Randall-Sundrum

Jan Heisig
Talk at Tuesday's "Werkstatt Seminar"

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References

Original RS papers:

[1] L. Randall and R. Sundrum *A Large Mass Hierarchy from a Small Extra Dimension*, hep-ph/9905221

[2] L. Randall and R. Sundrum *An Alternative to Compactification*, hep-th/9906064

Lectures on RS:

[3] R. Sundrum *TASI 2004 Lectures: To the Fifth Dimension and Back*, hep-th/0508134

[4] Csaba Csaki *TASI Lectures on Extra Dimensions and Branes*, hep-ph/0404096

[5] V.A. Rubakov *Extra Dimensions: A Primer*, in *Les Houches Sessions LXXXIV*, Elsevier (2006)

The large extra dimensions paper (ADD):

[6] Nima Arkani-Hamed, Savas Dimopoulos, Gia Dvali *The Hierarchy Problem and New Dimensions at a Millimeter*, hep-ph/9803315

SM trapped on a brane

Last session we heard about the original KK idea and learned that *compact* extra dimensions give rise to 4D mass terms—the KK excitations. The reason for introducing extra dimensions in the framework of today's topic is completely different to the original KK idea. We don't want to unify gravity and the gauge couplings of the SM (which by the way is hardly achievable due to the non-abilian structure). On the contrary we banish the SM fields from our extra dimension and trap them on a $(3+1)$-dimensional brane (called 3-brane). We only allow the graviton to propagate in the bulk (that is the full higher dimensional space). Such a scenario is of course string/M-theory motivated since SM fields correspond to open strings whoes ends have to fulfill certain boundary conditions while gravitons correspond to closed strings.

This idea leads to interesting consequences in the gravity sector and opens new possibilities, e.g. for solving the hierarchy problem. Let's consider such an attempt.

Before we examine the Randall-Sundrum idea let's briefly consider the "large" extra dimensions model (ADD)—as a warm up and also to compare it to RS later on. By the way, "large" means that the size of the compact extra dimension is big compared to the inverse Planck scale ${M_{\rm Pl}}^{-1}$.

Large extra dimensions (ADD)

In this set-up the higher dimensional space is made up of the direct product of the 4D Minkowski space and and d flat compact extra dimensions of size y_c ($\mathbb{M} \times \mathbb{T}^d$). The canonically normalized $(4+d)$D Einstein-Hilbert action reads

$$S = M^{(d+2)} \int_0^{y_c} d^d y \int d^4 x \sqrt{-G}\, \mathcal{R}\,, \tag{1}$$

where M, G_{MN} and $\mathcal{R}$ is the $(4+d)$D Planck mass, metric and Ricci scalar, respectively. Due to the trivial metric this can be easily reduced to a 4D effective action by performing the integral over the extra dimensions y

$$S_{\rm eff} = M^{(d+2)} y_c^d \int d^4 x \sqrt{-g^{(4)}} \mathcal{R}^{(4)}\,. \tag{2}$$

Thus, the 4D Planck mass $M_{\rm Pl}$ is related to the higher dimensional one as

$${M_{\rm Pl}}^2 = M^{(d+2)} y_c^d\,. \tag{3}$$

That is, you can in principle pretend to live in 4+d dimensions and choose d and y_c in such a way that the fundamental Planck scale M is of order 1 TeV which would straightforwardly solve the hierarchy problem.

Let's see in a very simple consideration what this means for Newton's gravity. This can be easily read off from the Gaussian law in $4 + d$ dimensions:

$$V(r)_{\text{Newton}} \sim \frac{1}{M^{(d+2)}} \begin{cases} r^{-(d+1)} & \text{for } (r \ll y_c) \\ y_c^{-d}\, r^{-1} & \text{for } (r \gg y_c) \end{cases} . \tag{4}$$

Since there has been no deviation found from Newton's law (lower case of (4)) down to distances of about 0.2 mm, $d \geq 3$ and $y_c \lesssim 10^{-8}$ m has to be chosen. However, there are constraints from cosmology and astrophysics which set a direct bound on the achieved fundamental scale M in this scenario which is why ADD is not as interesting anymore.

Incorporating brane tensions—the Randall-Sundrum model

Although in the ADD model we put the SM on a 3-brane the effect on the space-time is not reflected by the action (1). So, we simply neglected this contribution.

Now, one of the basic ingredients of the Randall-Sundrum approach is to include the brane tension (energy per unit 3-volume on the brane) and also allow for a 5D cosmological constant. The goal is to obtain 4D Poicare invariance but retain a non-trivial 5D metric.

The RS set-up

Consider one extra dimension which is the compact space $S_1/\mathbf{Z}_2$, that is, you impose the symmetries

- Periodicity: $y \to y + 2y_c$
- Orbifold symmetry: $y \to -y$

in the coordinate of the extra dimension y.

We take two 3-branes which we will call the (+)-brane and the (−)-brane and localize them at the fixed point $y = 0$ and $y = y_c$, respectively. Assuming that the thickness of the branes is small compared to y_c we approximate them as δ-functions in the y-space. The 5D action of this set-up reads

$$S = \int_{-y_c}^{y_c} d^d y \int d^4 x \Big\{ \sqrt{-G}\,(M^3\,\mathcal{R} - \Lambda) \\ -\sqrt{-g_+} T_+ \delta(y) - \sqrt{-g_-} T_- \delta(y - y_c) + (\text{SM on one brane}) \Big\}\,, \tag{5}$$

where Λ is the 5D cosmological constant, $T_\pm$ is the tension on the $(\pm)$-brane and

$$g^{\pm}_{\mu\nu}(x^\mu) = G_{MN}\left(x^\mu\,, y = \begin{Bmatrix} 0 \\ y_c \end{Bmatrix}\right)\,\delta^M \mu\,\delta^N \nu\,. \tag{6}$$

Variation with respect to G_{MN} gives the standard Einstein equation with Λ plus junction conditions on the branes:

$$\left(\mathcal{R}_{MN} - \frac{1}{2} G_{MN} \mathcal{R}\right) = -\frac{1}{2M^3}\Big\{\Lambda\, G_{MN} + \frac{\delta^\mu_M \delta^\nu_N}{\sqrt{-G}}\big[\,T_+ \sqrt{-g_+}\, g^+_{\mu\nu}\,\delta(y) \\ + T_- \sqrt{-g_-}\, g^-_{\mu\nu}\,\delta(y - y_c)\,\big]\Big\}\,. \tag{7}$$

In the rest of this talk I will solve (7) first for the *background metric* and then for *pertubations* in the $\mu\nu$ componets (the graviton).

Solutions for the background metric

The appealing thing about this set-up is that it allows for a 4D Poicare invariance background solution. To see this we choose a 4D flat ansatz

$$ds^2 = e^{-2\sigma(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{8}$$

where $e^{-2\sigma(\phi)}$ is just a fancy way of writing a positive function of the real variable y. It is called the warp-factor. This metric is of course non-factorizable unlike in the ADD case. Plugging into (7) gives us

$$yy: \qquad \sigma' = \sqrt{\frac{-\Lambda}{12M^3}}, \tag{9}$$

$$\text{“}\mu\nu - yy\text{”}: \qquad \sigma'' = \frac{1}{6M^3}\left(T_+\,\delta(y) + T_-\,\delta(y - y_c)\right) \tag{10}$$

Imposing the additional requirement of the orbifold symmetry the solution to (9) is

$$\sigma = |y|\sqrt{\frac{-\Lambda}{12M^3}}\,. \tag{11}$$

The second derivative is

$$\sigma'' = 2\sqrt{\frac{-\Lambda}{12M^3}}\left(T_+\,\delta(y) - T_-\,\delta(y-y_c)\right) . \tag{12}$$

Matching (12) with (10) leads to

$$T_- = -T_+ = -\sqrt{-12M^3\,\Lambda} \equiv -12M^3 k\,, \tag{13}$$

where we introduce the single dimensionful parameter k. We want to state four remarkable things:

- There is a solution
- Λ must be negative, so the resulting background metric is AdS_5 with radius $\sim k^{-1}$
- T_-, T_+ and Λ has to be fine-tuned (in some way similar to cosmology to get $\Omega = 1$)
- T_- has negative tension

Now, the background metric reads

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \tag{14}$$

Since every slice of this AdS_5 space is 4D flat one can decompose every field into four dimensional plane waves

$$\Phi \sim e^{ipx}\phi_p(y)\,. \tag{15}$$

Since we now know the background metric and some properties of it we will examine pertubations. But before solving the linearized Einstein equations, let's calculate the effective 4D action analogous to (2):

$$\begin{aligned} S_{\text{eff}} &\supset \int d^4x \int_{-y_c}^{y_c} dy\, M^3 e^{-4k|y|}\sqrt{-g^{(4)}}\, e^{2k|y|}\mathcal{R}^{(4)} \\ &= \left\{\frac{M^3}{k}(1-e^{-2ky_c})\right\}\int d^4x\sqrt{-g^{(4)}}\mathcal{R}^{(4)}\,. \end{aligned} \tag{16}$$

Thus, the expression in curly brackets is the 4D Planck mass M_{Pl}. Here—in contrast to (2)—we see that the size of the extra dimension has little effect on the ratio between M and M_{Pl}. As we will show now, you can choose $M \sim k \sim M_{\text{Pl}}$ but still solve the hierarchy-problem.

Solving the hierarchy problem with RS

Assuming that the SM is trapped on the negative-tension-brane and considering the fundamental Higgs field H one obtains

$$S \supset \int d^4x \sqrt{-g_-} \left\{ g_-^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2 \right\}, \tag{17}$$

where v_0 is a mass parameter. Plugging in the 4D metric $g_{\mu\nu}^{(4)} = e^{-2ky_c} g_{\mu\nu}^-$ ($g_{\mu\nu}^{(4)} = \eta_{\mu\nu}$ in flat space) and performing a wave-function renormalization $H \to e^{ky_c} H$ we obtain

$$S_{\text{eff}} \supset \int d^4x \sqrt{-g^{(4)}} \left\{ g^{(4)\mu\nu} D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - e^{-2ky_c} v_0^2)^2 \right\}. \tag{18}$$

Thus,

$$v = e^{-ky_c} v_0. \tag{19}$$

Any fundamental mass parameter will be redshifted on the $(-)$-brane according to the warp factor. To generate a mass parameter of order 1 TeV with $M \sim k \sim M_{\text{Pl}}$ one only needs $ky_c \sim 30$.

Pertubations

Now we consider perturbations about the background metric $\delta G_{\mu\nu}$. Thereby we set $\delta G_{yy} = \delta G_{\mu y} = 0$.

(In principle one has to consider δG_{yy} which would result in the appearance of a massless scalar mode. To then give this scalar mode a mass is the object of modulus stabilization. We do not consider this issue but simply assume that the modulus is stabilized in one or the other way. See Golderberger-Wise: hep-ph/9907447. At the present stage y_c can just be treated as a free parameter.)

Because of (15)

$$\delta G_{\mu\nu} = h_{\mu\nu}(x, y) = e^{ipx}\, h_{\mu\nu}(y)\,. \tag{20}$$

Without further sources of gravity (apart from T_-, T_+, Λ) it is possible to fix the gauge such that h is transverse and trace-free:

$$\partial_\mu h^\mu_\nu = h^\mu_\mu = 0\,. \tag{21}$$

With this all components of $h_{\mu\nu}$ obey

$$\begin{aligned} \text{bulk}: &\quad h'' - \left(4k - m^2 e^{2k|y|}\right) h = 0 \\ \text{branes}: &\quad h' + 2kh = 0 \quad \text{(junction condition)} \end{aligned} \tag{22}$$

For $m^2 = 0$ this can easily be solved

$$h_0(y) = e^{-2k|y|} \qquad (0 - \text{mode})\,. \tag{23}$$

The 0-mode is localized around the (0)-brane.

Clearly—due to the compact extra dimension—the KK-modes build up a discrete spectrum. Obtaining explicit solutions for the KK-modes $m \neq 0$ is quite involving and I will not go into detail here. But let me briefly sketch the calculation. Performing a coordinate transformation to conformal coordinates

$$ds^2 = \frac{1}{(k|z|+1)^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{24}$$

will turn (22) into a Schroedinger-like equation:

$$\left[-\partial_z^2 + V(z)\right] h(z) = m^2 h(z)\,, \qquad V(z) = \frac{15k^2}{8(k|z|+1)^2} - \frac{3k}{2}\delta(z) \tag{25}$$

$V(z)$ is called the volcano potential. The KK-modes have a small probability to "tunnel" to $z = 0$ because of the potential wall. The solutions are described by Bessel functions. One obtains a KK tower with a mass splitting of $\Delta m \sim z_c^{-1} \sim ke^{-ky_c}$. With $k \sim M_{\text{Pl}}$ and $ky_c \sim 30$ one can easily achieve $\Delta m \sim 1\,\text{TeV}$.

Randall-Sundrum II

Let us briefly illustrate another possibility which the warped metric gives rise to. By a second look at (16) one finds that this expression also makes sense in the limit $y_c \to \infty$ (in contrast to ADD). This statement is equivalent to the fact that the 0-mode (23) is normalizable even in the limit $y_c \to \infty$ (again in contrast to ADD).

Let's see how the KK-modes behave and what the effect of the 4D gravity would be if one shifted the (-)-brane to infinity. Due to the lack of a second junction condition at $y = y_c$ the spectrum becomes continuous. The Bessel functions that solve (25) can be approximated in the limit of large y (we now go back to our original coordinates) where they oscillate

$$h_m(y) \sim e^{-k|y|} \sin\left(\frac{m}{k}e^{2k|y|} + \varphi_m\right) \tag{26}$$

whereas they decrease towards small y and are suppressed at $y = 0$,

$$h_m(0) \sim \sqrt{\frac{m}{k}}\,. \tag{27}$$

From this, one can compute the contribution of the KK-modes to the 4D gravity. Pretending the SM is now trapped on the (+)-brane we can ask for the gravitational potential of two test masses with distance r on this brane. Each KK graviton produces a potential of Yukawa type. Therefore we obtain

$$\begin{aligned}\Delta V(r) &\sim \int_0^\infty dm\, \frac{|h_m(0)|^2}{M^3}\frac{e^{-mr}}{r} \\ &\sim \frac{k}{rM^3}\int_0^\infty dm\, \frac{me^{-mr}}{k^2} \sim \frac{1}{{M_{\text{Pl}}}^2\, r}\frac{1}{k^2r^2}\,.\end{aligned} \tag{28}$$

Thus, the 4D gravitational potential reads

$$V(r) \sim \frac{1}{{M_{\text{Pl}}}^2\, r}\left(1+\frac{\text{const.}}{k^2r^2}\right)\,. \tag{29}$$

The correction has a power law. At distances exceeding the AdS radius $\sim k^{-1}$ the correction is negligible.

This is in strong contrast to ADD. There, for $y_c \to \infty$ the 0-mode becomes more and more diluted and on the other hand more and more KK-modes enter the game. In the end higher dimensional gravity is recovered.