QCD SATURATION IN THE SEMI-CLASSICAL APPROACH

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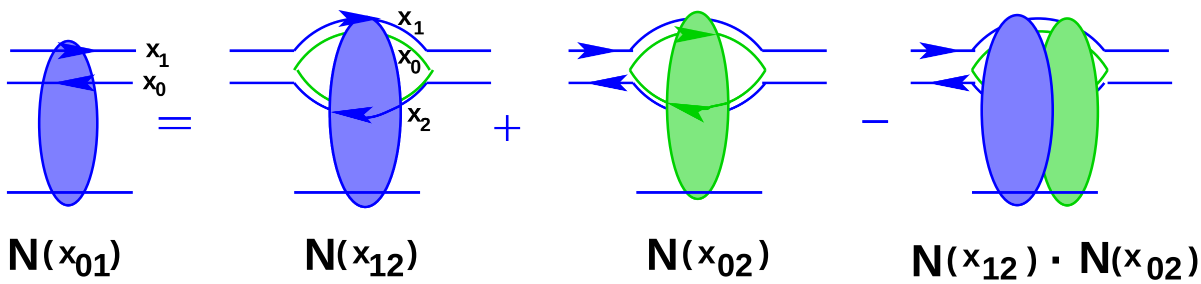
- **Practical needs to understand the large b_t behaviour to solve numerically the non-linear evolution equation;**

- **Theoretical understanding how to incorporate the non-perturbative corrections in the equation at large b_t;**

Non-linear evolution equation

Non – Linear colour dipole evolution equation



$$\frac{\partial N(\mathbf{x}_{01}, y; b_t)}{\partial y} = -\frac{2\,C_F\,\alpha_S}{\pi}\ \ln\left(\frac{\mathbf{x}_{01}^2}{\rho^2}\right)\ N(\mathbf{x}_{01}, y; b_t) +$$

$$\frac{2\,C_F\,\alpha_S}{\pi} \times \int_\rho d^2\mathbf{x}_2 \frac{\mathbf{x}_{01}^2}{\mathbf{x}_{02}^2\,\mathbf{x}_{12}^2}\Big(2\,N(\mathbf{x}_{02}, y; \mathrm{b}_t - \frac{1}{2}\mathbf{x}_{12})$$

$$-N(x_{02}, y; b_t - \frac{1}{2}x_{12})N(x_{12}, y; b_t - \frac{1}{2}x_{02})\Big)$$

In momentum representation at large b_t

$$\frac{\partial \tilde{N}(k, y; b_t)}{\partial y} = \hat{\alpha}_S\left(\hat{\chi}(\hat{\gamma}(k))\tilde{N}(k, y; b_t) - \tilde{N}^2(k, y; b_t)\right)$$

where

$$\hat{\gamma}(k) = 1 + \frac{\partial}{\partial \xi}$$

BFKL emission at large b_t

Emission term in momentum representation

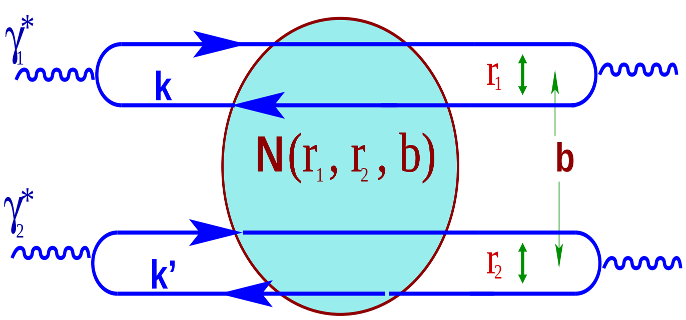
$$\frac{dN(\mathrm{x}_{01},b_t,y)}{dy} \;=\; -\boxed{\frac{2\,C_F\,\alpha_S}{\pi}\,\ln\left(\frac{\mathrm{x}_{01}^2}{\rho^2}\right)\;N(\mathrm{x},b_t,y)}$$
$$+\;\boxed{\frac{2\,C_F\,\alpha_S}{\pi}\;\int_\rho\;d^2\mathrm{x}_2\boxed{\frac{\mathrm{x}_{01}^2}{\mathrm{x}_{02}^2\,\mathrm{x}_{12}^2}}\;N(\mathrm{x}_{02},b_t,y)}$$

where

$$N(x_{01},b_t;y)\;=\;\mathrm{Im}\;a_{dipole}(x_{01},b_t;y)$$

Gluon Reggeization in momentum representation

wave function for dipole –> 2 dipoles decay



$$N(k,k';b_t)\;=\;\int\,\frac{d\gamma}{2\pi i}\,\phi_{in}\,e^{\alpha_S\chi(\gamma)y-(1-\gamma)\xi}$$

Lipatov (1986)

- $\chi(\gamma) = 2\psi(1) - \psi(1-\gamma) - \psi(\gamma)$

- $\xi \; = \; \ln(k^2\, k'^2\, b_t^4)$

- $\phi_{in}(\gamma, k'; b_t)$ **from BA:**

$$\phi_{in}(k', y; b_t) \; = \; \pi \alpha_S^2 \frac{N_c^2 - 1}{3\, N_c^2}\, (m_\pi\, b_t)^4\, K_4(2\, m_\pi\, b_t)\, \frac{1}{\gamma}$$

- **For** $1/2m_\pi \gg \; b_t \; \gg \; 1/k, 1/k'$

$$\phi_{in} \;\; \propto \;\; \frac{1}{b_t^4}$$

- **For** $b_t \; \gg \; 1/2m_\pi$

$$\phi_{in} \;\; \propto \;\; e^{-\,2m_\pi b_t}$$

- **npQCD corrections are included in BA (Levin & Ryskin (1990), Ferreiro, Iancu, Itakura & McLerran (2002))**

but not

in the equation (Kovner & Wiedemann (2002))

Semi-classical approach (a general discussion)

- $$N(k,k',y;b_t) = e^{S} = e^{\omega(\xi,y;b_t)\,y\,-\,(\,1-\gamma(\xi,y;b_t)\,)\,\xi\,+\,\beta(k',b_t)}$$

- $S_\xi = \gamma \qquad S_y = \omega$

- $\omega(\xi,y;b_t)$ and $\gamma(\xi,y;b_t)$ are smooth function of y and ξ

- $$\omega \;-\; \hat{\alpha}_S\,\chi(\gamma) \;+\; \hat{\alpha}_S\,e^{S} \;=\; 0$$

- The set of equations on trajectories:

$$(1.)\ \frac{d\,\xi}{d\,t} \;=\; -\,\hat{\alpha}_S\,\frac{d\,\chi(\gamma)}{d\,\gamma} \quad (2.)\ \frac{d\,y}{d\,t} \;=\; 1 \Longrightarrow \text{trajectories}$$

$$(3.) \qquad \frac{d\,S}{d\,t} \;=\; \hat{\alpha}_S\,(1-\gamma)\,\frac{d\,\chi(\gamma)}{d\,\gamma} + \omega$$

$$(4.) \qquad \frac{d\,\gamma}{d\,t} \;=\; \hat{\alpha}_S\,(\,1-\gamma\,)\,e^{S}\;, \quad (5.) \qquad \frac{d\,\omega}{d\,t} \;=\; -\,\hat{\alpha}_S\,\omega\,e^{S}$$

$$\frac{d(\gamma - 1)}{(\gamma - 1)} = \frac{d\,\omega}{\omega}$$

$$\omega = B_0\,\omega$$

-

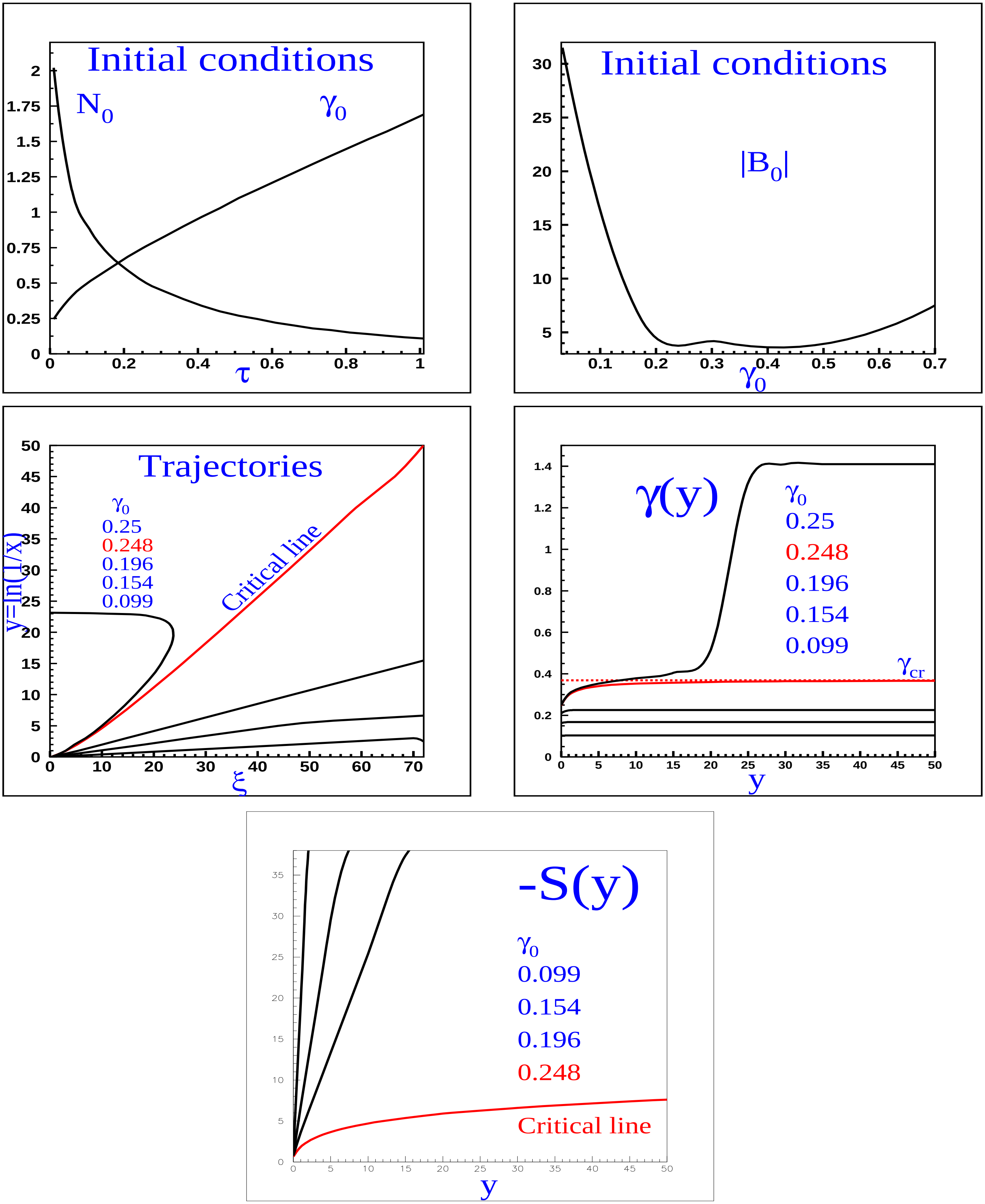
$$\frac{d\,\gamma}{d\,y} = \alpha_S\,(\,1 - \gamma\,)\,(\,B_0\,(\,1 - \gamma\,) + \chi(\gamma)\,)$$

- **Critical trajectory:** (GLR (1982), Bartels & Levin (1992), Mueller &Triantafyllopoulos (2002))

$$\frac{d\,\chi(\gamma_{cr})}{d\,\gamma_{cr}} = -\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \qquad \gamma_{cr} \approx 0.37$$

$$\xi_{cr} = -\,\alpha_S\,\frac{d\,\chi(\gamma_{cr})}{d\,\gamma_{cr}}\,(y - y_0) + \xi_0(b_t)$$

$$N = \frac{N_0}{N_0\,\alpha_S\,(y - y_0) + 1}$$


Initial conditions
N_0
γ_0
τ
Initial conditions
$|B_0|$
γ_0
Trajectories
$y=\ln(1/x)$
γ_0
0.25
0.248
0.196
0.154
0.099
Critical line
ξ
$\gamma(y)$
γ_0
0.25
0.248
0.196
0.154
0.099
γ_{cr}
y
-S(y)
γ_0
0.099
0.154
0.196
0.248
Critical line
y

b_t dependence

$$\xi \;=\; -\alpha_S \int_{y_0}^{y} d\,y' \,\frac{d\,\chi(\gamma(y'))}{d\gamma} \;+\; \xi_0(b_t)\;.$$

$$Q^2_s(y, k'^2; b_t) \;=\; Q^2_s(y = y_0, k'^2; b_t)\, e^{\alpha_S\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,(y-y_0)}$$

$$=\; k'^2\, m^4_\pi\, b^4_t\, K_4(2\,m_\pi\, b_t)\, \tau_0(\gamma = \gamma_{cr})\;\, e^{\alpha_S\,\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,(y-y_0)}$$

$\Longrightarrow C\;\frac{k'^2}{b^4_t}\,(\frac{1}{x})^{\lambda}$ $\quad$ $if\;\; 1/2m_\pi \;>\; b_t \;>\; 1/k'$

$\Longrightarrow C\;\frac{m^4_\pi}{k'^2}\,(\frac{1}{x})^{\lambda}\, e^{-2\,m_\pi\, b_t}$ $\quad$ $if\;\; b_t \;>\; 1/2m_\pi$

Saturation(Colour Glass Condensate) domain

Geometrical scaling:

$$N = \int \frac{d\omega}{2\pi i} \phi_{in} e^{\omega\, y + \gamma(\omega)\,\xi}$$

$$\gamma(\omega) = C\,\omega$$

for $k^2 \leq Q_s^2(x; b_t)$

$$N = N\left(z = \alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\, y - \xi\right)$$

$$N = N\left(k^2/Q_s^2(x; b_t)\right)$$

Physics:

GLR (1982), McLerran & Venugopalan (1994) Stasto, Golec-Biernat& Kwiecinski(2002)

Theory:

Bartels & Levin (1994), Ferreiro, Iancu, Itakura & McLerran (2002)

$$N(y,\xi;b_t) = \frac{1}{2}\int_{\xi}^{\xi_{sat}} d\,\xi'\,\left(1 - e^{-\phi(\xi',y)}\right)$$

$$N(y,\xi;b_t) = N\left(z = \alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\, y - \xi\right)$$

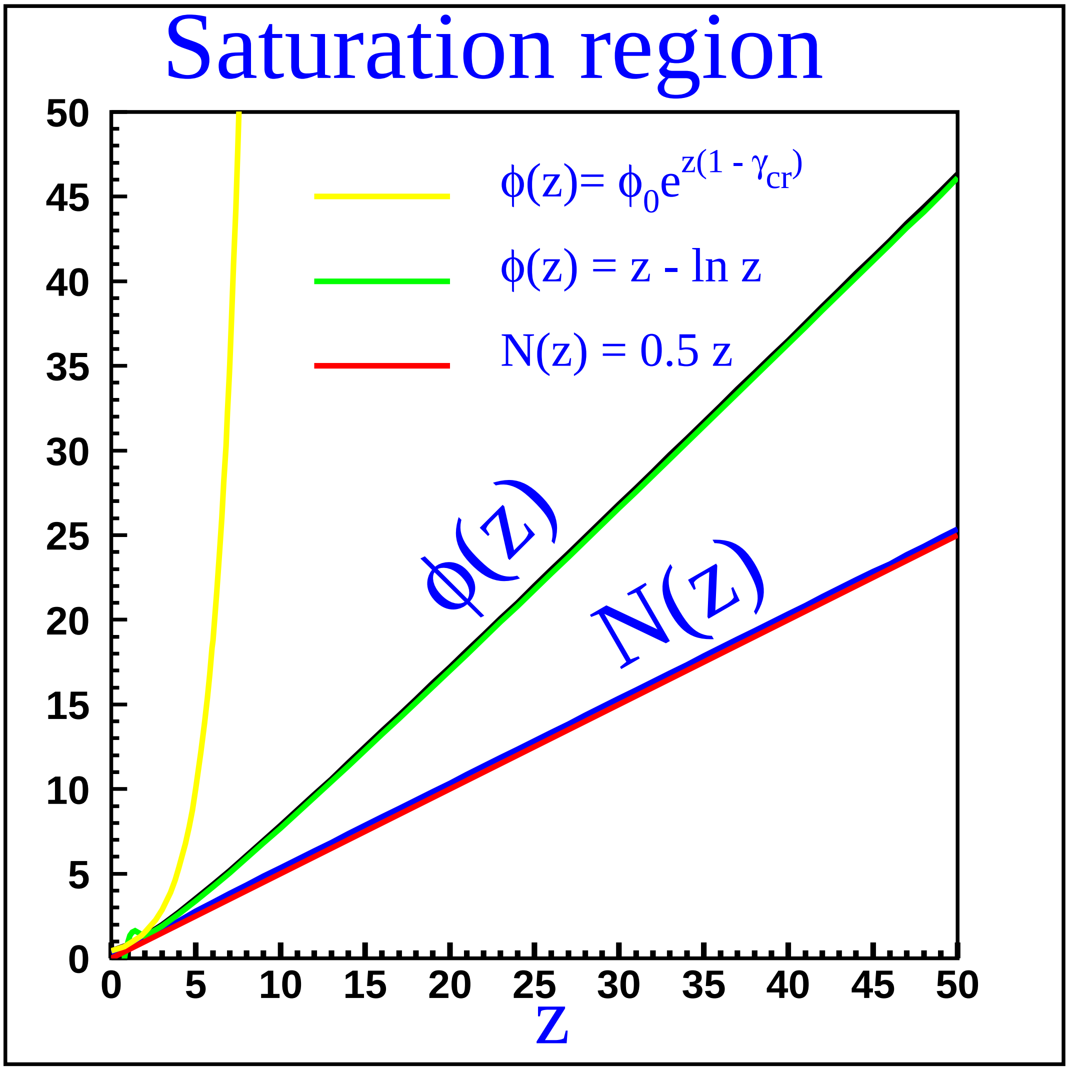
ϕ **is a smooth function of** z' **.**

Finally:

$$\frac{d\,\phi(z)}{dz} \;=\; p(z);$$

$$p(z)\,\frac{dp(\phi)}{d\phi} \;+\; \alpha_S\,\frac{dL(p)}{dp}\,p\,\frac{dp(\phi)}{d\phi} \;=\; \alpha_S\left(1 \;-\; e^{-\,\phi}\right)\;;$$

$$L(f) \;=\; \frac{(1-f)\,\chi(f) \;-\; 1}{f}\,;$$



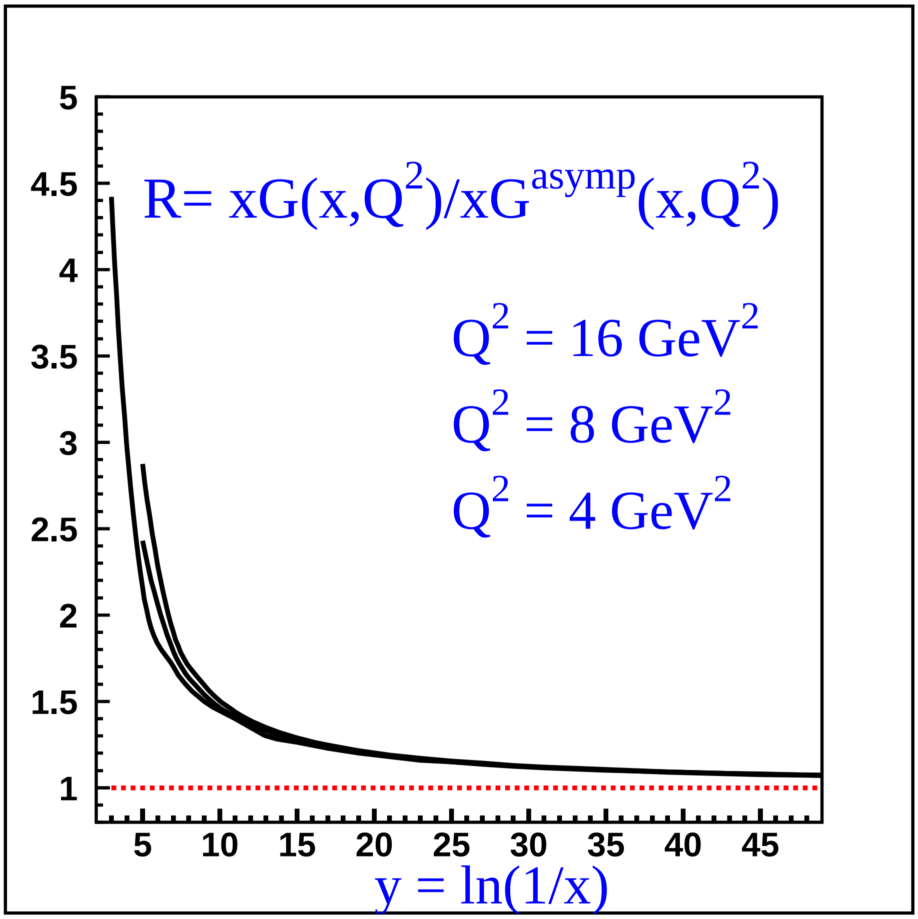
$$\sigma_{tot}(dipole) = 2 \int d^2 b_t\, N(z = \ln(Q_s^2(y; b_t) \; - \; \xi)$$

$$= 2 \int d^2 b_t\, N(z = \ln(Q_s^2(y; b_t = 0) \; - 2 m_\pi\, b_t \; - \; \xi)$$

$$\Longrightarrow |_{x \to 0} \quad \frac{2\pi}{3(2m_\pi)^2} \ln^3(Q_s^2(x)/k^2) \; \propto \; \boxed{\frac{2\pi}{3(2m_\pi)^2} \ln^3(1/x)}$$

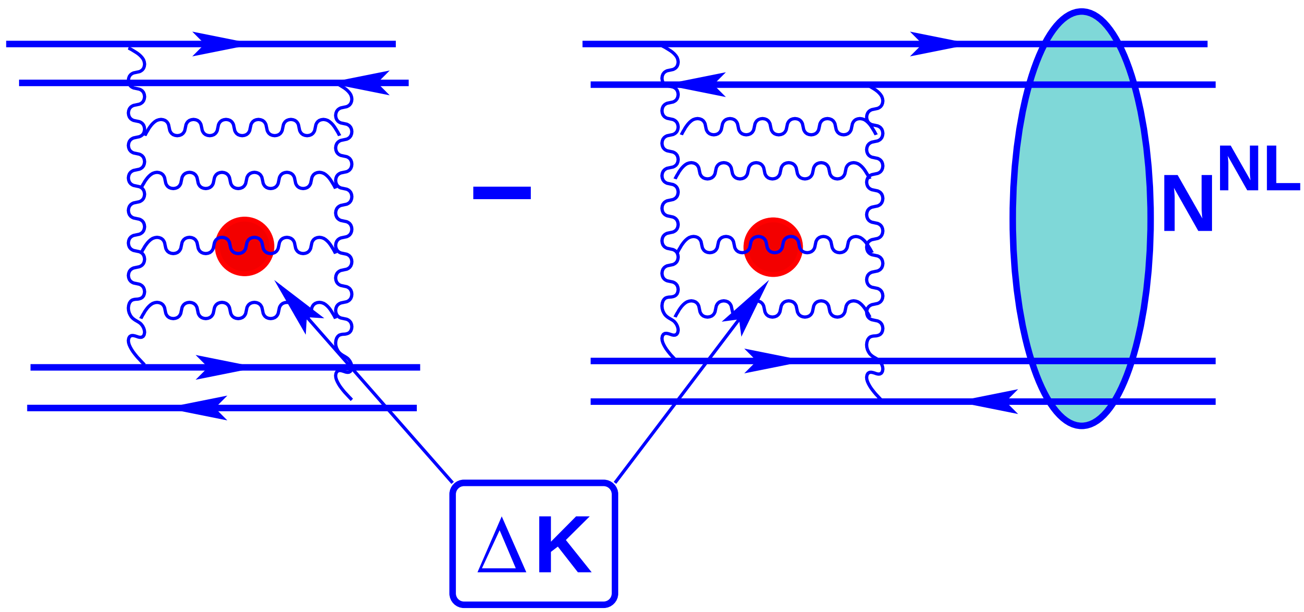
$$xG(x, Q^2) = \frac{4 N_c \pi}{\alpha_S} \int^{Q^2} d\,k^2 \sigma \propto \frac{Q^2}{(2m_\pi)^2} \ln^3 Q_s^2(x)$$

$$\Longrightarrow \frac{Q^2}{(2m_\pi)^2} \ln^3 (1/x)$$



Why npQCD only in initial condition?

$$K \;=\; K(BFKL) \;+\; \Delta K(npQCD)$$



Conclusions.

$$\alpha_S = \text{Const}$$

In the saturation region

$$N\left(r_t^2 Q_s^2(x;b_t)\right) = N\left(r_t^2 Q_s^2(x) e^{-2\,m_\pi\,b_t}\right)$$

$$\sigma_{dipole}(r_t,x) \leq \frac{2\,\pi}{(2\,m_\pi)^2}\,\ln^2\left(Q_s^2(x)\,r_t^2\right)$$

$$\sigma_{dipole}(r_t, x) \;\leq\; \frac{2\,\pi}{(2\,m_\pi)^2}\left(\alpha_S \frac{\chi(\gamma_{cr})}{1-\gamma_{cr}}\,\ln(1/x)\right)^2$$

BUT NOT

$$\sigma_{dipole}(r_t, x) \;\leq \frac{2\,\pi}{(2\,m_\pi)^2}\left(\alpha_S \chi(\frac{1}{2})\,\ln(1/x)\right)^2$$

E.L. & Ryskin (1990);
Ferreiro, Iancu, Itamura & McLerran (2002)

Running α_S

1. $\frac{d\,\xi}{d\,t} = F_\gamma = -\,\hat{\alpha}_S(\xi)\,\frac{d\,\chi(\gamma)}{d\,\gamma}$

2. $\frac{d\,y}{d\,t} = F_\omega = 1\;;$

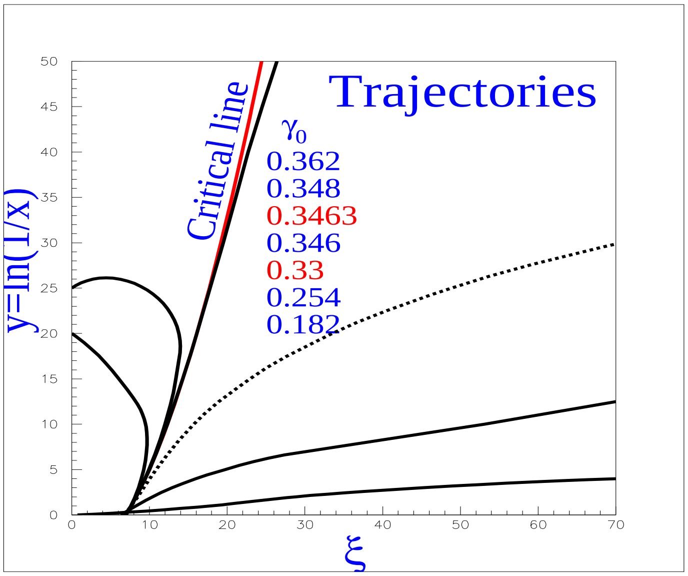
3. $\frac{d\,S}{d\,t} = \gamma\,F_\gamma + \omega\,F_\omega = \hat{\alpha}_S(\xi)\,(1-\gamma)\,\frac{d\,\chi(\gamma)}{d\,\gamma} + \omega\;;$

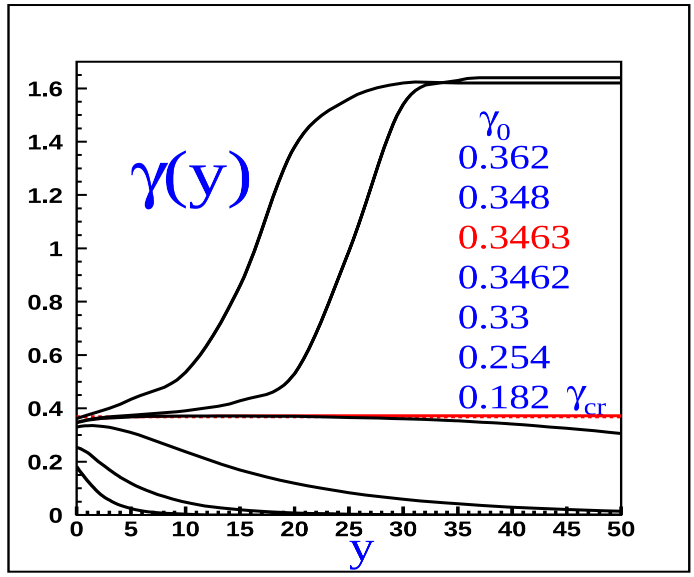
4. $\frac{d\,\gamma}{d\,t} = -\,(F_\xi + \gamma\,F_S)$

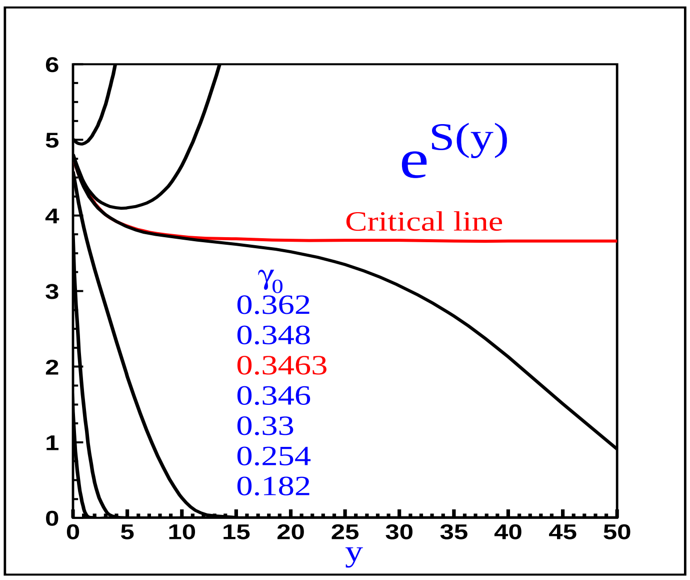
$= \hat{\alpha}_S^2(\xi)\,(\,1-\gamma\,)\,e^S - \hat{\alpha}_S^2(\xi)\,\frac{b}{4\,\pi}\,\chi(\gamma) + \hat{\alpha}_S^3(\xi)\,\frac{b}{2\pi}\,e^S\;;$

5. $\frac{d\,\omega}{d\,t} = -\,(F_y + \omega\,F_S) = -\,\hat{\alpha}_S^2(\xi)\,\omega\,e^S\;;$

Running α_S







Saturation momentum:

$$(\xi - \bar{\xi}(k', b_t))^2 - (\xi_0(b_t) - \bar{\xi}(k', b_t))^2 = \frac{4\,\pi}{b}\,\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}\,(y - y_0)\ ,$$

Running α_S

$$Q_s^2(y, b_t) \;=\; \Lambda^2 \, e^{\sqrt{(\,\xi_0(b_t)\;-\;\bar{\xi}(k', b_t)\,)^2 \;+\; \frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y - y_0\,)}}$$

Q_s **does not depend on** b_t **for**

$$b_t \;\geq\; (1/2m_\pi)\,\sqrt{\frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y\;-\;y_0\,)}$$

BUT for

$$b_t \;\geq\; (1/2m_\pi)\,\sqrt{\frac{4\,\pi}{b}\chi'_{\gamma_{cr}}(\gamma_{cr})\,(y\;-\;y_0\,)}$$

$$Q_s^2 \longrightarrow e^{-2\,m_\pi\,b_t}$$

$$\sigma_{dipole}(r_t, x) \;\leq\; \frac{2\,\pi}{(2\,m_\pi)^2}\;\ln^2\left(Q_s^2(x)\,r_t^2\right)$$

$$\sigma_{dipole}(r_t, x) \;\leq\; \frac{2\,\pi}{(2\,m_\pi)^2}\left(\frac{4\,\pi\,\chi(\gamma_{cr})}{b\,(1-\gamma_{cr})}\;\ln(1/x)\right)$$

BUT NOT

$$\sigma_{dipole}(r_t, x) \;\leq\; \frac{2\,\pi}{(2\,m_\pi)^2}\left(\alpha_S\,\chi(\frac{1}{2})\;\ln(1/x)\right)^2$$

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