From simple ep-Scattering to Structure Functions and HERA Physics

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Abstract

My work here at DESY consists of the study of the inner structure of a proton. Because of the lack of previous knowledge in quantum field theory like QED, the first part was to get to know basic concepts in relativistic quantum mechanics to be able to calculate transition rates and cross sections of electromagnetic reactions. So this report gives a road map of my study I did here, and it proves that I'm really familiar with these methods now. I didn't mention everything I learnt like higher order correction and radition corrections, because it wasn't needed for the last part of my work.

At the beginning I explain the transition matrix, necessary to calculate cross sections. Then I present the calculation for the simplest of the scattering process. The Mott cross section describes an electron scattering off a Coulomb potential, that can also be interpreted as a spin 0 particle with infinite mass. The next step is to take the mass and the spin of a proton into account, so it is treated as a Dirac particle. The major fault of this model is that the proton has no structure.

Then I introduce the structure functions for electromagnetic and weak interactions, which describe the inner structure of a proton. This is needed in deep inelastic scattering. The next step is the quark parton model that leads to the Bjorken scaling, Callan-Gross relation and quark distribution functions. At the end I explain the experimental results from HERA considering the structure function F_2.

1 Transition Matrix [3]

Let a particle be described by a wave function ψ_i that comes from minus infinity and is disturbed by a potential at a certain time and coordinate (or in a finite space-time-range) then is ψ_{scat} the spherical wave that is produced at the scattering point and ψ_f is the wave function of the scattered particle much later after the perturbation. So one can define the transition matrix S by the relation

$$\psi_{scat} = S\,\psi_i. \tag{1}$$

Then the transition matrix element is given by

$$S_{fi} = \int d^3x\,\psi_f^\dagger(x)\,S\,\psi_i(x) = 2\,E_i\,\delta_{fi} + S_{fi}^{(1)} + S_{fi}^{(2)} + ... \tag{2}$$

According to the fact that QED is a pertubative theory, S_{fi} can be expanded in a sum which is the iterative solution of the propagator. Here are the first and second order matrix elements

$$S_{fi}^{(1)} = -i\,e \int d^4x\,\bar{\psi}_f(x) \not{A}(x)\,\psi_i(x), \tag{3}$$

$$S_{fi}^{(2)} = i\,e^2 \int d^4x \int d^4x'\,\bar{\psi}_f(x') \not{A}(x')\,G(x'-x) \not{A}(x)\,\psi_i(x). \tag{4}$$

2 Cross Sections of Scattering Processes

2.1 Coulomb Scattering of Electrons [1]

With this knowledge one can calculate the first order transition matrix element and the cross section of an electron that is scattered by a Coulomb potential. Then the incoming and outgoing electron is described by the plane wave solution of the Dirac equation

$$\psi_i(x) = \sqrt{\frac{m}{E_i\,V}}\,u(p_i,s_i)\,\exp\left(-i\,p_i\cdot x\right), \quad \psi_f(x) = \sqrt{\frac{m}{E_f\,V}}\,\bar{u}(p_f,s_f)\,\exp\left(i\,p_f\cdot x\right) \tag{5}$$

and the four-potential is given by

$$A_0(x) = \frac{-Z\,e}{4\pi\,|\mathbf{x}|}, \quad \mathbf{A}(x) = 0. \tag{6}$$

Inserting this into equation (3) the transition matrix becomes

$$\begin{aligned} S_{fi} &= \frac{-Z\,e}{4\pi\,V}\sqrt{\frac{m^2}{E_f\,E_i}}\,\bar{u}(p_f,s_f)\,\gamma^0\,u(p_i,s_i)\int d^4x\,\frac{\exp\left(i\,\left(p_f-p_i\right)\cdot x\right)}{|x|} \\ &= \frac{-Z\,e}{4\pi\,V}\sqrt{\frac{m^2}{E_f\,E_i}}\,\frac{\bar{u}(p_f,s_f)\,\gamma^0\,u(p_i,s_i)}{|\mathbf{q}|^2}\,2\pi\,\delta\left(E_f-E_i\right), \end{aligned} \tag{7}$$

where the time-integration delivers the delta-function, the space-integration is the Fourier-transform of the Coulomb potential and $\mathbf{q}$ is the momentum transfer. The process can be interpreted as the interaction of an electron and a virtual photon, so $\mathbf{q}$ is the momentum of the virtual photon. Then the transition probability per particle is given by

$$|S_{fi}|^2 \frac{V d^3 p_f}{(2\pi)^3} = \frac{Z^2\,(4\pi\,\alpha)^2}{E_i\,V} \frac{\left|\bar{u}(p_f, s_f)\,\gamma^0\,u(p_i, s_i)\right|^2}{|\mathbf{q}|^4} \frac{d^3 p_f}{(2\pi)^3\,E_f}\,(2\pi\,\delta\,(E_f - E_i))^2\,. \tag{8}$$

The square of the δ-function can be calculated with the relation

$$(2\pi\,\delta\,(E_f - E_i))^2 = 2\pi\,\delta\,(0)\;2\pi\,\delta\,(E_f - E_i)\,. \tag{9}$$

Ome can assume that the interaction has taken place in a finite time intervall, then the δ-function is approximately described by

$$2\pi\,\delta\,(E_f - E_i) = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt\;\exp\left(i\,(E_f - E_i)\right) \quad \Rightarrow \quad 2\pi\,\delta\,(0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = T. \tag{10}$$

That means that the energy is not conservered exactly, but it satisfies the uncertainty principle.

One can compute the differential cross section. To that end, one writes the integral in spherical coordinates and divides by the differential solid angle. So with equation (7) dividing by the incident flux the cross section becomes

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \int \frac{4\,(Z\,\alpha\,m)^2}{|\mathbf{v_i}|\,E_i} \frac{\left|\bar{u}(p_f, s_f)\,\gamma^0\,u(p_i, s_i)\right|^2}{|\mathbf{q}|^4}\,\delta\,(E_f - E_i)\,\frac{p_f^2\,dp_f}{E_f} \\ &= \frac{4\,(Z\,\alpha\,m)^2}{|\mathbf{q}|^4}\,\left|\bar{u}(p_f, s_f)\,\gamma^0\,u(p_i, s_i)\right|^2. \end{aligned} \tag{11}$$

For the calculation we used

$$p = \sqrt{E^2 - m^2} \quad \Rightarrow \quad p\,dp = E\,dE \tag{12}$$

From the energy-momentum-conservation it follows that $E_f = E_i$ and $p_f = p_i$, therefore the sign of the current cancels out. Many experiments are done with unpolarized electrons and the polarization of the final particles aren't measured so it is reasonable to average over the spins

$$\begin{aligned}
\frac{d\bar\sigma}{d\Omega} &= \frac{4\,(Z\,\alpha\,m)^2}{2\,|\mathbf{q}|^4} \sum_{\pm s_f, s_i} \left|\bar u(p_f, s_f)\,\gamma^0\,u(p_i, s_i)\right|^2 \\
&= \frac{4\,(Z\,\alpha\,m)^2}{2\,|\mathbf{q}|^4} \sum_{\pm s_f, s_i} \bar u(p_f, s_f)\,\gamma^0\,u(p_i, s_i)\,\bar u(p_i, s_i)\,\gamma^0\,u(p_f, s_f) \\
&= \frac{4\,(Z\,\alpha\,m)^2}{2\,|\mathbf{q}|^4} \sum_{\pm s_f} \bar u(p_f, s_f)\,\gamma^0\,(\not p_i + m)\,\gamma^0\,u(p_f, s_f) \qquad (13)\\
&= \frac{4\,(Z\,\alpha\,m)^2}{2\,|\mathbf{q}|^4} \sum_{j,k=1}^{4} \sum_{\pm s_f} \left(\gamma^0\,(\not p_i + m)\,\gamma^0\right)_{jk}\,u(p_f, s_f)_k\,\bar u(p_f, s_f)_j \\
&= \frac{4\,(Z\,\alpha\,m)^2}{2\,|\mathbf{q}|^4}\,\mathrm{Tr}\left(\gamma^0\,(\not p_i + m)\,\gamma^0\,(\not p_f + m)\right).
\end{aligned}$$

One can make use of trace theorems to evaluate this expression

$$\frac{d\bar\sigma}{d\Omega} = \frac{Z^2\,\alpha^2}{2\,|\mathbf{q}|^4}\left(\mathrm{Tr}\,\gamma^0\,\not p_i\,\gamma^2\,\not p_f + m^2\,\mathrm{Tr}\left(\gamma^0\right)^2\right) = \frac{Z^2\,\alpha^2}{2\,|\mathbf{q}|^4}\left(8\,E_i\,E_f - 4\,p_i\cdot p_f + 4\,m^2\right). \qquad (14)$$

This can be expressed by the energy and scattering angle θ, namely

$$\mathbf{q}^2 = p_i^2 + p_f^2 - 2\,p_i\cdot p_f = 2\,m^2 - 2\,E^2 + 2\,\mathbf{p}^2\,\cos(\theta) = -4\,\mathbf{p}^2\sin^2\left(\frac{\theta}{2}\right) \qquad (15)$$

and

$$p_i\cdot p_f = E^2 - \mathbf{p}^2\left(1 - 2\,\sin^2\left(\frac{\theta}{2}\right)\right) = m^2 + 2\,\mathbf{p}^2\,\sin^2\left(\frac{\theta}{2}\right). \qquad (16)$$

Finally with $\mathbf{p}^2 = \beta^2\,E^2$ the cross section becomes

$$\frac{d\bar\sigma}{d\Omega} = \frac{Z^2\,\alpha^2}{4\,\mathbf{p}^2\,\beta^2\,\sin^4\left(\frac{\theta}{2}\right)}\left(1 - \beta^2\,\sin^2\left(\frac{\theta}{2}\right)\right). \qquad (17)$$

This is the so-called Mott cross section. The coulomb-potential can be interpreted as a pointlike particle with infinity mass and spin 0.

2.2 Electron Scattering from a Dirac Proton [1]

The difference to the previous calculation is that the proton is a free particle with spin but without a structure, it's just treated as a pointlike particle like an electron. The structure of the proton is considered later on. Hence the calculations become more difficult and the Mott cross section gets a correction term. First one has to compute the four-potential one gets from the Maxwell-equation

$$\Box A^\mu(x) = J^\mu(x) \qquad (18)$$

where the four-current is known from the Dirac-theory as

$$J^\mu(x) = e\,\bar{\psi}_f(x)\,\gamma^\mu\,\psi_i(x). \tag{19}$$

It doesn't matter if the electron or the proton causes the four-potential the result is the same as the two particles are included completely symmetrically in the formulas. So one assumes that the four-potential is generated by the proton, and in what follows the proton properties are described by capital letters or are marked by an index p.

One way to solve this equation is the Green function or propagator method. The Green function is the solution if the inhomogenity is the δ-function. The advantage of this method is that once one has the Green function one can calculate the solution of the differential equation by integrating over the product of the Green function and the inhomogenity. To get the Green function it is useful to put in the Fourier representation in the differential equation for the Green function.

$$\Box \int \frac{d^4q}{(2\pi)^4}\,\exp\left(-i\,q\cdot(x-y)\right) G(q) = \int \frac{d^4q}{(2\pi)^4}\,\exp\left(-i\,q\cdot(x-y)\right) \tag{20}$$

$$\Rightarrow \quad G(x-y) = \int \frac{d^4q}{(2\pi)^4}\,\exp\left(-i\,q\cdot(x-y)\right)\frac{-1}{q^2+i\epsilon} \tag{21}$$

The differential operator can be applied under the integral, because it doesn't act on the integral-variable and the equation is true if the integrands of the integrals are equal. So one easily gets the inverse transform of the Green function. Let's go for the four-potential solution. First some short cuts for a better overview in the formulas.

$$k_e := \sqrt{\frac{m^2}{E_f\,E_i}\,\frac{1}{V}} \qquad k_p := \sqrt{\frac{M^2}{E_f^p\,E_i^p}\,\frac{1}{V}} \tag{22}$$

$$M_{fi} := \bar{u}(p_f,p_f)\,\gamma_\mu\,u(p_i,s_i)\,\frac{e^2}{(p_f-p_i)^2+i\epsilon}\,\bar{u}(P_f,S_f)\,\gamma^\mu\,u(P_i,S_i) \tag{23}$$

One of the two integrals for the four-potential can be performed:

$$\begin{aligned}
A^\mu(x) &= \int d^4y\,G(x-y)\,J^\mu(y)\\
&= -\int d^4y \int \frac{d^4q}{(2\pi)^4}\,\frac{-1}{q^2+i\epsilon}\,k_p\,\exp\left(i\,y\cdot(q+P_f-P_i) - i\,q\cdot x\right)\bar{u}(P_f,S_f)\,\gamma^\mu\,u(P_i,S_i)\\
&= -k_p \int d^4q\,\exp\left(-i\,q\cdot x\right)\delta^4(q+P_f-P_i)\,\frac{-1}{q^2+i\epsilon}\,\bar{u}(P_f,S_f)\,\gamma^\mu\,u(P_i,S_i)
\end{aligned} \tag{24}$$

Now the transition matrix element can be calculated and the calculations are analogous to the one in the previous chapter but requiring a little bit more effort. From equation (3) and further calculations give

$$
\begin{aligned}
S_{fi} &= -i \int d^4x\, e\, \bar{\psi}_f(x)\, \gamma_\mu\, \psi_i(x)\, A^\mu(x) \\
&= -i \int d^4x \int d^4q\, k_e\, k_p\, \exp\left(-i\, x \cdot (p_f - p_i - q)\right) \delta^4(q + P_f - P_i)\, M_{fi} \\
&= -i\, (2\pi)^4\, k_e\, k_p \int d^4q\, \delta^4(q + P_f - P_i)\, \delta^4(p_f - p_i - q)\, M_{fi} \\
&= -i\, (2\pi)^4\, k_e\, k_p\, \delta^4(P_f - P_i + p_f - p_i)\, M_{fi}.
\end{aligned}
\tag{25}
$$

Using trace theorems one can calculate the spin average lorentz-invariant transition amplitude

$$
\begin{aligned}
\left|\bar{M}_{fi}\right|^2 &= \frac{1}{4} \sum_{s_f, s_i, S_f, S_i} \left|\, \bar{u}(p_f, s_f)\, \gamma^\mu\, u(p_i, s_i)\, \bar{u}(P_f, S_f)\, \gamma_\mu\, u(P_i, S_i)\right|^2 \\
&= \frac{e^2}{4q^4}\, \mathrm{Tr}\, \frac{\not{p}_f + m}{2m}\, \gamma^\mu\, \frac{\not{p}_i + m}{2m}\, \gamma^\nu\; \mathrm{Tr}\, \frac{\not{P}_f + m}{2M}\, \gamma_\mu\, \frac{\not{P}_i + m}{2M}\, \gamma_\nu \\
&= \frac{e^2}{64\, m^2\, M^2\, q^4}\, \mathrm{Tr}\left(\not{p}_f\, \gamma^\mu\, \not{p}_i\, \gamma^\nu + m^2\, \gamma^\mu\, \gamma^\nu\right)\, \mathrm{Tr}\left(\not{P}_f\, \gamma_\mu \not{P}_i\, \gamma_\nu + M^2\, \gamma_\mu\, \gamma_\nu\right) \\
&= \frac{e^2}{4\, m^2\, M^2\, q^4} \left(p_f^\mu\, p_i^\nu + p_i^\mu\, p_f^\nu - g^{\mu\nu}\, \left(p_f \cdot p_i - m^2\right)\right) \left(P_{f\mu}\, P_{i\nu} + P_{i\mu}\, P_{f\nu} - g_{\mu\nu}\, \left(P_f \cdot P_i - m^2\right)\right) \\
&= \frac{e^2}{2\, m^2\, M^2\, q^4} \left(P_f \cdot p_f\, P_i \cdot p_i + P_f \cdot p_i\, P_i \cdot p_f - m^2\, P_f \cdot P_i - M^2\, p_f \cdot p_i + 2\, M^2\, m^2\right).
\end{aligned}
\tag{26}
$$

With this results you can write down the spin-averaged differential cross section

$$
\begin{aligned}
d\bar{\sigma} &= \int V^2\, \frac{d^3p_f}{(2\pi)^3}\, \frac{d^3P_f}{(2\pi)^3}\, \frac{V}{|J_{inc}|}\, \frac{\left|\bar{S}_{fi}\right|}{V\,T} \\
&= \int \frac{d^3p_f}{(2\pi)^3}\, \frac{d^3P_f}{(2\pi)^3}\, \frac{m\,M}{E_f\, E_f^p}\, \frac{m\,M}{E_i\, E_i^p}\, \frac{(2\pi)^4\, \delta^4\left(P_f - P_i + p_f - p_i\right)}{|\mathbf{v_i} - \mathbf{V_i}|}\, \left|\bar{M}_{fi}\right| \\
&= \int \frac{d^3p_f}{(2\pi)^3}\, \frac{m\,M}{E_f\, E_f^p}\, \frac{m\,M\, (2\pi)^4\, \delta^4\left(P_f - P_i + p_f - p_i\right)}{\sqrt{(p_i \cdot P_i)^2 - m^2\, M^2}}\, \left|\bar{M}_{fi}\right|.
\end{aligned}
\tag{27}
$$

Note, that the square of the δ-function delivers an additional factor V as it is four dimensional . To compute the cross section one can go in the frame of reference in which the initial proton is at rest, so the four-momenta become

$$
p_f = (E', \mathbf{p}'), \quad p_i = (E, \mathbf{p}), \quad P_i = (M, 0) \tag{28}
$$

and it is useful to take advantage of the identity

$$
\frac{d^3p}{2\,E} = \int_0^\infty dp_0\, \delta\left(p_\mu\, p^\mu - m^2\right) d^3p = \int_{-\infty}^\infty d^4p\, \delta\left(p_\mu\, p^\mu - m^2\right)\, \theta(P_f^0) \tag{29}
$$

and use the spherical coordinates again. One can proceed with equation (27) to derive

$$
\begin{aligned}
\frac{d\bar{\sigma}}{d\Omega'} &= \frac{2}{|\mathbf{p}|} \int \frac{m^2\, M\, p'\, dE'}{(2\pi)^2}\, \left|\bar{M}_{fi}\right|^2\, \delta\left(P_f^2 - M^2\right)\, \theta(P_f^0)\, \delta^4\left(P_f - p' - P_i - p\right) \\
&= \frac{m^2\, M}{4\pi^2}\, \frac{p'}{p}\, \frac{\left|\bar{M}_{fi}\right|^2}{M + E - \frac{p\,E'}{p'}\, \cos\theta}.
\end{aligned}
\tag{30}
$$

Now one evaluates the transition amplitude in this frame of reference in the relativistic case. Because of the energy-momentum conservation one can eliminate P_f, from which it follows

$$\begin{aligned}
\left|\bar{M}_{fi}\right|^2 &= \frac{8\pi^2\,\alpha^2}{m^2\,M^2\,q^4}\,\left(2\,P_i\cdot p_f\,P_i\cdot p_i + p_i\cdot p_f\,\left(P_i\cdot p_i - P_i\cdot p_f - M^2\right)\right) \\
&= \frac{\pi^2\,\alpha^2}{2\,m^2\,E\,E'\,\sin^4\left(\frac{\theta}{2}\right)}\,\left(2\,M^2\,E\,E' + E\,E' - E\,E'\,\cos(\theta)\,\left(M\,\left(E-E'\right)-M^2\right)\right) \\
&= \frac{\pi^2\,\alpha^2}{2\,m^2\,E\,E'\,\sin^4\left(\frac{\theta}{2}\right)}\,\left(2 - 2\,\sin^2\left(\frac{\theta}{2}\right)\left(\frac{q^2}{2\,M^2}+1\right)\right) \\
&= \frac{\pi^2\,\alpha^2}{m^2\,E\,E'\,\sin^4\left(\frac{\theta}{2}\right)}\,\left(\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2\,M^2}\sin^2\left(\frac{\theta}{2}\right)\right).
\end{aligned} \tag{31}$$

In this calculation it was used

$$q^2 = -4\,E\,E'\,\sin^2\left(\frac{\theta}{2}\right), \quad E\,E'\,\sin^2\left(\frac{\theta}{2}\right) = M(E-E'). \tag{32}$$

From equations (30) and (31) one gets the cross section for relativistic electrons ($E \approx p$)

$$\frac{d\bar{\sigma}}{d\Omega'} = \frac{\alpha^2}{4\,E^2}\,\frac{\cos^2\left(\frac{\theta}{2}\right) - \frac{q^2}{2\,M^2}\,\sin^2\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)\,\left(1+\frac{2\,E}{M}\,\cos^2\left(\frac{\theta}{2}\right)\right)}. \tag{33}$$

Let us compare this result with the Mott cross section for relativistic electrons

$$\frac{d\bar{\sigma}}{d\Omega'} = \left(\frac{d\bar{\sigma}}{d\Omega'}\right)_{Mott}\,\frac{1-\frac{q^2}{2\,M^2}\,\tan^2\left(\frac{\theta}{2}\right)}{1+\frac{2\,E}{M}\,\sin^2\left(\frac{\theta}{2}\right)} \tag{34}$$

In comparison with the Mott cross section there is a correction term that is due to the spin-spin interaction and the finite mass of the proton.

3 Structure Functions

3.1 Deep inelastic Scattering

If one wants to know something about the inner structure of a proton, one needs to go to high energy electrons that scatter off protons. Deep inelastic scattering (DIS) means that there is no particle conservation. Due to the scattering event, annihiliation and creation of particles taken place, so the reaction is described by $(e+p \to e+X)$, where X is an unknown hadron system with an overall quantum numbers of a proton that depends on the energy of the system. In DIS, one encounters structure functions instead of of the form factors, which describe the charge and magnetic moment distribution.

3.2 Electromagnetic Interaction [2]

Equation (26) can be expressed as

$$\left|\bar{M}_{fi}\right|^2 = \frac{e^2}{16\, m^2\, M^2}\, L^{\mu\nu}\, P_{\mu\nu}, \tag{35}$$

where

$$L^{\mu\nu} = \frac{1}{2}\, \mathrm{Tr}(\not{p}_f + m)\, \gamma^\mu\, (\not{p}_i + m)\, \gamma^\nu \tag{36}$$

is the lepton tensor and $P^{\mu\nu}$ is the proton tensor which is similar to the lepton tensor.

A hadronic tensor depends on the momenta q, p and has the form

$$W^{\mu\nu} = -W_1\, g^{\mu\nu} + W_2\, \frac{p^\mu\, p^\nu}{M^2} + W_3\, \frac{q^\mu\, q^\nu}{M^2} + W_4\, \frac{p^\mu\, q^\nu + p^\nu\, q^\mu}{M^2}. \tag{37}$$

As the electromagnetic current is conserved, it follows

$$q_\mu\, W^{\mu\nu} = -W_1\, q^\nu + W_2\, \frac{q\cdot p\, p^\nu}{M^2} + W_3\, \frac{q^2\, q\nu}{M^2} + W_4\, \frac{q\cdot p\, q^\nu + q^2\, p^\nu}{M^2} = 0, \tag{38}$$

and similarly $q_\nu\, W^{\mu\nu} = 0$. With these and noting that $q \cdot p = -\frac{q^2}{2}$ it becomes

$$W_3 = \frac{M^2}{q^2}\, W_1 + \frac{1}{4}\, W_2, \quad W_4 = \frac{1}{2}\, W_2, \tag{39}$$

yielding

$$W^{\mu\nu} = W_1\, \left(-g^{\mu\nu} + \frac{q^\mu\, q^\nu}{q^2}\right) + \frac{W_2}{M^2}\, \left(p^\mu + \frac{q^\mu}{2}\right)\left(p^\nu + \frac{q^\nu}{2}\right). \tag{40}$$

The W_1 and W_2 are the struture functions, so the cross section is

$$\frac{d^2\sigma}{dQ^2 dW^2} = \frac{2\pi\, \alpha^2\, M}{(s - M^2)^2\, Q^2}\, \left(2W_1(W^2, Q^2) + W_2(W^2, Q^2)\, \left(\frac{\left(s - M^2\right)\, \left(s - W^2 - Q^2\right)}{M^2\, Q^2} - 1\right)\right), \tag{41}$$

or in terms of the variables $x = \frac{Q^2}{2\, M\, \nu}$ and $y = \frac{2\, M\, \nu}{s - M^2}$

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi\, \alpha^2\, M}{(s - M^2)^2\, x}\, \left(2W_1(x, y) + W_2(x, y)\, \left(\frac{\left(s - M^2\right)\, (1 - y)}{M^2\, x\, y} - 1\right)\right). \tag{42}$$

The cross section can be evaluated in the lab frame and expressed by the electron energies E, E' and the scattering angle θ and the proton energy E_p

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{\alpha^2\, M}{8\, E^2\, E_p\, \sin^4\left(\frac{\theta}{2}\right)}\, \left(2\, W_1 \sin^2\left(\frac{\theta}{2}\right) + W_2 \frac{4\, E_p^2}{M^2}\, \cos^2\left(\frac{\theta}{2}\right)\right). \tag{43}$$

3.3 Weak charged current Interaction [2]

Now let us consider a reaction where a neutrino is involved and a $W^{\pm}$ is exchanged. As the weak interaction is only coupled to left-handed particles and the weak current is not conserved, so $q^{\mu}\, W_{\mu\nu} \neq 0$ and $q^{\mu}\, L_{\mu\nu} \neq 0$ the tensors have to be modified

$$\begin{aligned} L^{\mu\nu} &= \bar{u}(k)\, \gamma^{\mu}(1-\gamma_5)\, u(k')\, \bar{u}(k')\, \gamma^{\nu}\, (1-\gamma_5)\, u(k) \\ &= 8\, \left(k^{\mu}\, k'^{\nu} + k^{\nu}\, k'^{\mu} - g^{\mu\nu}\, k\cdot k' + i\, \epsilon^{\mu\nu\rho\sigma}\, k_{\rho}\, k_{\sigma}\right) \end{aligned} \tag{44}$$

$$W^{\mu\nu} = W_1\, \left(-g^{\mu\nu} + \frac{q^{\mu}\, q^{\nu}}{q^2}\right) + \frac{W_2}{M^2}\, \left(p^{\mu} + \frac{q^{\mu}}{2}\right) \left(p^{\nu} + \frac{q^{\nu}}{2}\right) - \frac{i}{M}\, W_3\, \epsilon^{\mu\nu\rho\sigma}\, p_{\rho}\, q_{\sigma} \tag{45}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. The cross section becomes

$$\frac{d^2\sigma^{\pm}}{dxdy} = \frac{G_F^2\, (s-M^2)}{2\pi\, \left(1+\frac{Q^2}{M_W^2}\right)^2} \left(x\, y^2\, M\, W_1(x,y) + \left(1 - y - \frac{x\, y\, M^2}{s-M^2}\right)\, \nu\, W_2(x,y) \pm x\, y\, \left(1-\frac{y}{2}\right)\, \nu\, W_3(x,y)\right) \tag{46}$$

where the $\pm$ refers to $W^{\pm}$ for the charged current, G_F is the Fermi coupling constant and M_W is the mass of the W-boson. In the proton rest frame the cross section is

$$\frac{d^2\sigma^{\pm}}{d\Omega dE'} = \frac{G_F^2\, E'^2}{2\pi^2\, \left(1+\frac{Q^2}{M_W^2}\right)^2} \left(2\, W_1\, \sin^2\left(\frac{\theta}{2}\right) + W_2\, \cos^2\left(\frac{\theta}{2}\right) \mp \frac{E+E'}{M}\, W_3\, \sin^2\left(\frac{\theta}{2}\right)\right) \tag{47}$$

4 Quark Parton Model

The parton model due to Feynman says that the proton is made up of more fundamental constituents called partons. At deep inelastic scattering there is a big energy transfer ν leading to high q^2. As a result the virtual photon is able to resolve the inner structure of the nucleon. In the simplified version, the virtual photon (in electromagnetic interactions) scatters off a single parton and the other partons in the nucleon are still unaffected. In the frame where the proton has a large momentum, one can neglect the mass of the proton and can interpret the proton as a current of parallel moving partons. In the HERA lab frame this condition is nearly fullfilled. A parton carries the fraction of the momentum

$$p_i = x\, p. \tag{48}$$

4.1 Bjorken Scaling [3]

Bjorken predicted that in deep inelastic electron proton scattering (for large Q^2 and ν) the structure functions just depend on one variable x. Mathematicly it means

$$W_1(Q^2,\nu) \rightarrow F_1(x), \quad \text{for } Q^2,\nu \rightarrow \infty \tag{49}$$

$$\frac{\nu}{M}\,W_2(Q^2,\nu) \rightarrow F_2(x), \quad \text{but } x = \frac{Q^2}{2\,M\,\nu} \text{ finite} \tag{50}$$

Partons are spin 1/2 particles with charge Q_j in units of e so oine can use equation (33) in terms of x, y and summed over all partons, yielding

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\,\alpha^2}{Q^4}\,s\,x\left(1-y+\frac{y^2}{2}\right)\sum_j Q_j^2\,f_j(x), \tag{51}$$

where $f_j(x)$ is a density function. $f_j(x)\,dx$ describes the probability to find parton j with relative momentum between x and $x+dx$.

From equation (42) and using Bjorken scaling the cross section becomes

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\,\alpha^2}{Q^4}\,s\,\left(x\,y^2\,F_1(x) + (1-y)\,F_2(x)\right). \tag{52}$$

Comparing equations (51) and (52) one gets the Callan-Gross relation

$$F_2(x) = x\sum_j Q_j^2\,f_j(x), \quad F_2(x) = 2\,x\,F_1(x). \tag{53}$$

4.2 Quark Distributions [3]

It is convenient to rename the density function $f_j(x)$ to the quark flavours, then $u(x)$ is the density distribution of the up-quarks and similarly for the other quarks. The nucleon consists of valence and sea quarks. Thus, proton has two up-quarks and one down-quark. In addition, it has sea quarks and gluons. The sea is a dynamical process that happens in the proton due to the gluons that can create quark and anti-quark pairs. Taking into account only up, down, strange and charm quarks from equation (53) follows that

$$F_2^{ep}(x) = \frac{4}{9}\,x\,\left(u(x)+\bar{u}(x)+c(x)+\bar{c}(x)\right) + \frac{1}{9}\,x\,\left(d(x)+\bar{d}(x)+s(x)+\bar{s}(x)\right). \tag{54}$$

By replacing $u \leftrightarrow d$ and $\bar{u} \leftrightarrow \bar{d}$ one gets the structure function for the neutron because of the isospin invariance.

The density functions are related to the properties of a proton so that

$$\int u_v(x)\,dx = 2, \quad \int d_v(x)\,dx = 1. \tag{55}$$

The energy-momentum sum rule implies

$$\int x\,\big(u(x)+\bar{u}(x)+c(x)+\bar{c}(x)+d(x)+\bar{d}(x)+s(x)+\bar{s}(x)\big)\;dx = 1-\epsilon. \tag{56}$$

If there are just quarks (and anti-quarks) in a proton then ϵ has to be 0. Experiments have shown that ϵ is about 0.5 that means that the quarks carry only half the amount of momentum of a nucleon. Therefore, there must be other particles, these are identified with gluons. There are some other sum rules that are not further mentioned here.

5 HERA Physics

At the elecrton-proton collider HERA, experiments measured precisely, the structure functions of protons. To determine the variables Q^2, x, y one can measure the energies of the incident and final electron E, E' and scattering angle θ and the proton energy E_P, then one gets the relations

$$\begin{aligned} Q^2 &= 4\,E\,E' \\ x &= \frac{E'\,\sin^2\left(\frac{\theta}{2}\right)}{E_P\,\left(1-\frac{E'}{E}\,\sin^2\left(\frac{\theta}{2}\right)\right)} \\ y &= 1-\frac{E'}{E}\,\sin^2\left(\frac{\theta}{2}\right). \end{aligned} \tag{57}$$

In the case that the final lepton is a neutrino, one needs to reformulate the equations to the Jet energy E_J and angle θ_J

$$\begin{aligned} Q^2 &= \frac{E_J^2\,\sin^2\left(\frac{\theta_J}{2}\right)}{1-\frac{E_J}{E}\,\sin^2\left(\frac{\theta_J}{2}\right)} \\ x &= \frac{E_J\,\cos^2\left(\frac{\theta_J}{2}\right)}{E_P\,\left(1-\frac{E_J}{E}\,\sin^2\left(\frac{\theta_J}{2}\right)\right)} \\ y &= \frac{E_J}{E}\,\sin^2\left(\frac{\theta_J}{2}\right) \end{aligned} \tag{58}$$

because of the very difficult measurement of neutrinos. [2]

At HERA the structure function F_2 has been measured overs the range $0.00005 < x < 1$ and $0.5\,GeV^2 < Q^2 < 30000\,GeV^2$ [4]. Figure (1) shows the measurement of the Bjorken scaling where at large and small x the scaling is violated but in between the structure function shows nearly no dependence on Q^2. The behaviour of the Bjorken scaling can be explained by incoherent scattering at single partons while the violation is due to the QCD. In QCD quarks steadily radiate and absorb gluons which can create a quark and anti-quark pair. The

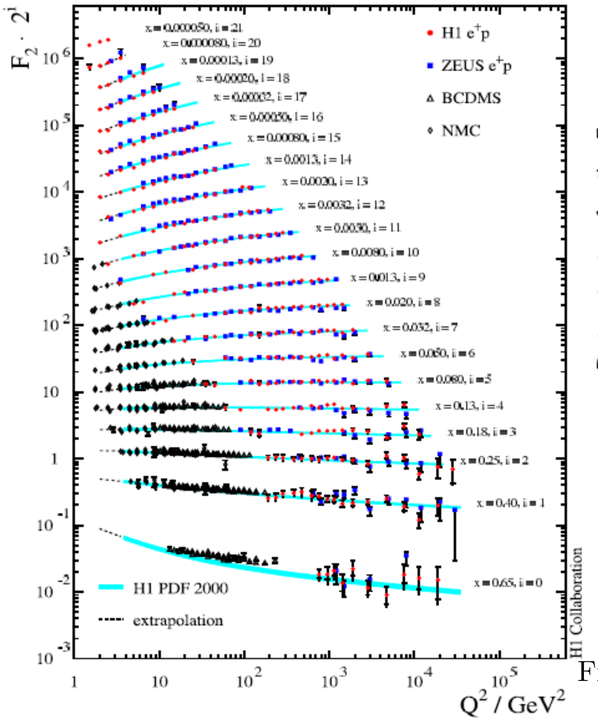

Figure 1: Measurements of $F_2(x, Q^2)$ [4]

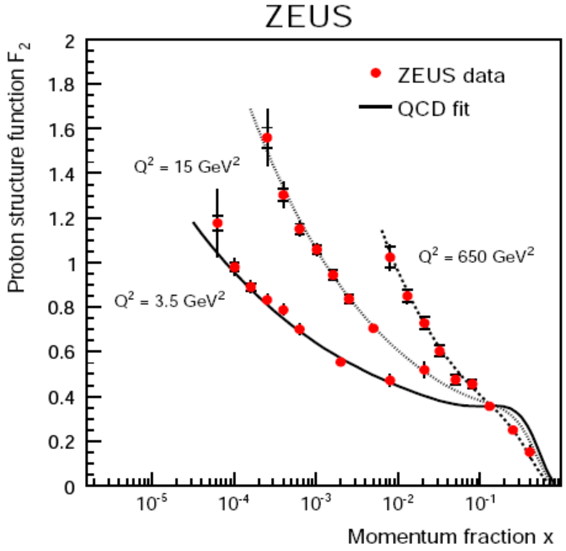

Figure 2: Low-x behaviour of $F_2(x, Q^2)$ at different values of Q^2 [4]

influence of these sea quarks depends on the resolution which is related to the wavelength of the virtual photon $\lambda_\nu \propto \frac{1}{Q}$. The better the resolution is, the more probable it is to observe a quark that is surrounded by a gluon and quark anti-quark cloud. The virtual photon detects a sea quark which carries a much smaller momentum then the valence quarks. That is exactly what the figure shows.

The dependence of the structure function F_2 on x is shown in figure (2). The structure function gives the momentum distribution in the proton. There is not just one particle that carries the entire momentum but there are several particles that carry fractional amounts of momentum. The more quarks and anti-quark pairs and gluons are there in the proton the bigger is the rise of the structure function for small momentum fraction, x.

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