,. \tag{6}$$

By solving (6) we obtain

$$x^{(I)}(\tau) = x_0 \tanh \sqrt{2g}x_0\left[\tau - \tau_0\right] \tag{7}$$

The solution is also called instanton as tunnneling is also instantan in real time, see Fig. 2 A mored advanced topic in this area is the so called Θ-vacuum. It is actually constructed from ground states refering to different winding numbers that appear if we consider a periodic potential. As I don't want to go into more detail, even though this is a very interesting topic, I shall advise the reader once more to check the references [5], [6] or [9] for more information.

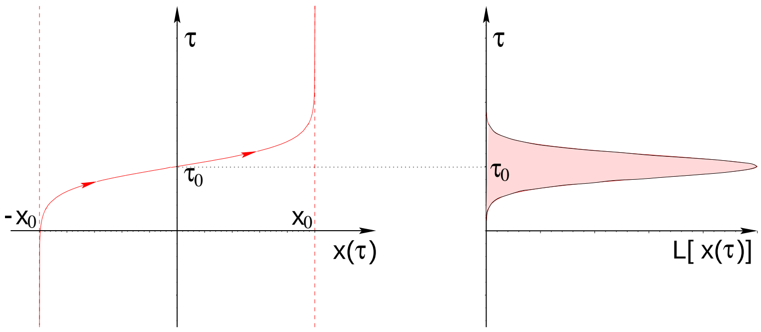

Figure 2: shape of the one instanton solution. [Figure taken from F.Schrempp, general colloquium 1999]

3 First Steps/Calculations

To get a better feeling for the subject some (simple) calculations are performed (detailed versions of some of them can be found in the appendix). We therefore restrict ourselves to the SU(2) gauge group. A good point to start is the (euclidean) action, given by:

$$S = \frac{1}{2} \int d^4x Tr \left\{ F^{\mu\nu} F_{\mu\nu} \right\} \tag{8}$$

Essential for the action is, that it is invariant under gauge transformations. So we proof this to ourselves by taking the most general form of a gauge transformation on the field A_μ

$$A_\mu \longrightarrow A^{'}_\mu = U A_\mu U^{-1} + U(\partial U^{-1}) \tag{9}$$

And the definition of the field-strength-tensor $F_{\mu\nu}$, neclecting constants.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{10}$$

Now it is straight forward to show, that $F_{\mu\nu}$ itself transforms covariant, which yields:

$$F_{\mu\nu} \longrightarrow F^{'}_{\mu\nu} = U F_{\mu\nu} U^{-1} \tag{11}$$

Concerning the action S we then obtain

$$\begin{aligned}
S^{'} &= \frac{1}{2} \int d^4x Tr \left\{ F^{'}_{\mu\nu} F^{\mu\nu '} \right\} \\
&= \frac{1}{2} \int d^4x Tr \left\{ U F_{\mu\nu} U^{-1} U F^{\mu\nu} U^{-1} \right\} \\
&= \frac{1}{2} \int d^4x Tr \left\{ U F_{\mu\nu} F^{\mu\nu} U^{-1} \right\} \\
&= \frac{1}{2} \int d^4x Tr \left\{ F_{\mu\nu} F^{\mu\nu} \right\} \\
&= S
\end{aligned}$$

Therefore the action is invariant under gauge transformations of the field A_μ, as given above.
From the action one is able to derive the Lagrangian L, as it is definded by the following relation

$$S = \int \mathscr{L} d^4x \tag{12}$$

So we get

$$\mathscr{L} = \frac{1}{2} Tr\{F_{\mu\nu}F^{\mu\nu}\} \tag{13}$$

In the folloing we assume knowing already the (one) instanton solution A_μ for the self-dual equation

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \,. \tag{14}$$

As A_μ is object to gauge transformations these solutions can have different forms according to the different gauges. In our case we consider the so called regular $A_\mu^{I,reg}$ and the singular gauge $A_\mu^{I,sing}$.

$$A_\mu^{I,reg} = \frac{2}{g} \frac{\sigma_{\mu\nu} x_\mu}{x^2 + \rho^2} \tag{15}$$

$$A_\mu^{I,sing} = \frac{2}{g} \frac{\bar{\sigma}_{\mu\nu} x_\mu}{x^2 + \rho^2} \frac{\rho^2}{x^2} \tag{16}$$

where $\sigma_{\mu\nu} = \frac{1}{4i} (\sigma_\mu \bar{\sigma_\nu} - \sigma_\nu \bar{\sigma_\mu})$ with $\sigma_\mu = (-i\sigma_j, 1)$, $\bar{\sigma_\nu} = (i\sigma_l, 1)$ and $\sigma_{\mathrm{J},l}$ being the Pauli matrices. The parameter ρ is also called radius of the instanton and a shift $x- > x + x_0$ is referred to a change in the position of the instanton.

Now we insert these instanton gauge fields in the field strength tensor $F_{\mu\nu}$ in order to calculate the Lagrangian density. As the Lagrangian density is an observable, we expect it to be the same for both gauges. And indeed, as is shown in the appendix, we obtain the same result for both gauges! Namely:

$$\mathscr{L} = \frac{12}{g} \frac{\rho^4}{(\rho^2 + x^2)^4} \tag{17}$$

We want to go on and calculate the action S. Therefore we have to solve a fourdimensional integral d^4x. But as the action S depents on x^2 only, we can go to four dimensional polar coordinates. So $x^2 \longrightarrow r^2$ and $d^4x \longrightarrow \frac{\pi^2}{6} r^3 dr$ (see also appendix). Integrating this one dimensional integral yields then the following term for the action S

$$S = 8\frac{\pi^2}{g^2} \tag{18}$$

This shows, that the action S is invariant under a change of the radius ρ and the position x_0 of the instanton, even though the Lagrangeain density is not independent of these variables. To illustrate this some plots were made, see Fig. 3.

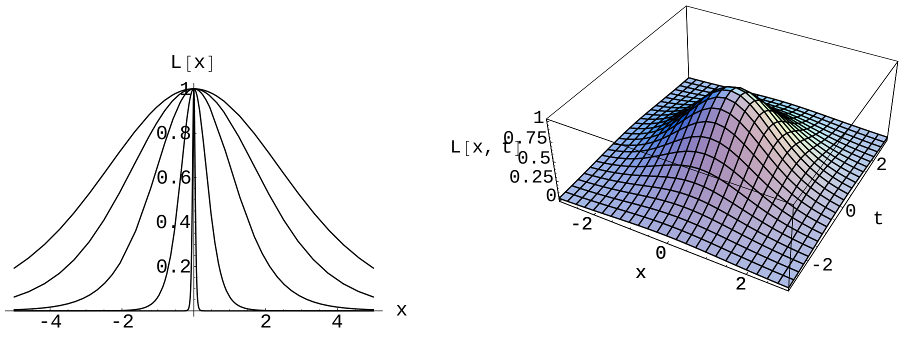

Figure 3:
On the left: two dimensional Plot of the Lagrange density for different values of ρ
On the right: 3d-Plot of the Lagrange density.

4 Instantons and Conformal Symmetry

When studying instantons an interesting feature is the so called instanton size distribution [7]. From lattice calculations we recieve Fig. 4. This result shows, that the size of the instantons is kind of fixed around a certain peak value of about $0.5fm$. This is in contrast to instanton perturbation theory, where we recieve a divergent behavior when going to large values of ρ.This is shown in Fig. 4 as a solid line. Of course this divergence can not be the right physical result, as it would mean, that we expect the probability to find instantons at a large size ρ to become infinity. So we concentrate on the distribution in Fig. 4, that was obtained via lattice simulations [4]. It is easy to notice that there exists a symmetry in this distribution, and indeed the black symbols in Fig. 4 refer to an inversion $\rho \longrightarrow \frac{\rho_p^2 eak}{\rho}$ of the instanton size ρ. According to this a symmetry under conformal inversion of the instanton size ρ lies at hand, which might be able to help us understanding, why the instanton perturbation theory breakes down at some value of ρ. So in this section we will study the conformal symmetry group and the effects of conformal transformations on instantons in more detail.

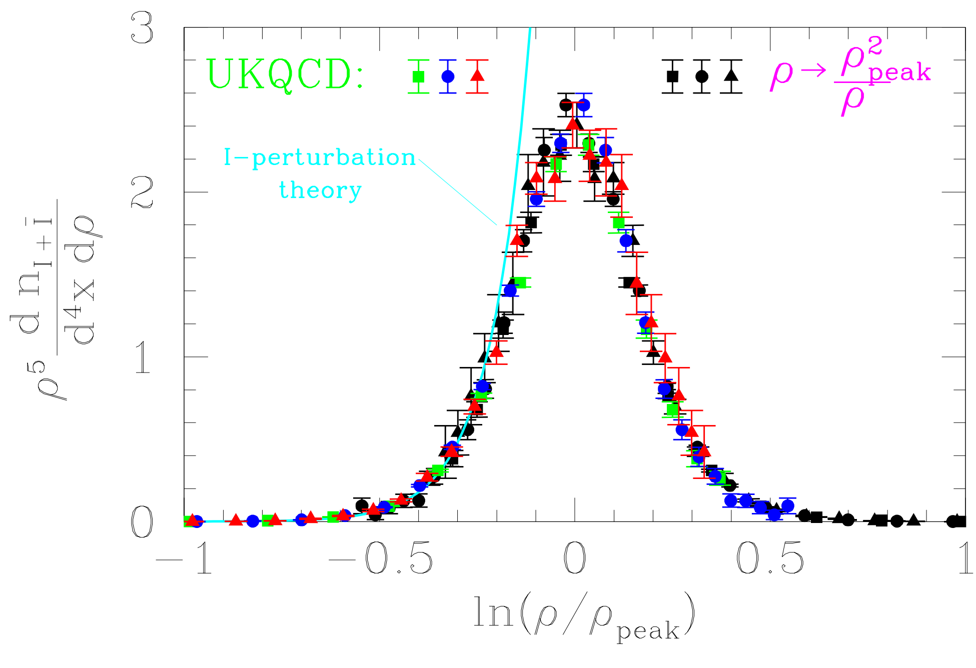

Figure 4: The colored symbols are the actual data from lattice calculations. Here a scale $ln(\frac{\rho}{\rho_p eak})$ is chosen, due to the fact, that it changes sign, if we go from ρ to $\frac{\rho_{peak}^2}{\rho}$. The black symbols are therefore the transformed data. [4]

4.1 Conformal Symmetry

A conformal transformation of the coordinates leaves the line element invariant up to a scale factor $\sigma(x)$ [7]

$$ds^2 = \sigma(x) ds^{'2} \tag{19}$$

In the following, as we want to consider active space-time transformations, the condition holds that

$$g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} = \sigma(x) g_{\alpha\beta}\left(x'\right) \tag{20}$$

The conformal symmetry group is made up by 15 generators. These generators devided into the Poincare group generators P, dilations D and special conformal transformations K.

$$P : x^\mu \longrightarrow x^{\mu'} = M^{\mu\nu} x_\nu + a_\mu \tag{21}$$

$$D : x^\mu \longrightarrow x^{\mu'} = \lambda x^\mu \tag{22}$$

$$K : x^\mu \longrightarrow x^{\mu'} = \frac{c^2}{b^2} \frac{x^\mu + \frac{a^\mu}{b^2} x^2}{1 + \frac{2ax}{b^2} + \frac{x^2 a^2}{b^4}} \tag{23}$$

another conformal transformation is the inversion

$$I_{b^2} : x^\mu \longrightarrow x^{\mu'} = \frac{b^2}{x^2} x^\mu \tag{24}$$

which cannot be a generator due to the fact, that a generator has to be 'close to unity', i.e. every generator can be written in terms of an infinitesimal transformation. But nevertheless can we use the inversion to make up the dilatation D and the special conformal transformations K. This is also shown in the appendix.

4.2 Instantons and Inversion

Now we want to look what happens to an instanton under an inversion. So we apply I to the regular gauge $A_\mu^{I,reg}$. We then obtain

$$A'_\mu(x', \rho) = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu^{I,reg}(x, \rho) \tag{25}$$

$$= \frac{x^2}{b^2}(\delta_\mu^\nu - \frac{2x_\mu x^\nu}{x^2}) \frac{2}{g} \frac{x^\sigma}{\rho^2 + x^2} \eta_{a\nu\sigma} \frac{\sigma^a}{2} \tag{26}$$

$$= A_\mu^{\bar{I},sing}(x', \rho') \tag{27}$$

with $\rho' \equiv \frac{b^2}{\rho}$. So the inversion relates the instantons with size ρ to the ones with size $\frac{1}{\rho}$. This is an remarkable effect. It was already discussed by Jackiw and Rebbi in 1976 [3].

Nowadays lattice calculations conform this symmetry were nicely, as we saw in Fig. 4.Now we look at the instanton perturbation theory again. We see, that it fits quite well for small values of ρ but diverges for larger values. Actually we could even try to make out the point up to which instanton pertubation theory is valid. This point seems to coincidence with the position ρ_{peak}, so we really have to ask ourselves if this is just a coincidence or what is the physics behind this remarkable structure.

5 Conclusions and Outlook

In the text we gave a short introduction to instantons and mentioned some of their applications. Of course we are not able to cover every aspect in as much detail, as we would like to. But nevertheless some remarkable features of instantones were shown. We especially concentrated on instantons and symmetries, which led to the discovery of a remarkable relation between instantons at large ρ and instantons at small ρ. This relation can surely be investigated further. So for instance the following questions arise:
Is there a physical interpretation of this relation? To what extend can we use the relation to 'repair' the instanton pertubation theory? How does this relation behave on the quantum level?
Some of these questions have already been studied, e.g. in [7], others still remain interesting topics for future projects. Also the question remains if we can 'see' instantons and there still is hope to find some during the analysing of the remaining HERA data or in the upcoming measurements at LHC [8]. In the end it becomes clear how important symmetries in physics are and that they can appear on every level and lead to new discoveries or a better understanding of the world, that surrounds us.

6 Appendix

- Invariance of the action under local gauge transformations:
Transformation of the covariant derivative $D_\mu = (\partial_\mu - igA_\mu)$:

$$\Psi \longrightarrow \Psi' = U\Psi$$
$$A_\mu \longrightarrow A'_\mu = UA_\mu U^{-1} + \frac{i}{g}U(\partial U^{-1})$$

$$\begin{aligned}
(D_\mu \Psi)' &= \left(\partial_\mu - igA'_\mu\right) U\Psi \\
&= (\partial_\mu U)\,\Psi + U\,(\partial_\mu \Psi) - igUA_\mu U^{-1}U\Psi - i^2 U\left(\partial_\mu U^{-1}\right) U\Psi \\
&= (\partial_\mu U)\,\Psi + U\,(\partial_\mu \Psi) - igUA_\mu U^{-1}U\Psi - (\partial_\mu U)\,\Psi \\
&= U\,(\partial_\mu \Psi) - igUA_\mu U^{-1}U\Psi \\
&= U\,(\partial_\mu - igA_\mu)\,\Psi \\
&= UD_\mu \Psi
\end{aligned}$$

for D_μ holds $UD_\mu\Psi = D'_\mu U\Psi$ and therefore:

$$D'_\mu = UD_\mu U^{-1}$$

Then with $F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$ follows $F_{\mu\nu} \longrightarrow F'_{\mu\nu} = UF_{\mu\nu}U^{-1}$.

- Calculation of the Instanton Lagrangian
we start with $F_{\mu\nu}$ and use the form we obtain when deriving it from $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ [9]:

$$F_{\mu\nu} = \left(\partial_\nu a + 2x_\nu a^2\right) b^\dagger \partial_\mu b + \left(a - x^2 a^2\right)(\partial_\mu)\, b^\dagger\,(\partial_\nu b - \partial_\nu)\, b^\dagger \partial_\mu b$$

with

$$a = \frac{1}{x^2+\rho^2}$$
$$b = x_\alpha \tau^\alpha$$
$$\tau^0 = \mathbb{1}\ , \quad \tau_j = i\sigma_j$$

where $j = 1,2,3$ and σ_j are the Pauli matrices. we get

$$\begin{aligned}
F_{\mu\nu} &= \left(\partial_\nu a + 2x_\nu a^2\right) b^\dagger \partial_\mu b + \left(a - x^2 a^2\right)\left(\partial_\mu\right) b^\dagger \left(\partial_\nu b - \partial_\nu\right) b^\dagger \partial_\mu b \\
&= \left(\partial_\nu \left(\frac{1}{x^2+\rho^2}\right) + 2x_\nu \left(\frac{1}{x^2+\rho^2}\right)^2\right) \left(x_\alpha \tau^\alpha\right)^\dagger \partial_\mu \left(x_\beta \tau^\beta\right) + \\
&\left(\frac{1}{x^2+\rho^2} - \frac{x^2}{\left(x^2+\rho^2\right)^2}\right) \left(\partial_\mu \left(x_\gamma \tau^\gamma\right)^\dagger \partial_\nu \left(x_\delta \tau^\delta\right) - \partial_\nu \left(x_\delta \tau^\delta\right)^\dagger \partial_\mu \left(x_\gamma \tau^\gamma\right)\right) \\
&= \left(\frac{x^2+\rho^2-x^2}{\left(x^2+\rho^2\right)^2}\right) \left(\delta_{\mu\gamma} \bar{\tau}^\gamma \delta_{\nu\delta} \tau^\delta - \delta_{\nu\delta} \bar{\tau}^\delta \delta_{\mu\gamma} \tau^\gamma\right) \\
&= \frac{\rho^2}{\left(x^2+\rho^2\right)^2} \left(\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu\right)
\end{aligned}$$

So the lagrangian is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} Tr\left\{F_{\mu\nu} F^{\mu\nu}\right\} \\
&= \frac{1}{2} Tr\left\{\frac{\rho^4}{\left(x^2+\rho^2\right)^4} \left(\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu\right)\left(\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu\right)\right\} \\
&= \frac{1}{2} \frac{\rho^4}{\left(x^2+\rho^2\right)^4} Tr\left\{\left(\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu\right)\left(\bar{\tau}^\mu \tau^\nu - \bar{\tau}^\nu \tau^\mu\right)\right\}
\end{aligned}$$

The only thing, that is left to determine is the trace. But this trace simply results in a constant $C = 12$

- "Proof", that the lagrangian is the same in singular and regular gauge: We proof, that $A_\mu^{I,reg} = \frac{2}{g} \frac{\sigma_{\mu\nu} x_\mu}{x^2+\rho^2}$ can be obtained via a gauge transformation from $A_\mu^{I,sing} = \frac{2}{g} \frac{\sigma_{\mu\nu} x_\mu}{x^2+\rho^2} \frac{\rho^2}{x^2}$ [10] and, as the lagrangian is invariant under gauge transformations (see above), it stays the same for both gauges. With

$$U = \frac{\sigma_\mu x_\mu}{\sqrt{x^2}}$$

The following holds

$$A_\mu^{I,sing} = U A_\mu^{I,reg} U^{-1} + \frac{i}{g} \left(U(\partial_\mu U^{-1}\right)$$

We see this with

$$\begin{aligned}
U A_\mu^{I,reg} U^{-1} &= \frac{i}{g} \frac{x^2}{x^2+\rho^2} \left(\partial_\mu U\right) U^{-1} \\
&= -\frac{2}{g} \frac{\bar{\sigma}_{\mu\nu} x_\nu}{x^2+\rho^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{i}{g}\left(\partial_\mu U\right)^{-1} U &= \frac{i}{g} U \left(U^{-1}\partial_\mu U\right)^\dagger U^{-1} \\
&= \frac{i}{g} U \left(\frac{-2i\sigma_{\mu\nu}x_\nu}{x^2}\right)^\dagger U^{-1} \\
&= \frac{1}{g} U \left(\frac{-2\sigma_{\mu\nu}x_\nu}{x^2}\right) U^{-1} \\
&= \frac{1}{g}\frac{2\bar{\sigma}_{\mu\nu}x_\nu}{x^2}
\end{aligned}$$

because now

$$\begin{aligned}
UA_\mu^{I,reg}U^{-1} + \frac{i}{g}\left(U(\partial_\mu U^{-1}\right) &= \frac{2}{g}\bar{\sigma}_{\mu\nu}x_\nu\left(-\frac{1}{x^2+\rho^2}+\frac{1}{x^2}\right) \\
&= \frac{2}{g}\frac{\bar{\sigma}_{\mu\nu}x_\nu}{x^2+\rho^2}\frac{\rho^2}{x^2} \\
&= A_\mu^{I,sing}
\end{aligned}$$

here $\sigma_{\mu\nu} = \frac{1}{4i}\left(\sigma_\mu\bar{\sigma}_{nu} - \sigma_\nu\bar{\sigma}_{mu}\right)$ and $\sigma_\mu = (-i\sigma_i, 1)$ and $\bar{\sigma}_\nu = (i\sigma_j, 1)$

- Trick for integrating S in four dimensional space-time: The n-dimensional volume elemet dV in n-dim polar coordinates is given by

$$dV = r^{n-1}\sin\theta_1\left(\sin\theta_2\right)^2\ldots\left(\sin\theta_n - 2\right)^{n-2}$$

But due to the fact, that the action S only depents on r^2, we can go to a onedimensional integral dr by looking at the volume of the n-dimensional shere:

$$V_n = \begin{cases} \frac{(2\pi)^{\frac{n}{2}}r^n}{2\cdot 4\cdot\dots\cdot n} & \text{if n is even} \\ \frac{2(2\pi)^{\frac{n-1}{2}}r^n}{1\cdot 3\cdot\dots\cdot n} & \text{if n is odd} \end{cases}$$

So for $n = 4$

$$V = \frac{\pi^2}{2}r^4$$

Now it is easy to derive dV

$$dV = \frac{\pi^2}{6}r^3 dr$$

With this we get for the action S:

$$S = \int \mathscr{L}(r^2)\frac{\pi^2}{6}r^3 dr$$

- Generation of special conformal translation K and dilatation D from inversions:

$$\begin{aligned}
K &= I_{c^2} T_a I_{b^2} \\
I_{c^2} T_a I_{b^2} &= I_{c^2} T_a \frac{b^2}{x^2} x^\mu \\
&= I_{c^2} \left(\frac{b^2}{x^2} x^\mu + a^\mu \right) \\
&= \frac{c^2 \left(\frac{b^2}{x^2} x^\mu + a^\mu \right)}{\left(\frac{b^2}{x^2} x^\mu + a^\mu \right)^2} \\
&= \frac{c^2 \left(\frac{b^2}{x^2} \left(x^\mu + \frac{x^2}{b^2} a^\mu \right) \right)}{\frac{b^4}{x^4} x^2 + \frac{2b^2 a^\mu x^\mu}{x^2} + a^2} \\
&= \frac{c^2 \left(\frac{b^2}{x^2} \left(x^\mu + \frac{x^2}{b^2} a^\mu \right) \right)}{\frac{b^4 + 2b^2 a^\mu x^\mu + a^2 x^2}{x^2}} \\
&= \frac{c^2 \left(x^\mu + \frac{x^\mu a^\mu}{b^2} \right)}{b^2 + 2x^\mu a^\mu + \frac{a^2}{b^2} x^2} \\
&= \frac{c^2 \left(x^\mu + \frac{x^\mu a^\mu}{b^2} \right)}{b^2 \left(1 + 2x^\mu \frac{a^\mu}{b^2} + \frac{a^2}{b^4} x^2 \right)} \\
&= K
\end{aligned}$$

$$\begin{aligned}
D_{\frac{a^2}{b^2}} &= I_{a^2} I_{b^2} \\
I_{a^2} I_{b^2} &= I_{a^2} \frac{b^2}{x^2} x^\mu \\
&= \frac{\frac{b^2}{x^2} x^\mu a^2}{\left(\frac{b^2}{x^2} x^\mu \right)} \\
&= \frac{\frac{b^2}{x^2} x^\mu a^2}{\left(\frac{b^4}{x^2} \right)} \\
&= \frac{a^2}{b^2} x^\mu \\
&= \lambda x^\mu \\
&= D_{\frac{a^2}{b^2}}
\end{aligned}$$

- conformal inversion of $A_\mu^{I,reg}$:

we use

$$x^2 x'^2 = b^4$$
$$x^2 = \frac{b^4}{x^2}$$
$$\rho' \equiv \frac{b^2}{\rho}$$
$$x_\mu x^\nu x^\sigma \eta_{a\nu\sigma} \frac{\sigma^a}{2} = 0$$

and obtain

$$\begin{aligned}
A'_\mu(x', \rho) &= \frac{\partial x^\nu}{\partial x'^\mu} A_\nu^{I,reg}(x, \rho) \\
&= \frac{x^2}{b^2}(\delta_\mu^\nu - \frac{2x_\mu x^\nu}{x^2}) \frac{2}{g} \frac{x^\sigma}{\rho^2 + x^2} \eta_{a\nu\sigma} \frac{\sigma^a}{2} \\
&= \frac{2}{g} \frac{1}{b^2} \left(\frac{x^2}{\rho^2 + x^2} - \frac{2x_\mu x^\nu}{\rho^2 + x^2} \right) x^\sigma \eta_{a\nu\sigma} \frac{\sigma^a}{2} \\
&= \frac{2}{g} \frac{1}{b^2} \left(\frac{x^2}{\rho^2 + x^2} \right) x^\sigma \eta_{a\nu\sigma} \frac{\sigma^a}{2} \\
&= \frac{2}{g} \frac{1}{b^2} \left(\frac{x^\sigma}{1 + \frac{\rho^2}{x^2}} \right) \eta_{a\nu\sigma} \frac{\sigma^a}{2} \\
&= \frac{2}{g} \frac{\frac{b^4}{\rho^2}}{\frac{b^4}{x^2}} \ \frac{\frac{b^2}{x^2}\ x^\sigma}{\frac{b^4}{\rho^2} + \frac{b^4 \rho^2}{x^2}} \\
&= A_\mu^{\bar{I},sing}(x', \rho')
\end{aligned}$$

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