DGLAP and BFKL equations in supersymmetric gauge theories.

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0.1 Introduction

Integrated and unintegrated parton distributions

$$f_a(x,Q^2) \;=\; \int_{k_\perp^2<Q^2} dk_\perp^2\, \varphi_a(x,k_\perp^2)\,.$$

for vector ($a=g$), spinor ($a=q$) and scalar ($a=s$) partons.

The DGLAP equation in the Lorentz spin representation

$$\frac{d}{d\ln Q^2} f_a(j,Q^2) = \sum_b \gamma_{ab}(j) f_b(j,Q^2).$$

$f_a(j,Q^2)$ are the Mellin momenta of parton distributions and $\gamma_{ab}(j)$ is the anomalous dimension matrix for the twist-2 operators.

The BFKL equation for the gluon distributions at small x

$$\frac{d}{d\ln(1/x)}\varphi_g(x,k_\perp^2) \;=\int d^2k'_\perp K(k_\perp,k'_\perp)\varphi_g(x,k_\perp^2)$$

The matrix elements of the local operators $O^a_{\mu_1,\dots,\mu_j}$ and $\tilde{O}^a_{\mu_1,\dots,\mu_j}$ are related to the distributions $f_a(x,Q^2)$ and $\Delta f_a(x,Q^2)$ for unpolarized and polarized partons

$$\begin{aligned}\int_0^1 dx\, x^{j-1} f_a(x,Q^2) \;&=\; <h|\tilde{n}^{\mu_1}\dots\tilde{n}^{\mu_j}\, O^a_{\mu_1,\dots,\mu_j}|h>\,, \qquad a=(q,g,\varphi)\,,\\ \int_0^1 dx\, x^{j-1}\Delta f_a(x,Q^2) \;&=\; <h|\tilde{n}^{\mu_1}\dots\tilde{n}^{\mu_j}\, \tilde{O}^a_{\mu_1,\dots,\mu_j}|h>\,, \qquad a=(q,g)\,. \quad (1)\end{aligned}$$

The vector $\tilde{n}^\mu \sim q^\mu + xp^\mu$ is lightlike $\tilde{n}^2=0$.

Generally one can introduce the mixed projections for this tensor

$$\tilde{n}^{\mu_1}\dots\tilde{n}^{\mu_{1+\omega}}\, O^a_{\mu_1,\dots,\mu_{1+\omega},\sigma_1,\dots,\sigma_{|n|}}\, l_\perp^{\sigma_1}\dots l_\perp^{\sigma_{|n|}},\, (l_\perp,p)=(l_\perp,q)=0,$$

where $|n|$ is conformal spin and $\omega=j-1$ is an eigenvalue of the BFKL kernel

$$\omega=\omega(\nu,|n|)\,,\;\; m=\frac{1}{2}+i\nu+\frac{n}{2},\; \tilde{m}=\frac{1}{2}+i\nu-\frac{n}{2}$$

with the Möbius-group invariant eigenfunction

$$< 0|\phi(\vec{\rho_1})\phi(\vec{\rho_2})O_{|n|,\nu}(\vec{\rho_0})|0 >= (\frac{\rho_{12}}{\rho_{10}\rho_{20}})^m(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*})^{\tilde{m}}.$$

The anomalous dimensions γ do not depend on various projections of $O_{\mu_1,...,\mu_J}$.

0.2 Remarkable properties of the BFKL dynamics in LLA

1. Möbius invariance (L.L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d_k}$$

Eigenvalues of two Casimir operators: $m(1-m)$ and $\tilde{m}(\tilde{m}-1)$ with

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,, \;\; \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}$$

for the principal series of the unitary representations.

2. Holomorphic separability of the BFKL hamiltonian (L.L (1989))

$$\widehat{K}_{12} = -\frac{g^2}{8\pi^2}\, H_{12}\,, \;\; H_{12} = h_{12} + h^*_{12}\,,$$

$$h_{12} = \ln p_1 + \ln p_2 + \frac{1}{p_1 p_2^*}\, \ln \rho_{12}\, p_1 p_2^* + \frac{1}{p_1^* p_2}\, \ln \rho_{12}\, p_1^* p_2 + 2\gamma_E\,, \;\; \rho_{12} = \rho_1 - \rho_2$$

3. Holomorphic factorization of the n-gluon wave function at large N_c (L.L. (1989))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_2^*),$$

$$H = \frac{h + h^*}{2}\,, \;\; h = \sum_{r=1}^{n} h_{r,r+1}\,.$$

4. Duality symmetry in multi-colour QCD (L.L. (1999))

$$\rho_{r,r+1} \to p_r \to \rho_{r-1,r}\,,$$

5. Integrability at large N_c (L.L. (1993)). Integrals of motion

$$[q_r, q_s] = [q_r, h] = 0\,, \quad q_r = \sum_{k_1 < k_2 < ... < k_r} \rho_{k_1 k_2} \rho_{k_2 k_3} ... \rho_{k_{r-1} k_r} \, p_{k_1} p_{k_2} ... p_{k_r}$$

6. Equivalence with the integrable Heisenberg model (L.L. (1994) and L.F., G.K.(1995)) with spins $\vec{M}_k$ being the generators of Möbius group

$$M_k^z = \rho_k \partial_k \,, \; M_k^+ = \partial_k \,, \; M_k^- = -\rho_k^2 \partial_k \,.$$

0.3 Next-to-leading corrections to the BFKL equation in SUSY

The BFKL kernel (V.F., L.L. (1998) and G.C., M.C. (1998) in QCD; A.K., L.L. (2000) in SUSY)

$$K(k_\perp, k'_\perp) = \overline{a}\, K_B(k_\perp, k'_\perp) + \overline{a}^2\, K_{NL}(k_\perp, k'_\perp)\,, \;\; \overline{a} = g^2 N_c/(16\pi^2)\,,$$

$\overline{a}$ is in the $\overline{MS}$ scheme. Eigenvalue $\omega = \omega(n,\gamma)$ in QCD contains non-analytic terms $\delta_{|n|,0}$ and $\delta_{|n|,2}$.

The BFKL singularity is a Mandelstam cut. The corrections from virtual fermions and bosons in SUSY are the same as in QED (up to a colour factor). $\delta_{|n|,2}$ is cancelled in all supersymmetric gauge theories, $\delta_{|n|,0}$ is cancelled only in the $N = 4$ SUSY.

The result for $N = 4$ (A.K. ,L.L. (2000))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\delta(n,\gamma)\,,\;\; \hat{a} = \overline{a} + \frac{1}{3}\overline{a}^2\,,\;\; \gamma = \frac{1}{2} + i\nu\,,$$

$\hat{a}$ is in the dimension reduction scheme, γ is the anomalous dimension.and $\delta(n,\gamma)$ has the property of the hermitian separability (A.K., L.L. (2002))

$$\delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\omega}{2\hat{a}}\Big(\rho(M) + \rho(M^*)\Big),\; M = \gamma + \frac{|n|}{2}$$

and

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\Big[\Psi'(\frac{z+1}{2}) - \Psi'(\frac{z}{2})\Big]\,,$$

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)(\Psi(1) - \Psi(M))\,,$$

$$\Phi(M) = \sum_{k=0}^{\infty}\frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty}\frac{(-1)^k}{k+M}\left(\Psi'(k+1) \;-\; \frac{\Psi(k+1)-\Psi(1)}{k+M}\right)\,.$$

But the property of the holomorphic separability is violated

$$\delta(n,\gamma) = \delta(m)+\delta(\widetilde{m})+(1+(-1)^n)\,\varepsilon(n)\,\pi^2\,\left(\frac{\Psi(m)\,\cos(m\,\pi)}{\sin^2(m\,\pi)} - \frac{\Psi(\widetilde{m})\,\cos(\widetilde{m}\,\pi)}{\sin^2(\widetilde{m}\,\pi)}\right).$$

$$\delta(m) = \frac{\phi(m)+\phi(1-m)}{2} - \frac{\omega}{2\hat{a}}\,\frac{\rho(m)+\rho(1-m)}{2}\,,$$

$$m = \gamma + \frac{n}{2}\,,\;\; \widetilde{m} = \gamma - \frac{n}{2}\,.$$

0.4 Analytic continuation of the eigenvalue to negative $|n|$

For $\overrightarrow{\rho_{1'}} \to \overrightarrow{\rho_{2'}}$ the solution of the inhomogeneous BFKL equation is simplified

$$< \phi(\overrightarrow{\rho_1})\, \phi(\overrightarrow{\rho_2})\, \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) >\sim \sum_n C(\nu_\omega, |n|)\, \frac{E_{\nu_\omega,|n|}(\overrightarrow{\rho_{11'}}, \overrightarrow{\rho_{21'}})}{\omega'(|n|, \nu_\omega)} |\rho_{1'2'}|^{1+2i\nu_\omega} \left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{|n|/2}$$

where $E_{\nu_\omega,|n|}(\overrightarrow{\rho_{10}}, \overrightarrow{\rho_{20}})$ is the Polyakov three-point function and ν_ω is a solution of the algebraic equation

$$\omega = \omega(|n|, \nu)\,.$$

The simple interpretation in terms of the Wilson operator-product expansion

$$\lim_{\rho_{1'} \to \rho_{2'}} \phi(\overrightarrow{\rho_{1'}})\, \phi(\overrightarrow{\rho_{2'}}) \;=\; \sum_n \frac{C(\nu_\omega, |n|)}{\omega'(|n|, \nu_\omega)} |\rho_{1'2'}|^{2\Gamma_\omega} \left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{|n|/2} O_{\nu_\omega,|n|}(\overrightarrow{\rho_{1'}})\,,$$

where

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma(j)\,,\;\; \gamma(j)|_{\omega\to 0} = \frac{g^2 N_c}{4\pi^2 \omega}$$

and $\gamma(j)$ is an anomalous dimension of the operator with the conformal spin $|n| = 1, 2, ...$, having the twist higher than 2.

But it was assumed (A.K., L.L. (2000)) that after the analytic continuation of the BFKL anomalous dimension $\gamma(|n|, \omega)$ to the points $|n| = -r-1$ $(r = 0, 1, 2, ...)$ one can calculate singularities of the anomalous dimension of the twist-2 operators at negative integer j

$$j = 1 + \omega + |n| \to -r\,.$$

In LLA one can derive from the BFKL eigenvalue $\omega^0(n, \nu)$ in this limit

$$\gamma(j)|_{j\to -r} \;=\; \frac{g^2 N_c}{4\pi^2} \frac{1}{j+r}$$

for all $r = -1, 0, 1, ...$ This result is valid only for $N = 4$ SUSY. where we obtain :

$$\gamma(j) \;=\; \frac{g^2 N_c}{16\pi^2} \gamma^{LLA}(j), \quad \gamma^{LLA}(j) \;=\; 4(\Psi(1) - \Psi(j-1))$$

in an agreement with the direct calculation of $\gamma^{uni}(j)$ in this theory

In the NLO approximation there is a more complicated situation. Namely, the anomalous dimension $\gamma^{uni}(j)$ has the multiple poles $\Delta\gamma \sim \alpha^2 (j+r)^{-3}$ at even r, which is related to an appearance of the double-logarithmic corrections $\sim (\alpha \ln^2 s)^n s^{-r}$ in the Regge limit $s \to \infty$ at upper orders n of the perturbation theory.

0.5 Next-to-leading corrections to anomalous dimension matrix

The anomalous dimension matrices can be written as follows for the unpolarized and polarized cases

$$\gamma_{\mathbf{unpol}} = \begin{vmatrix} \gamma_{gg} & \gamma_{gq} & \gamma_{g\varphi} \\ \gamma_{qg} & \gamma_{qq} & \gamma_{q\varphi} \\ \gamma_{\varphi g} & \gamma_{\varphi q} & \gamma_{\varphi\varphi} \end{vmatrix}, \qquad \gamma_{\mathbf{pol}} = \begin{vmatrix} \tilde{\gamma}_{gg} & \tilde{\gamma}_{gq} \\ \tilde{\gamma}_{qg} & \tilde{\gamma}_{qq} \end{vmatrix}.$$

It is possible to construct 5 independent twist-two operators with a multiplicative renormalization in LLA. The corresponding parton distributions and their LLA anomalous dimensions are (L.L. (2000), A.K., L.L. (2002))

$$f_I(j) = f_g(j) + f_q(j) + f_\varphi(j)\,,\ \gamma_I^{(0)}(j) \equiv -4S_1(j-2)\,,$$
$$f_{II}(j) = -2(j-1)f_g(j) + f_q(j) + \frac{2}{3}(j+1)f_\varphi(j)\,,\ \gamma_{II}^{(0)}(j) \equiv -4S_1(j)\,,$$
$$f_{III}(j) = -\frac{j-1}{j+2}f_g(j) + f_q(j) - \frac{j+1}{j}f_\varphi(j)\,,\ \gamma_{III}^{(0)}(j) \equiv -4S_1(j+2)\,,$$
$$f_{IV}(j) = 2\Delta f_g(j) + \Delta f_q(j)\,,\ \gamma_{IV}^{(0)}(j) = -4S_1(j-1)\,,$$
$$f_V(j) = -(j-1)\Delta f_g(j) + \frac{j+2}{2}\Delta f_q(j)\,,\ \gamma_V^{(0)}(j) = -4S_1(j+1)$$

Thus, we have one supermultiplet of operators with the same anomalous dimension $\gamma^{LLA}(j)$ proportional to $\Psi(1) - \Psi(j-1)$. Note, that this result for $N=4$ SUSY leads to a solvability of the evolution equations for matrix elements of quasi-partonic operators due to the integrability of the corresponding Heisenberg spin model (L.L. (1997)).

The coefficients in these linear combinations for $N=4$ SUSY can be found from the super-conformal invariance and should be the same for all orders of the perturbation theory. But in next-to-leading approximation due to the violation of the conformal invariance (D.M. and A.B.) the result for the anomalous dimension in this basis has the triangle form (A.K.,L.L.,V.V. (2003)). Nevertheless its diagonal terms can be expressed in terms of the universal anomalous dimension by a shift of its argument (A.K., L.L., V.V. (2003))

$$\gamma(j) = -\frac{\alpha_s N_c}{\pi} S_1(j-2) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \hat{Q}(j-2)\,,$$

where

$$\hat{Q}(j) = -\frac{4}{3}\,S_1(j) + 16\,S_1(j)\,S_2(j) + 8\,S_3(j) - 8\,\tilde{S}_3(j) + 16\,\tilde{S}_{1,2}(j)\,,$$

$$S_k(j) = \sum_{i=1}^{j} \frac{1}{i^k}\,, \qquad \tilde{S}_k(j) = \sum_{i=1}^{j} \frac{(-1)^i}{i^k}\,,\ \ \tilde{S}_{k,l}(j) = \sum_{i=1}^{j} \frac{1}{i^k}\tilde{S}_l(i)$$

and we used the DRD scheme, but the coupling constant is expressed in the $\overline{MS}$ scheme. In the DRD scheme for α the term $-\frac{4}{3}\,S_1(j)$ in $\hat{Q}(j)$ is absent.

0.6 Relation between the DGLAP and BFKL equations

The Regge and Bjorken representations for the cross-section of hadron production in the virtual $\gamma^*\gamma^*$ collisions:

$$\sigma(s,Q^2,P^2)=\int\frac{d\omega}{2\pi i}\left(\frac{s}{|Q|\,|P|}\right)^{\omega}f_\omega(Q^2,P^2)=\int\frac{d\omega}{2\pi i}\left(\frac{s}{|Q|^2}\right)^{\omega}\varphi_\omega(Q^2,P^2)\,.$$

Therefore the anomalous dimensions for the DGLAP and BFKL equations are related as follows (V.F., L.L. (1998))

$$\gamma=\gamma_{BFKL}+\frac{\omega}{2}\,.$$

The equation for γ from the BFKL approach has the symmetry $\gamma\leftrightarrow J-\gamma$ ($J=|n|+1+\omega-\gamma$), which is a consequence of the hermitian separability

$$1=\frac{4\hat a}{\omega}\left(2\Psi(1)-\Psi(\gamma)-\Psi(J-\gamma)+\hat a\left(\phi(\gamma)+\phi(J-\gamma)\right)\right)$$
$$+2\hat a\left(\Psi'(\gamma)-\rho(\gamma)+\Psi'(J-\gamma)-\rho(J-\gamma)\right)\,.$$

Now we push $|n|\to -r-1$ and $\omega=j+r\to 0$ and calculate the singularities of the anomalous dimension $\gamma(j)$ at $j\to -r$

$$\gamma(j)=4\hat a\left(\frac{1}{j+r}+K(r)+L(r)\,(j+r)\right)+16\hat a^2\left(\frac{1}{(j+r)^3}+\frac{T(r)}{(j+r)^2}+\frac{R(r)}{j+r}\right)$$

for even r and

$$\gamma(j)=4\hat a\left(\frac{1}{j+r}+\widetilde K(r)+\tilde L(r)\,(j+r)\right)16\hat a^2\left(\frac{0}{(j+r)^3}+\frac{\widetilde T(r)}{(j+r)^2}+\frac{\widetilde R(r)}{j+r}\right)$$

for odd r. It is important, that one should find $\hat a^3$-corrections to calculate $K(r),L(r),T(r),R(r)$ and $\widetilde K(r),\tilde L(r),\widetilde T(r),\widetilde R(r)$.

On the other hand from the direct calculations of the universal anomalous dimension γ^{uni} we obtain

$$\gamma(j)=4\hat a\left(\frac{1}{j+r}-S_1(r+1)-\hat S_2(r+1)\,(j+r)\right)+16\hat a^2\left(\frac{1}{(j+r)^3}-\frac{2S_1(r+1)}{(j+r)^2}-\right.$$

for even r and

$$\gamma(j)=4\hat a\left(\frac{1}{j+r}-S_1(r+1)-\hat S_2(r+1)\,(j+r)\right)+16\hat a^2\,\frac{S_2(r+1)}{j+r}$$

for odd r.

Therefore the hypothesis (A.K., L.L. (2000)), that the BFKL equation in $N = 4$ SUSY contains a complete information about the DGLAP equation was verified in the next-to-leading approximation qualitatively.

0.7 AdS-CFT correspondence and the perturbation theory

According to AdS/CFT correspondence (Maldacena) the strong-coupling limit $\alpha_s N_c \to \infty$ is described by a classical supergravity in the anti-de Sitter space $AdS_5 \times S^5$. An interesting prediction was obtained for the anomalous dimension for twist-2 operators for $j \to \infty$ by A.Polyakov and others

$$\gamma(j) = a(z)\, \ln j\,,\ z = \frac{\alpha_s N_c}{\pi} \tag{2}$$

in the strong coupling regime

$$\lim_{z\to\infty} a = -z^{1/2} + \frac{3\ln 2}{8\pi} + \left(z^{-1/2}\right)\,. \tag{3}$$

On the other hand, all anomalous dimensions $\gamma_i(j)$ and $\tilde{\gamma}_i(j)$ coincide at large j and our results for $\gamma(j)$ allow one to find two first terms of the small-α_s expansion of the coefficient $\overline{\alpha}$

$$\lim_{z\to 0} \tilde{a} = -z + \frac{\pi^2 - 1}{12}\, z^2 + \ldots\,.$$

To go from this expansion to the strong coupling regime we perform a resummation of the perturbative result using the method similar to the Pade approximation and taking into account, that for large N_c the perturbation series has a finite radius of convergency. Namely, we present $\tilde{a}$ as a solution of the simple algebraic equation

$$z = -\tilde{a} + \frac{\pi^2 - 1}{12}\, \tilde{a}^2\,. \tag{4}$$

From this equation the following large-α_s behaviour of $\tilde{a}$ is obtained:

$$\tilde{a} \approx -1.1632\, z^{1/2} + 0.67647 + \mathcal{O}\left(z^{-1/2}\right) \tag{5}$$

in a rather good agreement with the above results based on the AdS/CFT correspondence.

Further, we have

$$\gamma(j) = (j-2)\,\gamma'(2)$$

and

$$\gamma'(2) = -1.645\, z + 1.216\, z^2$$

in the parturbation theory. Due to the BFKL equation we have for the leading singularity of the t-channel partial wave in j-plane

$$\gamma = 1/2 + i\nu + (j-1)/2 \to 1$$

for $j \to 2$. By the use of the above summation procedure we obtain for the Pomeron intercept

$$j = 2 - 1.1024\, z^{-1/2}$$

for large z. The correction $\sim z^{-1/2}$ from the AdS side is not estimated yet.

One can attemp also to calculate the intercept of the Pomeron at large z using its perturbative expansion $j-1 = c_1 z + c_2 z^2$ from the BFKL equation. After the Pade resummation $j-1 = c_1 z/(1-c_1 z/c_2)$ we obtain in the strong coupling regime $j \simeq 2.53$ in an approximate aggreement with the graviton spin $j=2$.

0.8 Conclusion

1. In $N = 4$ SUSY in LLA both BFKL and DGLAP dynamics are reduced to the integrable Heisenberg spin models in the multi-colour limit $N_c \to \infty$.

2. Next-to-leading corrections to the BFKL equation in the $N = 4$ model have the properties of the analyticity in the conformal spin and of hermitian separability.

3. The holomorphic separability for the BFKL kernel is violated.

4. In $N = 4$ SUSY all twist-2 operators belong to the same supermultiplet with the universal anomalous dimension $\gamma^{uni}(j)$, but the anomalous dimension matrix has a triangular form.

5. The leading singularities of $\gamma^{uni}(j)$ at $j \to -r$ are obtained from the BFKL equation.

6. The AdS-CFT correspondence and a resummation procedure allow to relate the small and strong coupling expansions in $N = 4$ SUSY.

7. Are the BFKL and DGLAP equations for many particles in $N = 4$ SUSY integrable in the next-to-leading approximation for $N_c \to \infty$?