Transverse Effect and Cancellation

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1. Introduction

- Usually at high energy, space-charge force was ignored for charged particle beams due to cancellation of E and B

$$
\begin{aligned}
F_x &= e(\vec{E} + \vec{\beta} \times \vec{B})_x \\
&= -e\left(\frac{\partial \Phi}{\partial x} - \beta_z \frac{\partial A_z}{\partial x}\right) \\
&= -e\frac{\partial}{\partial x}\int \frac{n(\vec{r}', t')(1-\beta_z \beta_z')}{|\vec{r}-\vec{r}'|} d\vec{r}' \\
&\approx -\frac{1}{\gamma^2}\frac{\partial \Phi}{\partial x}
\end{aligned}
$$

$\Rightarrow \vec{v} \to \cancel{\beta} c \vec{e}_z$

- In 1986, R. Talman pointed out *(PRL 14 1429 (1986)*)) that for coasting beams in circular accelerators, the cancellation was incomplete due to curvature of orbit:

$$
\begin{aligned}
F_r &= e(E_r + \frac{r\dot{\theta}}{c} B_z) \\
&= -e\frac{\partial \Phi}{\partial r} + e\beta_\theta \frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} \\
&= -e\frac{\partial (\Phi - \beta_\theta A_\theta)}{\partial r} + e\frac{\beta_\theta A_\theta}{r}
\end{aligned}
$$

Similar to the relativistic cancellation term

Centrifugal Space Charge Force

—— Energy independent

—— $\propto \frac{\lambda \ln(\sigma_\perp / R)}{R}$

$\mathrm{F}^{\mathrm{CFSC}}$: cause shift in design energy
shift in horizontal tune
nonlinear resonances
chromaticity

- **In 1990, E. Lee pointed out** *(Particle Accelerators* ***25*** *241 (1990))* **that for coasting beams in circular accelerators,** the energy deviation due to potential change **gives out effect that** cancels the F^{CFSC} **effect:**

$$\left(\frac{d\vec{P}}{dt}\right)_r = \vec{F}_r, \quad \Rightarrow \quad \frac{d}{dt}(\gamma m\dot{r}) - \gamma m r\dot{\theta}^2 = e(E_r + r\dot{\theta}_z B_z / c)$$

Equilibrium Orbit $\quad \gamma_e mc^2 = -\mathrm{Re}(E_r + B_z)_{\mathrm{elf}}$

Small Deviation from Design Orbit

$$-\delta\gamma\frac{c^2}{R} + \gamma\left(\delta\ddot{r} + \delta r\frac{c^2}{R^2}\right) = \frac{e}{m}\delta(E_r + B_z)_{\mathrm{elf}}$$

Energy Conservation

$$\boxed{\gamma mc^2 + e\Phi = \mathrm{const}}$$

$$\delta\ddot{r} + \frac{c^2}{R^2}\delta r = \frac{1}{\gamma m}\frac{d}{dr}\underbrace{\left[-\frac{e\phi}{R} + F_r\right]}_{G}\delta r$$

$$G = -\frac{e\phi}{R} + F_r = e\left[-\frac{\partial(\phi - \beta_\theta A_\theta)}{\partial r} + \frac{\beta_\theta A_\theta}{r} - \frac{\phi}{R}\right]$$

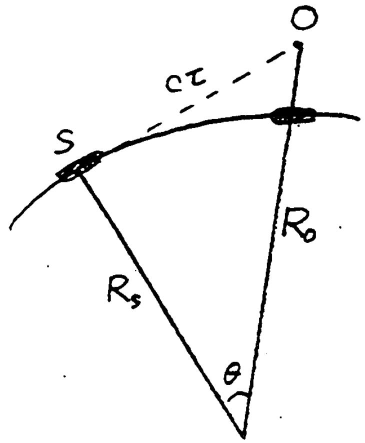
Singular

$$\frac{\partial F_r}{\partial r} \propto \frac{2\lambda}{R}\frac{1}{r-R}$$

$$\frac{\partial G}{\partial r} \propto -\frac{\lambda}{R^2}\left(1 + \ln\frac{x}{R}\right)$$

"Non-Inertial Space Charge Force"

- Single Particle Force from S to O



$$F_{\theta 0} = -e\left(\frac{1}{R_O}\frac{\partial \Phi}{\partial \theta} + \frac{1}{c}\frac{\partial A}{\partial t}\right)$$

$$= -e^2 \frac{\partial}{\partial s}\left[\frac{\dfrac{R_s}{\gamma_s^2 R_O} + \beta_s^2\left(\dfrac{R_s}{R_O} - 1\right) + \beta_s^2(1-\cos\theta)}{c\tau - \beta_s R_O \sin\theta}\right]$$

- Collective Longitudinal Force on an Off-Axis Test Particle by a Finite Uniform bunch

$$F_\theta(s) = \int_{s_r}^{s_f} F_{\theta 0}(s-s')\,\lambda(s')\,ds'$$

$$= -e^2\left[\frac{\dfrac{R_s}{\gamma_s^2 R_O} + \beta_s^2\dfrac{x}{R_s} + \beta_s^2(1-\cos\theta)}{c\tau - \beta_s R_O \sin\theta}\right]_{s_r}^{s_f}$$

space-charge force

"non-inertial space-charge force"

CSR force

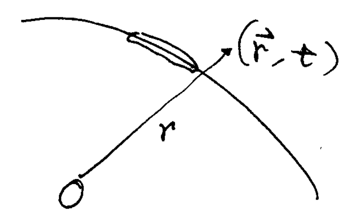
Collective Synchrotron Radiation Force

- In 1996, Ya Derbenev pointed out that for bunched beam, there also exists cancellation of transverse F^{CFSC} with effect of φ on transverse bunch dynamics.

- In 1999, R. Li pointed out that the noninertial space-charge force F^{NISC} contributes to the potential energy deviation $\frac{d\phi}{dt}$, so its effect on transverse dynamics cancels with that of F^{CFSC}.

- I'll review this cancellation study from my view of understanding.

2. Transverse CSR Force



$$\begin{cases} \vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{c\partial t} \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

$$\vec{F} = e\left(\vec{E} + \vec{\beta} \times \vec{B}\right) = -e\nabla\left(\phi - \vec{\beta}\cdot\vec{A}\right) - e\frac{d\vec{A}}{cdt}$$

<u>Transverse Force</u> [Derbenev]

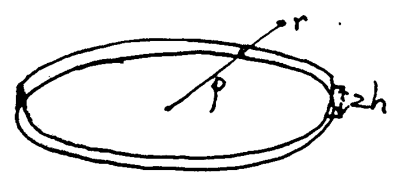
$$F_r = \underbrace{-e\frac{\partial}{\partial r}\left(\phi - \beta_0 A_\theta\right)}_{F_r^{(1)}} \underbrace{- e\left(\frac{\partial}{c\partial t} + \beta_0\frac{\partial}{r\partial\theta}\right)A_r}_{F_r^{(2)}} + \frac{e\beta_0 A_\theta}{r}$$

$\Downarrow$

Centrifugal space charge force

$$F^{CFSC} = \frac{e\beta_0 A_\theta}{r}$$

A Uniform-Vertical-Ribbon Beam on a Ring



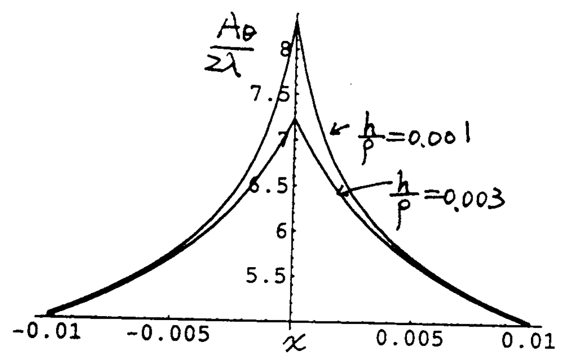
λ : longitudinal charge density

$x = r - \rho$: transverse offset

$2h$: vertical height of ribbon

$$A_\theta(x) = \frac{\lambda}{\rho}\int_0^h \frac{dz}{h}\int_0^{2\pi} d\Delta\theta \, \frac{\beta\cos\Delta\theta}{\sqrt{4\left(1+\frac{x}{\rho}\right)\sin^2\frac{\Delta\theta}{2} + \frac{x^2+z^2}{\rho^2}}}$$

- Nonlinear Behavior

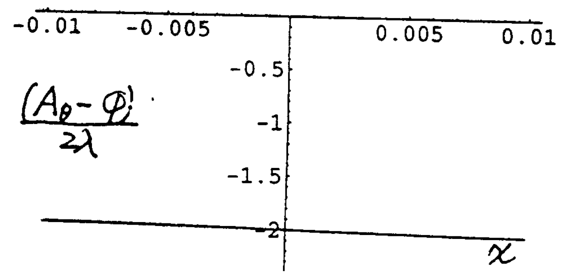


$$A_\theta(x) \simeq 2\lambda\left[1.426 - 0.5\,\ln\left(\frac{x^2+h^2}{\rho^2}\right) - \frac{x}{h}\,\mathrm{arctg}\frac{h}{x}\right]$$

- Dipole force on Axis:

$$A_\theta(x=0) = 2\lambda\left[1.426 - \ln\frac{h}{\rho}\right]$$

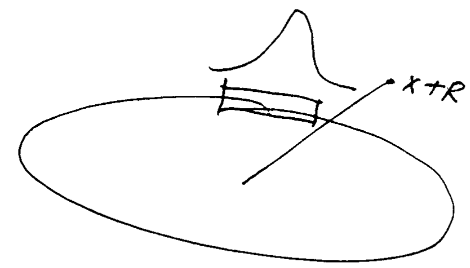
- Cancellation $\frac{A_\theta}{r} - \frac{\varphi}{\rho}$



Analytically, we have

$$\frac{\left[\frac{A_\theta}{r} - \frac{\varphi}{\rho}\right]}{2\lambda} \cong -2 + O\left[\frac{x}{\rho}\right]$$

F^{CFSC} for a Vertical-Ribbon-Gaussian Bunch



$$\varphi = \frac{s}{R}$$

$$F^{CFSC} = \frac{e\beta_0 A_\theta}{r}$$

$$A_\theta \simeq \Phi(s,x) = \int \frac{\rho(\vec{r}\,', t')}{|\vec{r} - \vec{r}\,'|} \, d\vec{r}\,'$$

$$= \frac{Ne}{2h}\int_{-h}^{h} dz' \int_{-\infty}^{\infty} d\Delta\theta \, \frac{\lambda(\varphi - \Delta\varphi(R\Delta\theta, x, z'))}{\sqrt{4\left(1+\frac{x}{R}\right)\sin^2\frac{\Delta\theta}{2} + \frac{x^2+z'^2}{R^2}}}$$

$$A_\theta = A_\theta^{(I)} + A_\theta^{(II)}$$

$$A_\theta^{(I)} = \frac{Ne}{h}\int_0^h dz' \int_0^\infty d\Delta\theta \, \frac{\lambda(\varphi - \Delta\varphi(R\Delta\theta,0,0)) + \lambda(\varphi - \Delta\varphi(-R\Delta\theta,0,0))}{\sqrt{4\left(1+\cancel{\frac{x}{R}}\right)\sin^2\frac{\Delta\theta}{2} + \frac{x^2+z'^2}{R^2}}}$$

Integrand has singularity

$$A_\theta^{(II)} = A_\theta - A_\theta^{(I)}$$

Integrand well behaved

has significant integrand only for local regime $\frac{\Delta s}{R} \lesssim \left(\frac{\sigma_z}{R}\right)^{\frac{3}{2}}$

$$A_\theta^{(II)} \ll A_\theta^{(I)}$$

Dominant Term

$$A_\theta^{(I)} \simeq \frac{1}{h}\int_0^h dz' \int_0^\infty d\Delta\theta \, \frac{\lambda\left(\varphi - \frac{\Delta\theta^3}{24}\right) + \lambda(\varphi + 2\Delta\theta)}{\sqrt{4\left(1+\frac{x}{R}\right)\sin^2\frac{\Delta\theta}{2} + \frac{x^2+z'^2}{R^2}}}$$

$$= -\frac{1}{h}\int_0^h dz' \int_0^\infty \ln \frac{2\Delta\theta}{\frac{\sqrt{x^2+z'^2}}{R}} \frac{\partial}{\partial\Delta\theta}\left[\lambda\left(\varphi - \frac{\Delta\theta^3}{24}\right) + \lambda(\varphi + 2\Delta\theta)\right] d\Delta\theta$$

$$= 2\lambda\left[\log\frac{R}{\sqrt{h^2+x^2}} + 1.87 + \frac{x}{h}\mathrm{ArcTan}\frac{h}{x}\right]. + \int_0^\infty \ln\varphi_1 \frac{\partial}{\partial\varphi}\left[\frac{\lambda(\varphi-\varphi_1)}{3} - \lambda(\varphi+\varphi_1)\right]_{\varphi_1}$$

— similar to G. Stupakov's analysis.

$$F^{CFSC} = \frac{e\beta_\theta A_\theta}{r} \simeq \frac{2\lambda}{r}\log\frac{R}{\sigma_\perp} + \text{corr.}$$

Driving Forces

■ Longitudinal Force

$$F_\theta \, \beta_\theta = e\left[-\boxed{\frac{d\varphi}{c\,dt}} - \frac{\partial}{c\,\partial t}(\varphi - \boldsymbol{\beta}\cdot\mathbf{A})\right]$$

non-inertial space charge force.

■ Radial Force $F_r = e\,(E_r - \beta_\theta B_z)$

—On circular path :

$$F_r = e\left[-\frac{\partial}{\partial r}(\varphi - \boldsymbol{\beta}\cdot\mathbf{A}) - \frac{dA_r}{c\,dt} + \boxed{\frac{\beta_\theta A_\theta}{r}}\right]$$

$$\Downarrow$$

F^{CFSC} [Talman]

centrifugal space-charge force

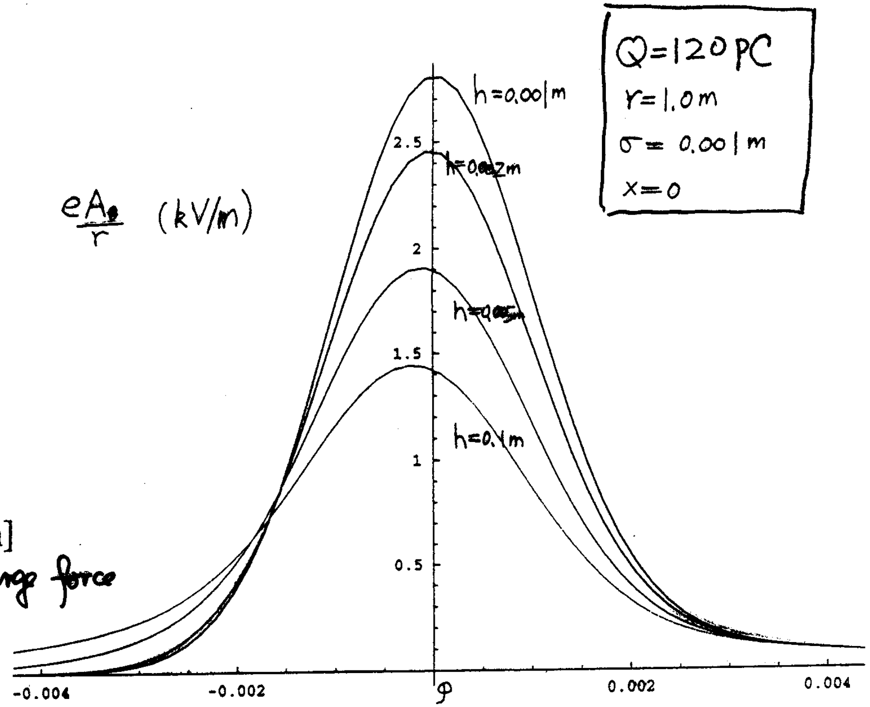
—On straight path

$$F_r = -e\frac{\partial}{\partial r}(\varphi - \boldsymbol{\beta}\cdot\mathbf{A}) = -\frac{e}{\gamma^2}\frac{\partial\varphi}{\partial r}$$

For a Rigid-Ribbon Beam (Longitudinally Gaussian distribution, vertically uniform distribution) :

$$A_\theta(\varphi, x) = \frac{1}{\sqrt{2\pi}\,\sigma\,(2h)}\int_{-h}^{h} dz \int_0^{2\pi} d\theta \, \frac{\lambda(\varphi - \Delta\varphi(\theta, x, z))}{\sqrt{4(1+x)\sin^2\frac{\theta}{2} + x^2 + z^2}}$$

$$\approx -\lambda(\varphi)\,\mathrm{Log}\left[\sqrt{x^2+h^2}\right] + \cdots$$



Features of F^{CFSC} :

- Logarithmic dependence on transverse size
- Needs extra numuerical care due to singularity and local retardation behavior
- Transverse motion actually depends on $(A_\theta - \beta_\theta\varphi)/r$ instead [Lee, Derbenev].

$$F_r = -e\underbrace{\frac{\partial}{\partial r}\left(\Phi - \beta_\theta A_\theta\right)}_{F_r^{(I)}} - e\underbrace{\left(\frac{\partial}{c\,\partial t} + \beta_\theta \frac{\partial}{r\,\partial\theta}\right)A_r}_{F_r^{(II)}} + e\frac{\beta_\theta A_\theta}{r}$$

$$\Downarrow$$

Centrifugal space charge force

$$\Phi - \beta_\theta A_\theta = \int \frac{\left(1-\beta^2\cos\Delta\theta\right)\lambda\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

$$1 - \beta^2\cos\Delta\theta = 1 - \beta^2 + \beta^2\left(1-\cos\Delta\theta\right)$$

$$\lambda\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right) = \lambda\left(\varphi - \frac{\Delta\theta^3}{24} + \underbrace{\Delta\theta\,\frac{x+x'}{2R} + \frac{(x-x')^2 + (y-y')^2}{2R^2\Delta\theta}}_{\ll \frac{\Delta\theta^3}{24}}\right)$$

for pencil beam

$$\frac{\sigma_s}{R} \gg \left(\frac{\sigma_\perp}{R}\right)^{2/3}$$

<u>According to Derbenev</u>

$$\boxed{F_r^{(I)} = -e\frac{\partial}{\partial r}\left(\Phi - \beta_\theta A_\theta\right) = \frac{2Ne^2}{R}\lambda(s) + 3g(s)x.}$$

$\downarrow$

Constant for coasting beam.

$$\frac{O\left(g(s)\,\sigma_\perp\right)}{O\left(Ne^2\frac{\lambda}{R}\right)} \sim \left(\frac{x}{R}\right)^{\frac{1}{3}}\left(\frac{x}{\sigma_s}\right)^{\frac{2}{3}} \ll 1.$$

$$\boxed{F_r^{(I)} \sim \left(\frac{\lambda}{R}\right)}$$

$$F_r^{(\mathrm{II})} = -e\left(\frac{\partial}{c\,\partial t} + \beta_\theta \frac{\partial}{r\,\partial\theta}\right) A_r$$

$$A_r = -e\int \frac{\beta \sin\Delta\theta\, \lambda\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'$$

$$\left(\frac{\partial}{c\,\partial t} + \beta_\theta \frac{\partial}{R\,\partial\theta}\right) A_r = 0$$

$$F_r^{(\mathrm{II})} = e\,\beta_\theta \frac{x}{R} \frac{\partial A_r}{\partial\theta} = \beta_\theta \frac{x}{R} \frac{Ne^2}{R} \frac{\partial}{\partial\varphi}\int \lambda\left(\varphi - \frac{\Delta\theta^3}{24}\right) d\varphi.$$

$$\boxed{\frac{O\left(F_r^{(\mathrm{II})}\right)}{O\left(F_r^{(\mathrm{I})}\right)} \sim \frac{\sigma_\perp}{R} \ll 1.}$$

3. Energy Variation

$$H = c\sqrt{(\vec{p} - e\vec{A})^2 + m^2c^2} + e\phi$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\Rightarrow \frac{d(\gamma mc^2 + e\phi)}{dt} = e\,\frac{\partial(\phi - \vec{\beta}\cdot\vec{A})}{\partial t}$$

$$\underbrace{\frac{d\,\gamma mc^2}{dt}}_{\text{rate of kinetic energy change}} = \underbrace{-e\frac{d\phi}{dt}}_{\text{rate of potential energy change}} + \underbrace{e\left(\frac{\partial\phi}{\partial t} - \vec{\beta}\cdot\frac{\partial\vec{A}}{\partial t}\right)}_{\text{radiation term due to collective interaction}}.$$

Potential Energy Change

$$e\frac{d\Phi}{dt} = e\left(\frac{\partial}{c\partial t} + \beta_\theta \frac{\partial}{r\partial\theta} + \beta_r \frac{\partial}{\partial r}\right)\Phi$$

(1) Coasting beam on equilibrium orbit, $r=\rho$, $\frac{\partial}{\partial t} = 0$.

$$e\delta\Phi = e\frac{\partial\Phi}{\partial r}\delta r = -eE_r\,\delta r$$

E. Lee's term which cancels with the effect of F^{CSE}

(2) Line charge in steady state (on-axis test particle)

$$e\frac{d\Phi}{c\,dt} = \left(\frac{\partial}{c\partial t} + \beta_\theta \frac{\partial}{\rho\partial\theta}\right)\int \frac{\lambda\left(\overset{\theta - \frac{\beta c t}{\rho}}{\overset{\uparrow}{\varphi}} - \frac{\Delta\theta^3}{24}\right)}{2\rho\sin\frac{\Delta\theta}{2}}\,\rho\, d\Delta\theta = 0$$

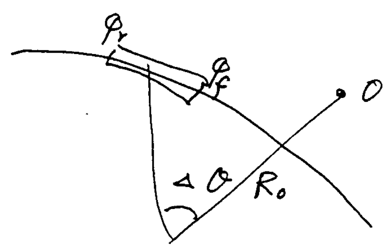
(3) Line charge in steady state (off-axis test particle)

$$e\frac{d\Phi}{c\,dt} = e\left(\frac{\partial}{c\,\partial t} + \beta_\theta \frac{\partial}{r\,\partial\theta}\right)\Phi = e\beta_\theta\left(\frac{\partial}{r\,\partial\theta} - \frac{\partial}{\rho\,\partial\theta}\right)\Phi$$

$$\underset{\substack{\text{uniform bunch} \\ [\varphi_r,\varphi_f]}}{=\!=} \quad -e^2\lambda\left[\frac{\beta_s \frac{x}{R}}{c\tau - \beta_s R_0 \sin\Delta\theta}\right]_{\varphi_r}^{\varphi_f}$$

$\downarrow$

non-inertial space charge force pointed out by B. Carlston



"Non-Inertial Space Charge Force"

- Collective Longitudinal Force on an Off-Axis Test Particle by a Finite Uniform bunch

Rate of Energy Change :

$$H = c\sqrt{(\vec{p}-e\vec{A})^2 + m^2c^4} + e\phi$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

$$\frac{d(\gamma mc^2 + e\phi)}{dt} = e\frac{\partial(\phi - \vec{\beta}\cdot\vec{A})}{\partial t}$$

$$\frac{d\,(\gamma mc^2)}{cdt} = \mathbf{F}\cdot\boldsymbol{\beta}_o = e\left(-\frac{d\Phi}{cdt} + \frac{\partial(\Phi - \beta_o \cdot \mathbf{A})}{c\,\partial t}\right)$$

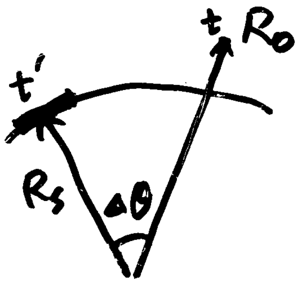
$$\underline{\underline{\text{for } \beta_o=\beta_s=\beta}}\; e^2\beta\lambda\left[\frac{\left(\frac{R_s}{R_o}-1\right) + (1-\beta^2\cos\Delta\theta)}{c\tau - \beta_s R_o \sin\Delta\theta}\right]_{\Delta\theta_r}^{\Delta\theta_f}$$

$$= e^2\beta\lambda\left[\frac{-\frac{x}{R_s} + \left(\frac{1}{\gamma^2} + \beta^2(1-\cos\Delta\theta)\right)}{c\tau - \beta_s R_o \sin\Delta\theta}\right]_{\Delta\theta_r}^{\Delta\theta_f}$$

"non-inertial space-charge force " [B . Carlsten]

usual space-charge force

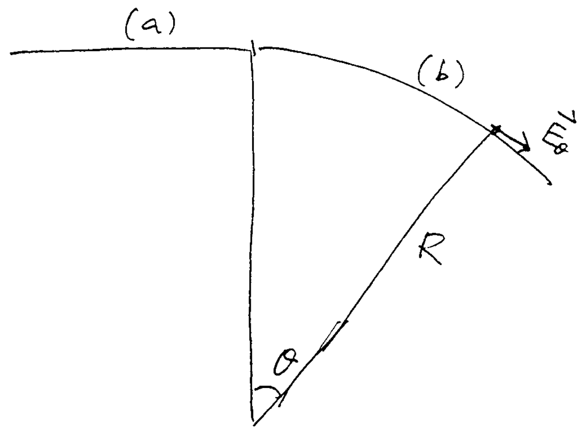
usual CSR force



(4) Transient CSR Force for Rigid-Line Bunch

$$\frac{d(\gamma m c^2)}{dt} = -e\frac{d\Phi}{dt} + e\frac{\partial(\Phi - \vec{\beta}\cdot\vec{A})}{\partial t} = \beta_0 c e E_\theta .$$

$$e E_\theta = e E_\theta^{(a)} + e E_\theta^{(b)}$$



$$E_\theta^{(a)} = \int_{\frac{R\theta^3}{24}}^{\infty} E_{\theta_0}^{(a)}(\theta, \Delta s)\, n(s-\Delta s)\, d\Delta s$$

$$E_\theta^{(b)} = \int_0^{\frac{R\theta^3}{24}} E_{\theta_0}^{(b)}(\theta, \Delta s)\, n(s-\Delta s)\, d\Delta s .$$

$$\boxed{E_{\theta_0}^{(a)} = -\frac{d\Phi_0}{dt} + \frac{\partial(\Phi_0 - \vec{\beta}\cdot\vec{A}_0)}{\partial t} = -\frac{\partial \hat{V}^{(a)}}{\partial \Delta s} .}$$

$$\hat{V}^{(a)} \simeq \begin{cases} 0 & \Delta s > \frac{R\theta^3}{6} \\ \frac{4e}{R\theta} & \Delta s < \frac{R\theta^3}{6} \end{cases}$$

$$e\hat{E}_\theta \simeq \frac{ZNe^2}{12.\ \cancel{\pi}\ (3R^2)^{\frac{1}{3}}} \left\{ \frac{2}{\theta}\left(\frac{3}{R}\right)^{\frac{1}{3}} \left[\lambda\left(s - \frac{R\theta^3}{6}\right) - \lambda\left(s - \frac{R\theta^3}{24}\right) \right] + \int_0^{\frac{R\theta^3}{24}} \frac{d\Delta s}{\Delta s^{\frac{1}{3}}} \frac{\partial \lambda(s-\Delta s)}{\partial s} \right\}$$

4. Cancellation

For a bunch moving on a circular orbit,

Eq. of Motion: $\frac{d\gamma m\vec{v}}{dt} = e(\vec{E} + \vec{\beta}\times\vec{B})$

Radial Component: $\frac{d(\gamma\beta_r)}{c\,dt} - \beta_\theta\left(\frac{\gamma\beta_\theta}{r} - \frac{\gamma_0\beta_0}{R}\right) = \frac{F_r}{mc^2}$

First order Eq: $x = r - R, \quad \Delta\gamma = \gamma - \gamma_0$

$$\boxed{\frac{d^2x}{c^2dt^2} + \frac{x}{R} = \frac{\Delta E}{RE_0} + \frac{F_r}{E_0}}$$

Transverse dynamics is driven by the combined effect of energy deviation and radial collective force.

From $$\frac{d\gamma mc^2}{dt} = e\left[-\frac{d\overline{\Phi}}{dt} + \frac{\partial(\overline{\Phi}-\vec{\beta}\cdot\vec{A})}{\partial t}\right],$$

<u>Energy deviation</u>

$$\Delta E = (\gamma(t)-\gamma_0)\,mc^2$$

$$= \underbrace{(\gamma(0)-\gamma_0)}_{\Delta\gamma_0}\,mc^2 - e\int_0^t \underbrace{\frac{d\overline{\Phi}}{dt}dt}_{\Phi(t)-\Phi(0)} + e\int_0^t \frac{\partial(\overline{\Phi}-\vec{\beta}\cdot\vec{A})}{\partial t}dt$$

$$\boxed{\Delta E = \left[\Delta\gamma_0 mc^2 + e\overline{\Phi}(0)\right] - e\overline{\Phi}(t) + \int_0^t \underbrace{\frac{\partial(\overline{\Phi}-\vec{\beta}\cdot\vec{A})}{c\,\partial t}}_{F_\theta^{eff}}\,c\,dt.}$$

<u>Eq. of Motion</u>

$$\frac{d^2x}{c^2dt^2} + \frac{x}{R^2} = \frac{\Delta E}{RE_0} + \frac{F_r}{E_0}$$

$$= \frac{\delta(0)}{R} + \frac{1}{E_0}\left(\frac{1}{R}\int_0^t F_\theta^{eff}\,c\,dt + G\right)$$

$$\delta(0) = \frac{\Delta\gamma_0 mc^2 + e\overline{\Phi}(0)}{E_0}$$

$$F_\theta^{eff} = e\int_0^t \frac{\partial(\overline{\Phi}-\vec{\beta}\cdot\vec{A})}{c\,\partial t}\,d\,ct$$

$$G = F_r - \frac{e\overline{\Phi}(t)}{R} = -\frac{\partial(\overline{\Phi}-\vec{\beta}\cdot\vec{A})}{\partial r} - e\frac{dA_r}{c\,dt} + \left(e\frac{\beta_\theta A_\theta}{r} - \frac{e\overline{\Phi}}{R}\right)$$

(1) Coasting Beam.

Energy Conservation: $\gamma mc^2 + e\Phi = \text{const.}$

$$\delta\ddot{r} + \frac{c^2}{R^2}\delta r = \frac{1}{\gamma m}\frac{d\left(\overbrace{-\frac{e\phi}{R} + F_r}^{G}\right)}{dr}\delta r$$

$$G = -\frac{e\phi}{R} + F_r = e\left[-\frac{\partial(\phi - \beta_\theta A_\theta)}{\partial r} + \frac{\beta_\theta A_\theta}{r} - \frac{\phi}{R}\right]$$

$$-e\frac{\partial(\phi - \beta_\theta A_\theta)}{\partial r} = \frac{2Ne^2\lambda}{R}$$

$$\neq e\left[\frac{\beta_\theta A_\theta}{r} - \frac{\phi}{R}\right] = e\left[\frac{\beta_\theta A_\theta - \phi}{R} - \frac{x}{R^2}\frac{\beta_\theta A_\theta}{R}\right]$$

$$= -\frac{4Ne^2\lambda}{R} + \frac{x}{R}\frac{Ne^2\lambda}{R}\ln\frac{a}{R}$$

$$F_\theta^{\,eff} = e\int_0^t \frac{\partial(\Phi - \vec{\beta}\cdot\vec{A})}{c\,\partial t}\,c\,dt = 0$$

$$O\left[\frac{dG}{dr}\right] \sim \frac{\lambda}{R^2}\left(1 + \ln\frac{x}{R}\right) \qquad \text{otherwise } O\left[\frac{dF}{dr}\right] \sim \frac{\lambda}{R^2}\frac{1}{x}$$

(2) Pencil Bunch in Steady State

$$\frac{d^2 x}{c^2 dt^2} + \frac{x}{R^2} = \frac{\delta(0)}{R} + \frac{1}{E_0}\left(\frac{1}{R}\int_0^t F_\theta^{eff}\, c\,dt + G \right)$$

$$F_\theta^{eff} = \frac{2Ne^2}{(3R^2\sigma_s)^{\frac{1}{3}}} \int_0^\infty \frac{d\varphi_1}{\varphi_1^{\frac{1}{3}}} \frac{\partial \lambda(\varphi - \varphi_1)}{\partial \varphi}.$$

$$G = F_r - \frac{e\overline{\Phi}}{R} = -e\frac{\partial(\overline{\Phi} - \vec{\beta}\cdot\vec{A})}{\partial r} - e\frac{dA_r}{c\,dt} + \frac{e\beta_\theta A_\theta}{r} - \frac{e\overline{\Phi}}{R}$$

$$F_r^{(I)} \sim \frac{2Ne^2\lambda}{R}$$

centripetal force

$$O(F_r^{(II)}) \sim \frac{\sigma_\perp}{R} O(F_r^{(I)})$$

$$-U_0(\phi) - \frac{x}{R}\frac{e\beta_\theta A_\theta}{R}.$$

$$\Vert$$

$$-\frac{2Ne^2}{(3R^5\sigma_s)^{\frac{1}{3}}} \int_0^\infty \frac{d\varphi_1}{\varphi_1^{1/3}} \lambda(\varphi - \varphi_1).$$

residual of cancellation

Driving Factors (steady-state, rigid bunch)

$$\boxed{\begin{aligned}&\sigma_s = 0.001\,m\\ &R = 1\,m\\ &Q = 60\,pC\end{aligned}}$$

$$\lambda(\varphi) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\varphi^2}{2}}, \quad \varphi = s/\sigma_s$$

■ Centripetal force

$$F_x = -e\,\frac{\partial(\varphi - \beta\bullet A)}{\partial r} = \frac{-2\,Ne^2\,\lambda(\varphi)}{R\,\sigma_s}$$

$$|F_x/F_s| \sim \left(\frac{\sigma_s}{R}\right)^{1/3}$$

■ Residual of Near-Cancellation

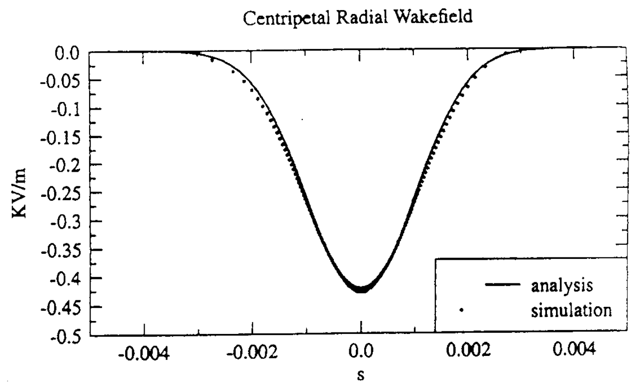
$$F_{res} = e\,\frac{A_\theta - \beta_\theta\,\varphi}{r} = -\frac{2\,Ne^2}{(3\,R^5\,\sigma_s)^{1/3}} \int_0^\infty \frac{d\varphi_1}{\varphi_1^{1/3}}\,\lambda(\varphi - \varphi_1)$$

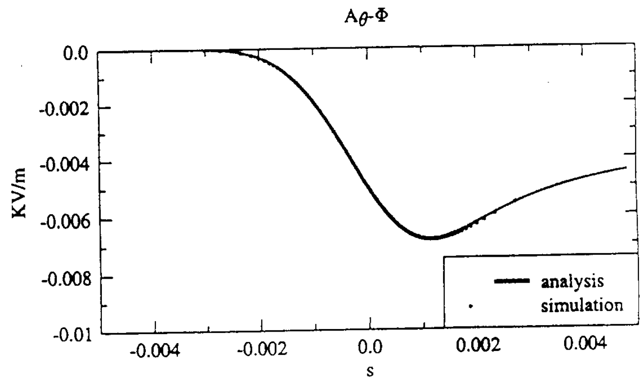
$$|F_{res}/F_s| \sim \frac{\sigma_s}{R}$$

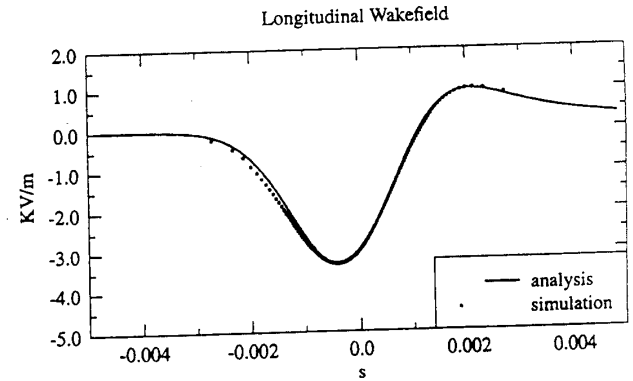
■ Longitudinal Force

$$F_s = e\,\frac{\partial(\varphi - \beta\bullet A)}{c\,\partial t} = \frac{2\,Ne^2}{(3\,R^2\,\sigma_s^{4})^{1/3}}\,\frac{\partial}{\partial\varphi}\int_0^\infty \frac{d\varphi_1}{\varphi_1^{1/3}}\,\lambda(\varphi - \varphi_1)$$

For $\frac{\sigma_s}{R} \ll 1$, $\left|\frac{F_x}{F_s}\right| \ll 1$, $\left|\frac{F_{res}}{F_s}\right| \ll 1$







— The dominant term in F_r is

$$F^{CFSC} \sim N e^2 \frac{\lambda}{R} \ln \frac{R}{\sigma_\perp}.$$

↑ Due to local interaction.

— $-\frac{e\phi}{R} + \frac{e\beta_0 A_\theta}{r}$ always cancel

$$\phi - \beta_0 A_\theta = \int \frac{(1-\beta^2 \cos\Delta\theta)\, n(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d\vec{r}'$$

$\rightarrow (1-\beta^2) + \beta^2(1-\cos\Delta\theta)$

$\rightarrow$ suppress local interaction.

(not only for pencil beam in steady-state, but more general)

$+ \frac{x}{R}\left(\frac{\lambda}{R} \ln \frac{R}{\sigma_\perp}\right)$ term

Comments

- The self-consistent bunch dynamics for $\rho(s, \delta, x, x'; t)$ is determined by

$$\frac{\partial \rho}{\partial t} + \dot{s}\frac{\partial \rho}{\partial s} + \dot{\delta}\frac{\partial \rho}{\partial \delta} + \dot{x}\frac{\partial \rho}{\partial x} + \ddot{x}\frac{\partial \rho}{\partial \dot{x}} = 0$$

$\Uparrow$ where $\frac{\Phi}{R} - \frac{A_\theta}{r}$ cancels

The inclusion of F_r, including F^{CFSC}, will change the phase space distribution.

- $\frac{\overline{\Phi}(0)}{E_0}$ acts as initial energy spread, may cause emittance increase for dispersive region, but has less effects for achromatic bend.

New Eq. of Motion : $\tilde{\phi} = \frac{e\phi}{mc^2}$

$$\frac{d(\gamma+\tilde{\phi})\vec{\beta}}{cdt} = \frac{e}{mc^2}\left[-\nabla(\phi-\vec{\beta}\cdot\vec{A}) - \frac{d}{cdt}(\vec{A}-\vec{\beta}\phi)\right]$$

In Circular coordinate :

$$\left\{\begin{aligned}
&\frac{d(\gamma+\tilde{\phi})\beta_r}{cdt} - \beta_\theta\left[\frac{(\gamma+\tilde{\phi})\beta_\theta}{r} - \frac{\gamma_0\beta_0}{\rho}\right] \\
&\qquad = \frac{e}{mc^2}\left[-\frac{\partial(\phi-\vec{\beta}\cdot\vec{A})}{\partial r} + \beta_\theta\frac{A_\theta-\beta_\theta\phi}{r} - \frac{d(A_r-\beta_r\phi)}{cdt}\right] \\
&\frac{d(\gamma+\tilde{\phi})\beta_\theta}{cdt} + \beta_r\left[\frac{(\gamma+\tilde{\phi})\beta_\theta}{r} - \frac{\gamma_0\beta_0}{\rho}\right] \\
&\qquad = \frac{e}{mc^2}\left[-\frac{\partial(\phi-\vec{\beta}\cdot\vec{A})}{r\partial\theta} - \beta_\theta\frac{A_r-\beta_r\phi}{r} - \frac{d(A_\theta-\beta_\theta\phi)}{cdt}\right] \\
&\frac{d(\gamma+\tilde{\phi})}{cdt} = \frac{e}{mc^2}\frac{\partial(\phi-\vec{\beta}\cdot\vec{A})}{c\partial t}
\end{aligned}\right.$$

First Order of Eq. for $x = r-\rho$:

$$\frac{d^2x}{c^2dt^2} + \frac{x}{\rho} = \frac{\Delta\gamma}{\rho\gamma_0} + \frac{e}{E}\left[\frac{1}{\rho}\int_0^t\frac{\partial(\phi-\vec{\beta}_0\cdot\vec{A})}{\partial t}dt'\right]$$

$$+\frac{e}{E}\left[-\frac{\partial(\phi-\vec{\beta}_0\cdot\vec{A})}{\partial r} + \beta_\theta\frac{A_\theta-\beta_\theta\phi}{r} - \frac{d(A_r-\beta_r\phi)}{cdt}\right]$$

Summary

- The counterpart of F^{CFSC} cancelling with effects of $\frac{e\Phi}{R}$ for coasting beam also exists for bunched beam.
- The residual of cancellation has negligible effect comparing to F_r^{eff}.
- For 1D line bunch model, $\frac{d\phi}{dt}$ is in the transient CSR field formula, one should be careful about its effect on transverse dynamics
- Simulation with both F_s and F_r included takes care of the cancellation implicitely.