Deep-inelastic structure functions at next-to-next-to-leading order QCD

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1. **Motivation**
2. **First Results**
3. **Summary**

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Motivation

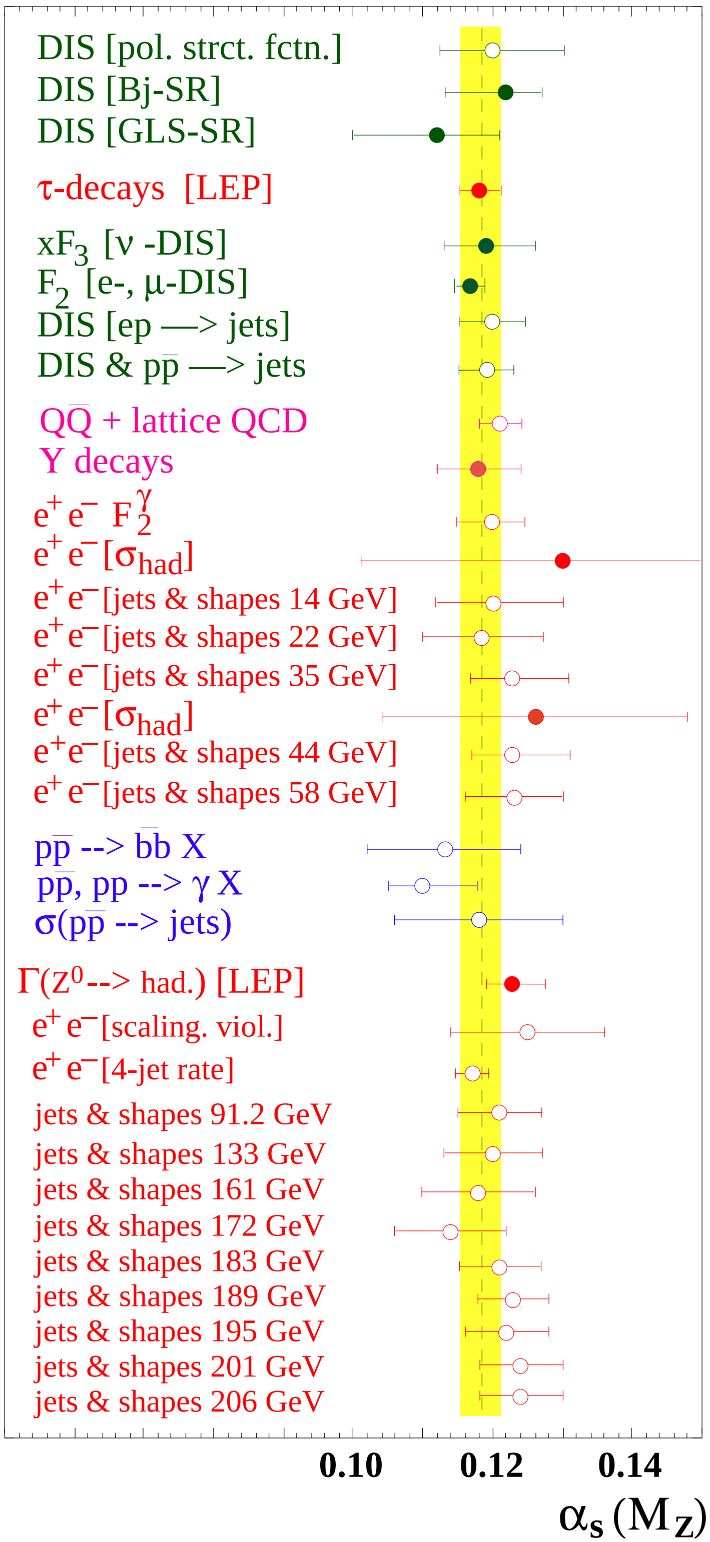
Structure of the proton

– Knowlegde on (un)-polarized parton distributions ?

– **How precise ?**

α_s from DIS

– **How precise ?**



Recent determinations of α_s

Bethke `hep-ex/0211012`

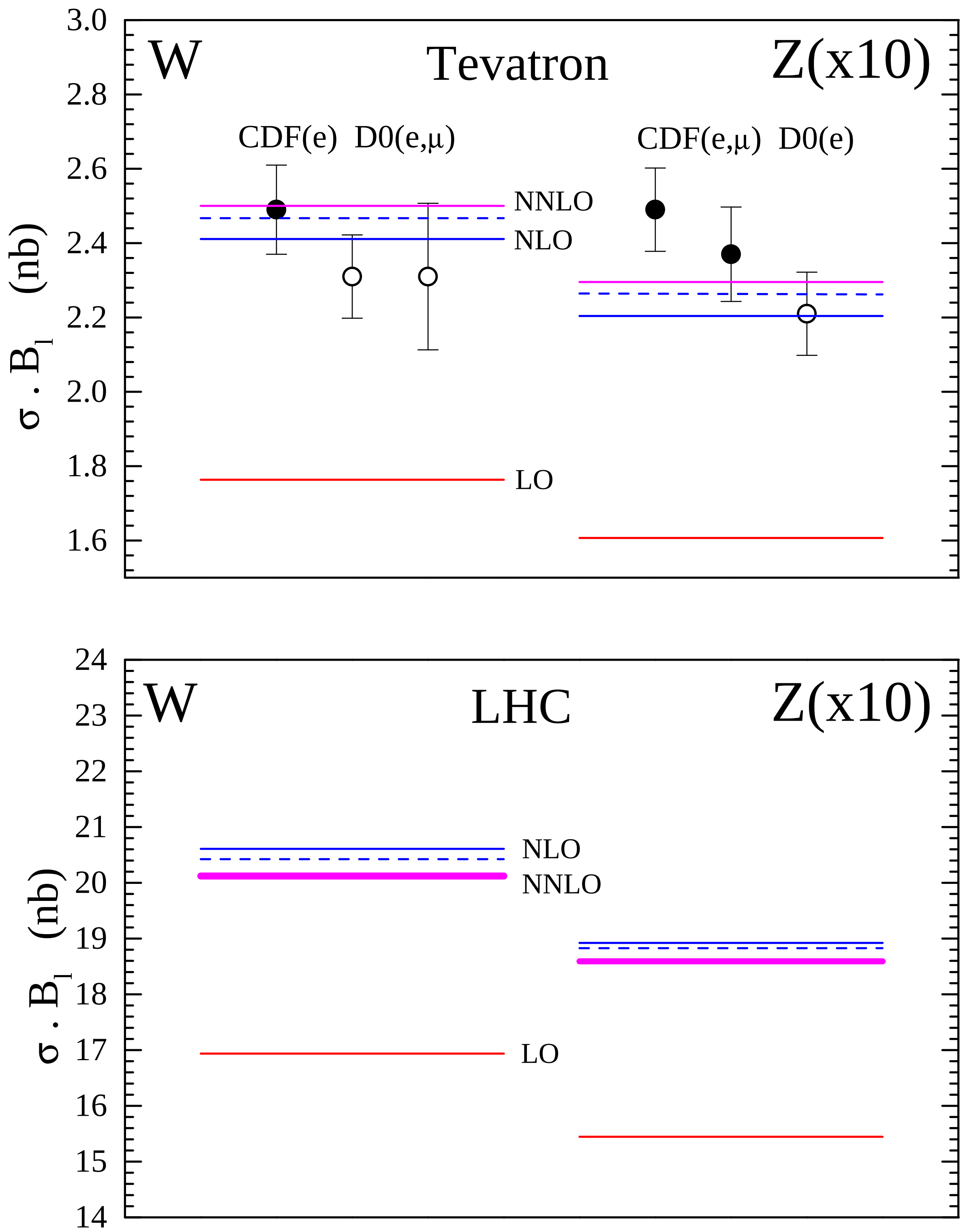
– **NLO** QCD analysis of HERA data for $F_2(x, Q^2)$
H1 coll. `hep-ph/0012053`

$$\alpha_s(M_Z^2) = 0.115 \pm 0.002(\mathrm{exp}) \pm 0.005(\mathrm{theo})$$

Future

– **NNLO** QCD analysis of HERA data for $F_2(x, Q^2)$ in 2006

$$\alpha_s(M_Z^2) = x \pm 0.001(\mathrm{exp}) \pm 0.001(\mathrm{theo})$$



Drell-Yan process

– Cross section at **NNLO**

Matsuura, van der Marck, van Neerven '88 ; Hamberg, van Neerven, Matsuura '91 ;

Kilgore, Harlander '01

– Use approximate **NNLO** PDF's

Martin, Roberts, Stirling, Thorne '02

based on approximate three-loop splitting functions

Vogt, van Neerven '01

fitted from fixed Mellin moments

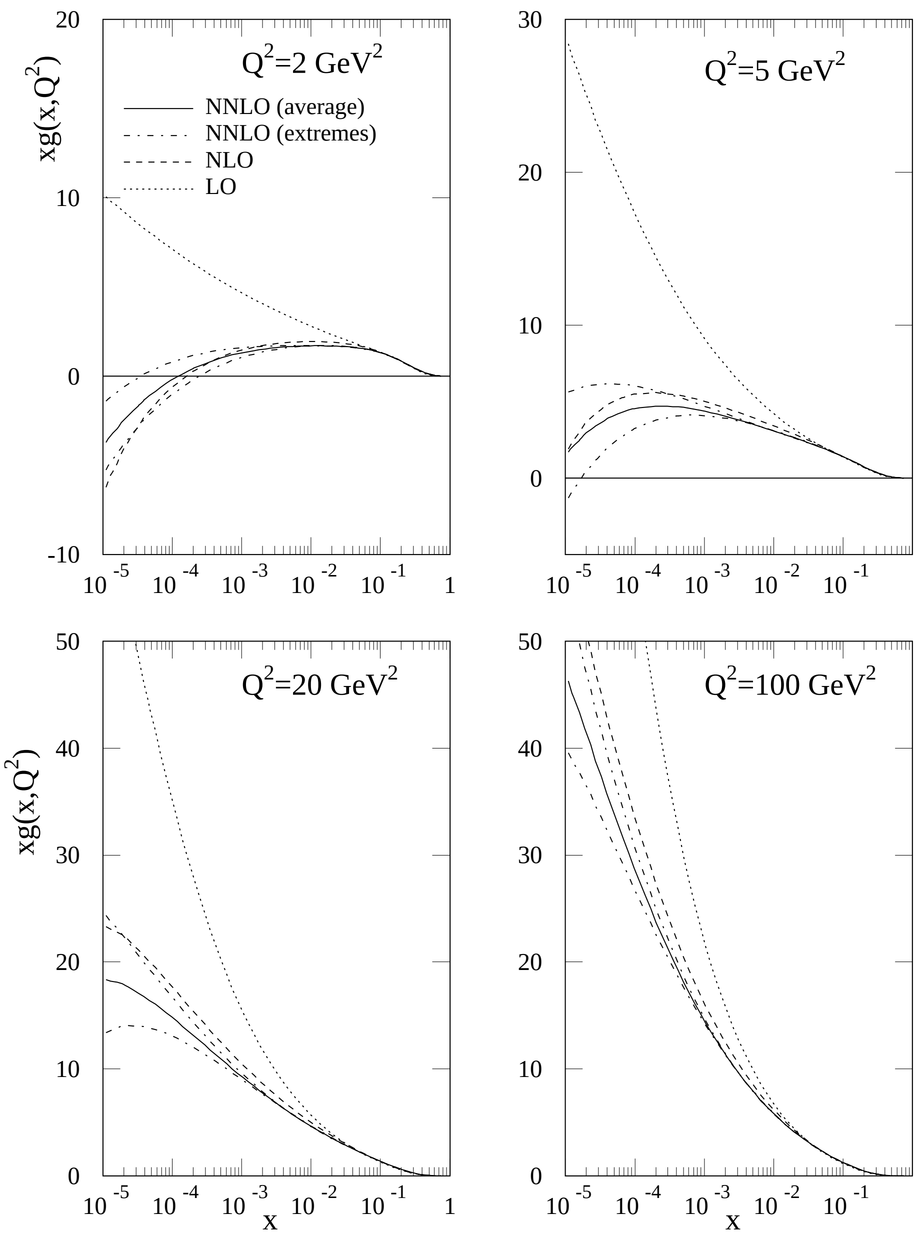
Larin, Nogueira, van Ritbergen, Vermaseren '97 ;

Retey, Vermaseren '00

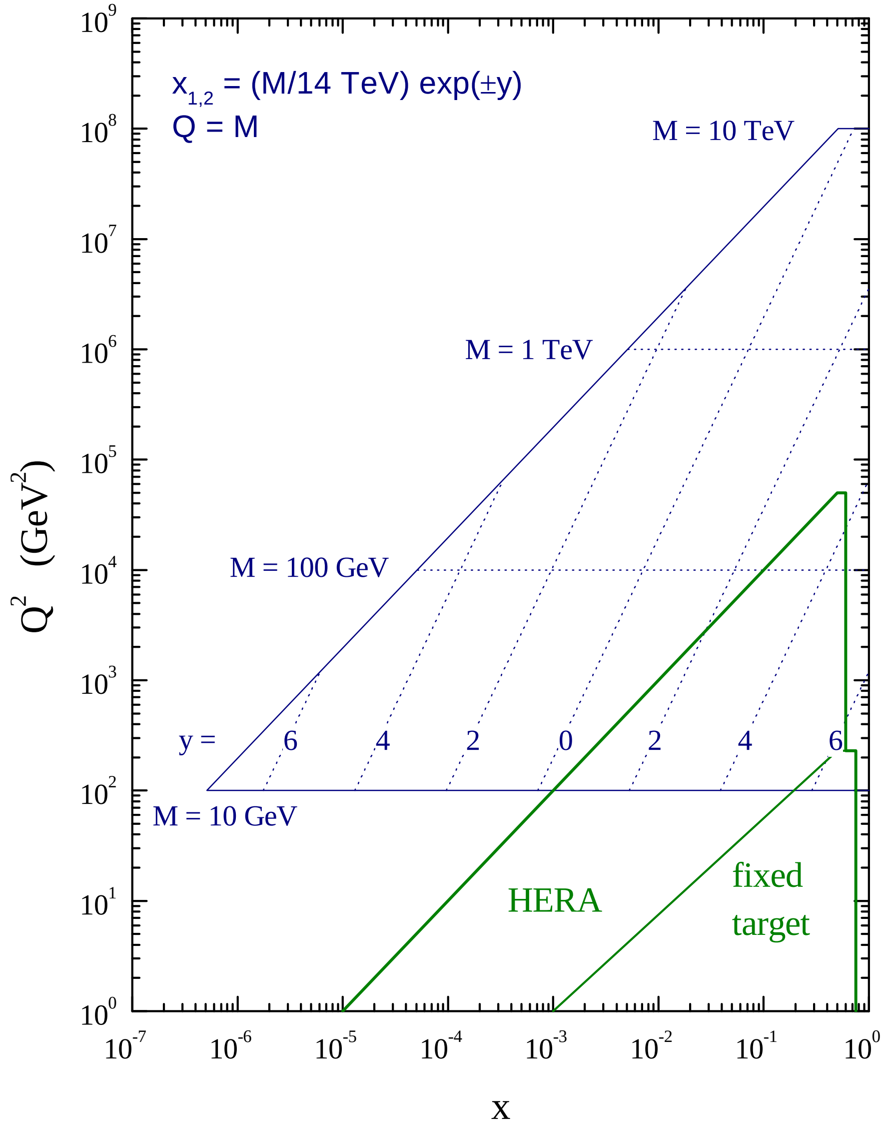
[figure from Stirling @ NuFact '02]

The gluon distribution

– **Is it precise enough for LHC ?**

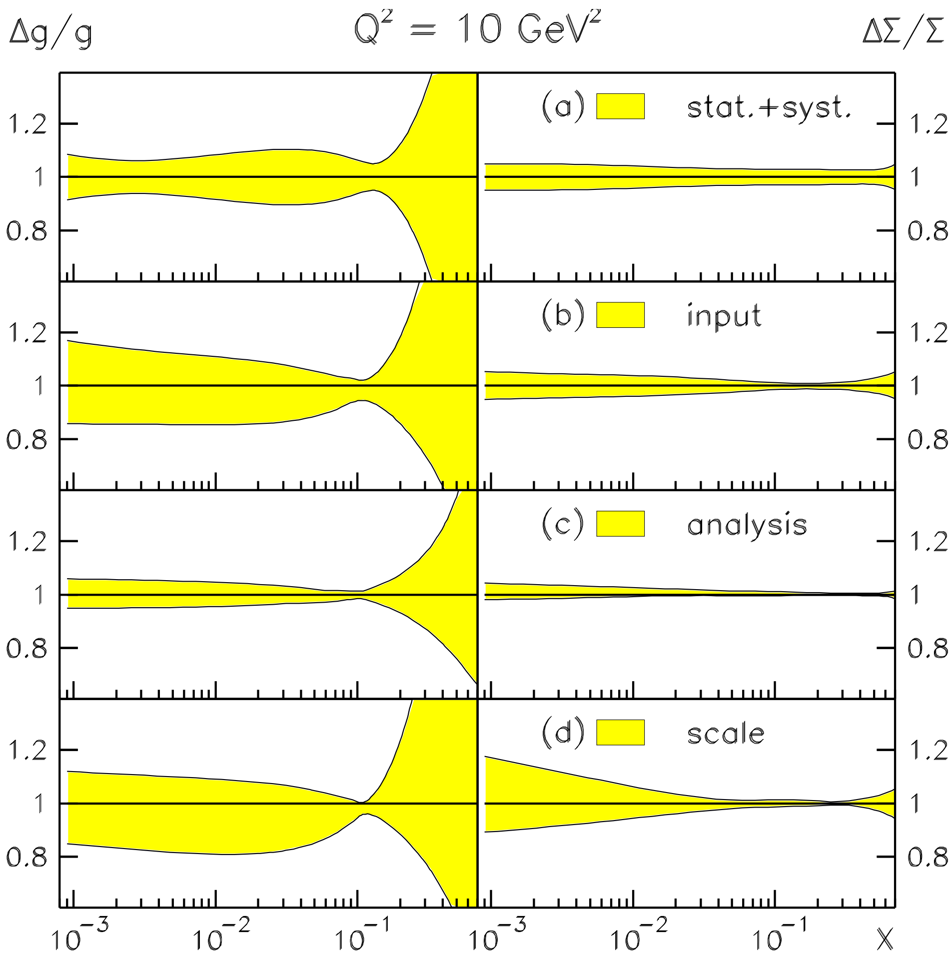


LHC parton kinematics



Martin, Roberts, Stirling, Thorne '02

PDF uncertainties at NLO



Botje '00

– Analysis 1999 DIS data with errors
Botje '00

– Scale variation

$$Q/\sqrt{2} \leq \mu \leq \sqrt{2}Q$$

– Gluons (g) :
stat. $\oplus$ syst. $\simeq$ input $\simeq$ scale error

– Quarks (Σ) :
scale error already dominates

Upshot

– NNLO needed
$\longrightarrow$ three-loop splitting functions

First results

Structure functions in Mellin space

– Use optical theorem and Operator product expansion to calculate Mellin moments

$$\int_0^1 dx\, x^{N-2} F_2(x,Q^2) \;=\; \sum_{i=\mathrm{ns,q,g}} C_{2,i}^N\left(\frac{Q^2}{\mu^2},\alpha_s\right) A_{\mathrm{P},N}^i(\mu^2)$$

– $C_{2,i}^N \longrightarrow$ coefficient functions in Mellin space
– $A_{\mathrm{P},N}^i \longrightarrow$ matrix element s of operators O^i of leading twist
– Scale dependence of renormalized operators $\longrightarrow$ anomalous dimensions $\gamma(\alpha_s,N)$

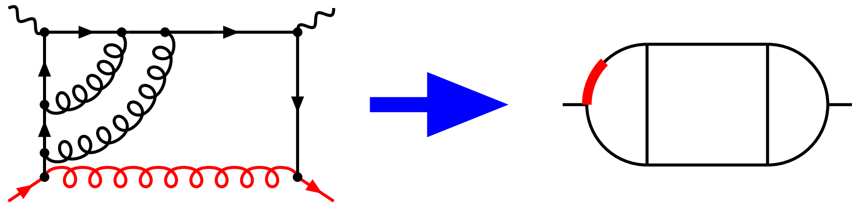
$$\frac{d}{d\ln\mu^2} O^{\mathrm{ren}} = -\gamma\, O^{\mathrm{ren}}$$

– **Upshot** : Mellin moments of F_2, F_3 and $F_L \longleftrightarrow$ parameters of OPE

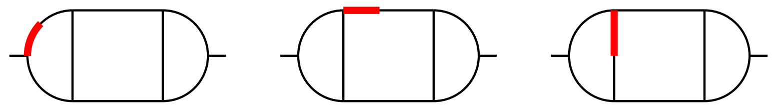
Task

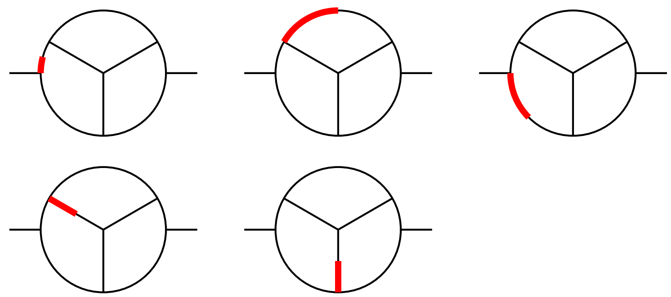
– Calculate coefficient functions C_2^N, C_3^N, C_L^N and anomalous dimensions γ in perturbative QCD

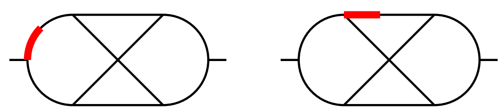
From integrals to Mellin moments



- Basic topologies with one p-dependent line
  - ladder

    
  - benz

    
  - nonplanar

    
- Solve integrals of given topology with algebraic relations
  $\longrightarrow$ map complicated diagrams onto simpler ones
- Recursion relations $\longrightarrow$ n-step difference equations

Mathematics of harmonic sums

– Harmonic sums $S_j(N)$ Gonzalez-Arroyo, Lopez, Ynduráin '79 ; Vermaseren '98 ; Blümlein, Kurth '98

– Recursive definition $S_{m_1,\ldots,m_k}(N) = \sum_{i=1}^{N} \frac{1}{i^{m_1}} S_{m_2,\ldots,m_k}(i)$

– Multiple polylogarithms
Goncharov '98 ; Borwein, Bradley, Broadhurst, Lisonek '99 ; Remiddi, Vermaseren '99

– Algebra of multiplication $S_j(N)S_k(N) \longrightarrow S_{\{j,k\}}(N)$

Recursion relations and algebra of harmonic sums $\longrightarrow$ Breakthrough in technology

Anomalous dimensions in Mellin space

– One-loop : Gross, Wilczek ’73

$$\gamma_{\mathrm{ns}}^{(0)}(N) \quad = \quad C_F\big(2(\mathbf{N}_- + \mathbf{N}_+)S_1 - 3\big)$$

– Two-loop : Floratos, Ross, Sachrajda ‘79 ; Gonzalez-Arroyo, Lopez, Ynduráin ‘79

$$\gamma_{\mathrm{ns}}^{(1)}(N) \;=\; 4C_AC_F\left(2\,\mathbf{N}_+S_3 - \frac{17}{24} - 2S_{-3} - \frac{28}{3}S_1 + (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{151}{18}S_1 + 2S_{1,-2} - \frac{11}{6}S_2\right]\right)$$
$$+4C_Fn_f\left(\frac{1}{12} + \frac{4}{3}S_1 - (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{11}{9}S_1 - \frac{1}{3}S_2\right]\right) + 4{C_F}^2\left(4S_{-3} + 2S_1 + 2S_2 - \frac{3}{8}\right.$$
$$\left.+\mathbf{N}_-\Big[S_2 + 2S_3\Big] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1 + 4S_{1,-2} + 2S_{1,2} + 2S_{2,1} + S_3\Big]\right)$$

– Compact notation :

$$\mathbf{N}_\pm\, f(N) \;=\; f(N \pm 1)\;, \qquad \mathbf{N}_{\pm\mathbf{i}}\, f(N) \;=\; f(N \pm i)$$

– Three-loop : fermionic contributions to nonsinglet anomalous dimension

$$
\begin{aligned}
\gamma_{\rm ns}^{(2)}(N) \;=\; & 16 C_A C_F n_f \Bigg(\frac{3}{2}\zeta_3 - \frac{5}{4} + \frac{10}{9}S_{-3} - \frac{10}{9}S_3 + \frac{4}{3}S_{1,-2} - \frac{2}{3}S_{-4} + 2S_{1,1} - \frac{25}{9}S_2 + \frac{257}{27}S_1 \\
& - \frac{2}{3}S_{-3,1} - \mathbf{N}_+\left[S_{2,1} - \frac{2}{3}S_{3,1} - \frac{2}{3}S_4\right] + (1-\mathbf{N}_+)\left[\frac{23}{18}S_3 - S_2\right] - (\mathbf{N}_- + \mathbf{N}_+)\Big[S_{1,1} + \frac{1237}{216}S_1 \\
& + \frac{11}{18}S_3 - \frac{317}{108}S_2 + \frac{16}{9}S_{1,-2} - \frac{2}{3}S_{1,-2,1} - \frac{1}{3}S_{1,-3} - \frac{1}{2}S_{1,3} - \frac{1}{2}S_{2,1} - \frac{1}{3}S_{2,-2} + S_1\zeta_3 + \frac{1}{2}S_{3,1}\Big]\Bigg) \\
& + 16 C_F {n_f}^2 \left(\frac{17}{144} - \frac{13}{27}S_1 + \frac{2}{9}S_2 + (\mathbf{N}_- + \mathbf{N}_+)\left[\frac{2}{9}S_1 - \frac{11}{54}S_2 + \frac{1}{18}S_3\right]\right) + 16 {C_F}^2 n_f \Bigg(\frac{23}{16} - \frac{3}{2}\zeta_3 \\
& + \frac{4}{3}S_{-3,1} - \frac{59}{36}S_2 + \frac{4}{3}S_{-4} - \frac{20}{9}S_{-3} + \frac{20}{9}S_1 - \frac{8}{3}S_{1,-2} - \frac{8}{3}S_{1,1} - \frac{4}{3}S_{1,2} + \mathbf{N}_+\Big[\frac{25}{9}S_3 - \frac{4}{3}S_{3,1} \\
& - \frac{1}{3}S_4\Big] + (1-\mathbf{N}_+)\left[\frac{67}{36}S_2 - \frac{4}{3}S_{2,1} + \frac{4}{3}S_3\right] + (\mathbf{N}_- + \mathbf{N}_+)\Big[S_1\zeta_3 - \frac{325}{144}S_1 - \frac{2}{3}S_{1,-3} + \frac{32}{9}S_{1,-2} \\
& - \frac{4}{3}S_{1,-2,1} + \frac{4}{3}S_{1,1} + \frac{16}{9}S_{1,2} - \frac{4}{3}S_{1,3} + \frac{11}{18}S_2 - \frac{2}{3}S_{2,-2} + \frac{10}{9}S_{2,1} + \frac{1}{2}S_4 - \frac{2}{3}S_{2,2} - \frac{8}{9}S_3\Big]\Bigg)
\end{aligned}
$$

Splitting functions in x-space

$$
\begin{aligned}
P_{\mathrm{ns}}^{(2)}(x) \;=\; & 16 C_A C_F n_f \Big(p_{\mathrm{qq}}(x) \Big[\frac{5}{9}\zeta_2 - \frac{209}{216} - \frac{3}{2}\zeta_3 + \frac{1}{3}\mathrm{Li}_3(x) - \frac{167}{108}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 - \frac{1}{4}\ln^2(x)\ln(1-x) \\
& - \frac{7}{12}\ln^2(x) - \frac{1}{18}\ln^3(x) - \frac{1}{2}\ln(x)\mathrm{Li}_2(x) \Big] + p_{\mathrm{qq}}(-x) \Big[\frac{1}{2}\zeta_3 - \frac{5}{9}\zeta_2 - \frac{2}{3}\ln(1+x)\zeta_2 + \frac{1}{6}\ln(x)\zeta_2 - \frac{10}{9}\ln(x)\ln(1+x) \\
& + \frac{5}{18}\ln^2(x) - \frac{1}{6}\ln^2(x)\ln(1+x) + \frac{1}{18}\ln^3(x) - \frac{10}{9}\mathrm{Li}_2(-x) - \frac{1}{3}\mathrm{Li}_3(-x) - \frac{1}{3}\mathrm{Li}_3(x) + \frac{2}{3}\mathrm{H}_{-1,0,1}(x) \Big] \\
& + (1+x)\Big[\frac{1}{6}\zeta_2 + \frac{1}{2}\ln(x) - \frac{1}{2}\mathrm{Li}_2(x) - \frac{2}{3}\mathrm{Li}_2(-x) - \frac{2}{3}\ln(x)\ln(1+x) + \frac{1}{24}\ln^2(x) \Big] + (1-x)\Big[\frac{1}{3}\zeta_2 - \frac{257}{54} \\
& + \ln(1-x) - \frac{17}{9}\ln(x) - \frac{1}{24}\ln^2(x) \Big] + \delta(1-x)\Big[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}{\zeta_2}^2 + \frac{25}{18}\zeta_3 \Big] \Big) + 16 C_F {n_f}^2 \Big(p_{\mathrm{qq}}(x) \Big[-\frac{1}{54} \\
& + \frac{5}{54}\ln(x) + \frac{1}{36}\ln^2(x) \Big] + (1-x)\Big[\frac{13}{54} + \frac{1}{9}\ln(x) \Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3 \Big] \Big) + 16 {C_F}^2 n_f \Big(p_{\mathrm{qq}}(x) \Big[\frac{5}{3}\zeta_3 - \frac{55}{48} \\
& - \frac{2}{3}\mathrm{Li}_3(x) + \frac{5}{24}\ln(x) + \frac{1}{3}\ln(x)\zeta_2 + \frac{10}{9}\ln(x)\ln(1-x) + \frac{1}{4}\ln^2(x) + \frac{2}{3}\ln^2(x)\ln(1-x) + \frac{2}{3}\ln(x)\mathrm{Li}_2(x) - \frac{1}{18}\ln^3(x) \Big] \\
& + p_{\mathrm{qq}}(-x)\Big[\frac{10}{9}\zeta_2 - \zeta_3 + \frac{4}{3}\ln(1+x)\zeta_2 - \frac{1}{3}\ln(x)\zeta_2 - \frac{5}{9}\ln^2(x) + \frac{20}{9}\ln(x)\ln(1+x) - \frac{1}{9}\ln^3(x) + \frac{1}{3}\ln^2(x)\ln(1+x) \\
& + \frac{20}{9}\mathrm{Li}_2(-x) + \frac{2}{3}\mathrm{Li}_3(-x) + \frac{2}{3}\mathrm{Li}_3(x) - \frac{4}{3}\mathrm{H}_{-1,0,1}(x) \Big] + (1+x)\Big[\frac{7}{36}\ln^2(x) - \frac{67}{72}\ln(x) + \frac{4}{3}\ln(x)\ln(1+x) \\
& + \frac{1}{12}\ln^3(x) + \frac{2}{3}\mathrm{Li}_2(x) + \frac{4}{3}\mathrm{Li}_2(-x) \Big] + (1-x)\Big[\frac{1}{9}\ln(x) - \frac{10}{9} - \frac{4}{3}\ln(1-x) + \frac{2}{3}\ln(x)\ln(1-x) - \frac{1}{3}\ln^2(x) \Big] \\
& - \delta(1-x)\Big[\frac{23}{16} - \frac{5}{12}\zeta_2 - \frac{29}{30}{\zeta_2}^2 + \frac{17}{6}\zeta_3 \Big] \Big)
\end{aligned}
$$

Easy-to-use parametrization

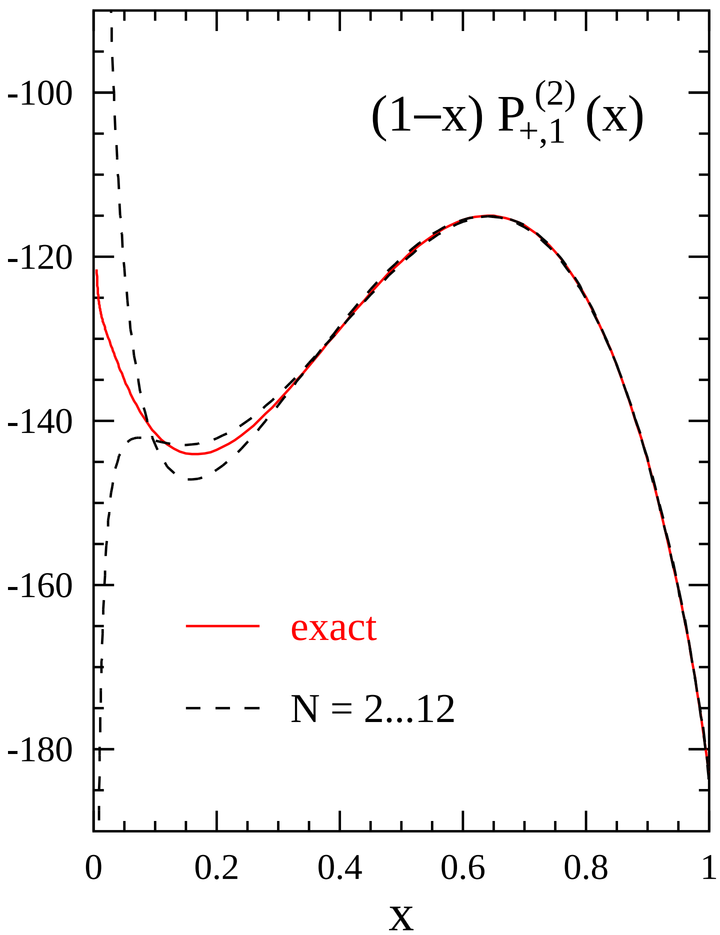
– Combine exact limits for $x \longrightarrow 0$ and $x \longrightarrow 1$ with smooth fit for intermediate x

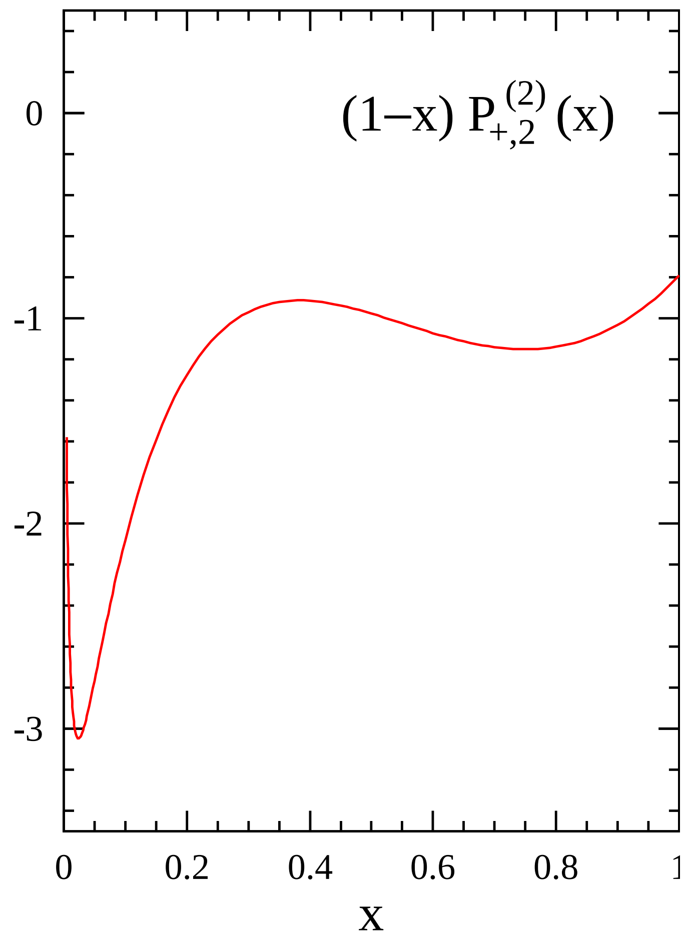
– Notation : end-point logarithms $L_0 = \ln(x)$, $L_1 = \ln(1-x)$, +-distributions $\mathcal{D}_i = \left[\frac{\ln^i(1-x)}{1-x}\right]_+$

$$
\begin{aligned}
P_{\mathrm{ns}}^{(2)}(x) \quad \cong \quad & n_f \Big(-183.187\,\mathcal{D}_0 - 173.927\,\delta(1-x) - 5120/81\,L_1 - 197.0 + 381.1\,x + 72.94\,x^2 \\
& \quad + 44.79\,x^3 - 1.497\,xL_0^3 - 56.66\,L_0L_1 - 152.6\,L_0 - 2608/81\,L_0^2 - 64/27\,L_0^3\Big) \\
+ \quad & n_f^2 \Big(-\mathcal{D}_0 - (51/16 + 3\,\zeta_3 - 5\,\zeta_2)\,\delta(1-x) + x\,(1-x)^{-1}L_0\,(3/2\,L_0 + 5) + 1 \\
& \quad + (1-x)\,(6 + 11/2\,L_0 + 3/4\,L_0^2)\Big)\ 64/81
\end{aligned}
$$

Comparison with estimates from fixed moments

van Neerven, Vogt ‘00





Three-loop coefficient functions

- OPE and optical theorem
  $\longrightarrow$ calculate simultaneously anomalous dimension $\gamma_{\rm ns}(N)$ and coefficient function $C^N_{2,\rm ns}$

- Three-loop coefficient functions are numerically the most relevant part of **NNNLO** corrections

- Easy-to-use parametrization

$$
\begin{aligned}
c^{(3)}_{2,\rm ns}(x) \quad \cong \quad & n_f \big(640/81\, \mathcal{D}_4 - 6592/81\, \mathcal{D}_3 + 220.573\, \mathcal{D}_2 + 294.906\, \mathcal{D}_1 - 729.359\, \mathcal{D}_0 \\
& \quad + 2572.597\, \delta(1-x) - 640/81\, L_1^4 + 167.2\, L_1^3 - 315.3\, L_1^2 + 4742\, L_1 \\
& \quad + 762.1 + 7020\, x + 989.4\, x^2 + L_0 L_1\, (326.6 + 65.93\, L_0 + 1923\, L_1) \\
& \quad + 260.1\, L_0 + 186.5\, L_0^2 + 12224/243\, L_0^3 + 728/243\, L_0^4 \big) \\
+ \quad & n_f^2 \big(64/81\, \mathcal{D}_3 - 464/81\, \mathcal{D}_2 + 7.67505\, \mathcal{D}_1 + 1.00830\, \mathcal{D}_0 - 103.2655\, \delta(1-x) \\
& \quad - 64/81\, L_1^3 + 15.46\, L_1^2 - 51.71\, L_1 + 59.00\, x + 70.66\, x^2 + L_0 L_1\, (-80.05 \\
& \quad - 10.49\, L_0 + 41.67\, L_1) - 8.050\, L_0 - 1984/243\, L_0^2 - 368/243\, L_0^3 \big)
\end{aligned}
$$

Applications to threshold resummation

– Coefficient function $C_{2,\mathrm{ns}}$
$\longrightarrow$ large double logarithmic corrections for $N \to \infty$ cf. $x \to 1$

$$\alpha_s^l \left[\frac{\ln^{2l-1}(1-x)}{1-x}\right]_+ \qquad \longleftrightarrow \qquad \alpha_s^l \ln^{2l}(N)$$

– Large double logarithms originate from soft and collinear regions in Feynman diagrams

– Use factorization properties in soft/collinear limit $\longrightarrow$ **resummation** of large logarithms
Collins, Soper '81 ; Sterman '87 ; Catani, Trentadue '89 ; Magnea, Sterman '90 ; Catani, Webber '98 ; etc. ...

$$C_{2,\mathrm{ns}}(\alpha_s, N) = g(Q^2) \exp[G_{\mathrm{DIS}}(\alpha_s, N)] =$$

$$(1 + \alpha_s g_{01} + \alpha_s^2 g_{02} + \ldots) \exp\left[\ln(N)\, g_1(\alpha_s \ln(N)) + g_2(\alpha_s \ln(N)) + \alpha_s g_3(\alpha_s \ln(N)) + \ldots\right]$$

Physical picture

– Functions g_l depend on coefficients $A_{i\leq l}$, $B_{i\leq l-1}$ and $D^{\mathrm{DIS}}_{i\leq l-1}$

– Factorization in soft and collinear limit
  - soft gluons collinear to initial state partons $\longrightarrow$ universal A_i
  - soft gluons collinear to final state partons $\longrightarrow$ universal B_i
  - soft gluons at large angles $\longrightarrow$ process-dependent D^{DIS}_i

Resummation of the next-to-next-to-leading logarithm

– Resumming to NNLL accuracy requires $g_3 \longrightarrow$ new coefficients A_3, B_2 and D^{DIS}_2
  - $g_1 \longrightarrow A_1$
  - $g_2 \longrightarrow A_1, A_2, B_1, D^{\mathrm{DIS}}_1$
  - $g_3 \longrightarrow A_1, A_2, A_3, B_1, B_2, D^{\mathrm{DIS}}_1, D^{\mathrm{DIS}}_2$

Matching

– Fermionic contributions to A_3 from $\gamma_{\rm ns}^{(2)}$
  – independent check Berger '02

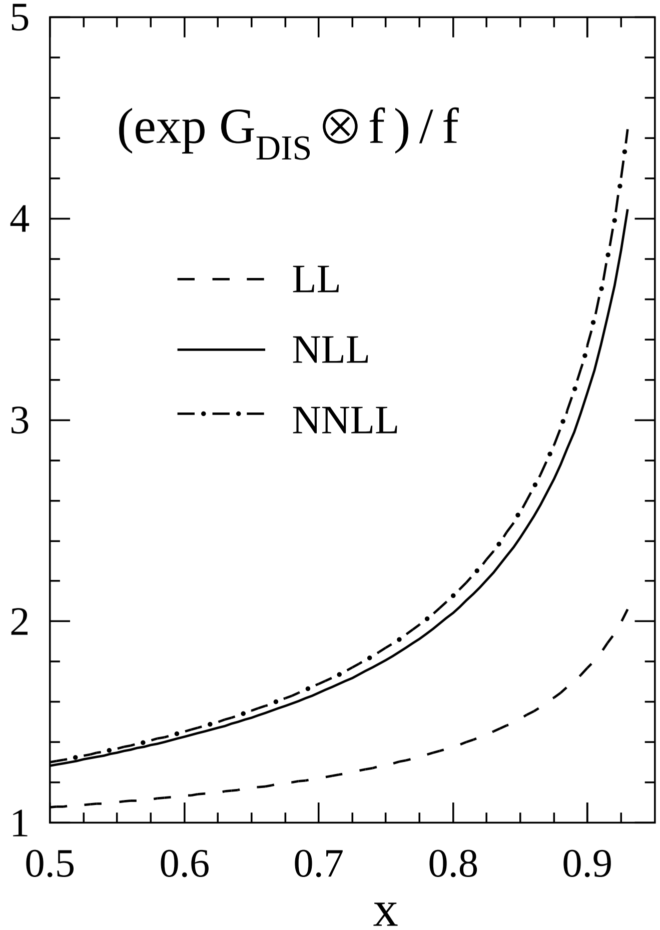
$$A_3\Big|_{n_f} = C_A C_F n_f \left[-\frac{836}{27}+\frac{160}{9}\zeta_2-\frac{112}{3}\zeta_3\right]+C_F^2 n_f\left[-\frac{110}{3}+32\,\zeta_3\right]+C_F n_f^2\left[-\frac{16}{27}\right]$$

– Determine $B_2, D_2^{\rm DIS}$ with help of three-loop coefficient function
  – $B_2 + D_2^{\rm DIS}$ previously known
  – $c_{2,\rm ns}^{(3)}$ involves $\beta_0(B_2+2\,D_2^{\rm DIS})$

$$\begin{aligned} B_2 &= C_F^2\left[-\frac{3}{2}-24\,\zeta_3+12\,\zeta_2\right]+C_F C_A\left[-\frac{3155}{54}+40\,\zeta_3+\frac{44}{3}\zeta_2\right]+C_F n_f\left[\frac{247}{27}-\frac{8}{3}\zeta_2\right] \\ D_2^{\rm DIS} &= 0 \end{aligned}$$

– Remarkable result $D_1^{\rm DIS} = D_2^{\rm DIS} = 0$
  – $D_i^{\rm DIS} = 0$ to all orders from renormalization group arguments Forte, Ridolfi '03

Numerical analysis



Vogt '01

– Convolution of resummed coefficient function $C_{2,\mathrm{ns}}$ with typical input distribution

– Put $\mu = Q$ and $\alpha_s = 0.2$

– Use numerical estimate for unknown A_3
Vogt '01

$$A_3 = (1178.8 \pm 11.5) - 183.18743\, n_f - \frac{64}{81} n_f^2$$

Upshot : large corrections for $x \to 1$
use **NNLO + NNLL resummed** perturbative QCD

Higher twist ? $F_2^{\mathrm{DATA}} = F_2^{\mathrm{QCD}} \left(1 + \frac{ht(x)}{Q^2} \right)$

Summary

Summary

- NNLO analysis of deep-inelastic structure functions $F_2, F_3 \longrightarrow$ **high precision**
  - match experimental accuracy in final HERA data
  - precise parton luminosity at LHC

- Mellin moments and nested sums $\longrightarrow$ **powerful technology**
  - innovative and efficient method to solve multi-loop integrals
    spin-off : $e^+e^- \longrightarrow 3$ jets at **NNLO** (n_f-terms) S.M., Uwer, Weinzierl '02

Outlook

- Complete calculation of deep-inelastic structure functions F_2, F_3 and F_L at **NNLO** (and beyond) S.M., Vermaseren, Vogt