WERKSTATTSEMINAR SS11: N=2 SUSY in d=5,6

Introduction / Motivation:

- Mechanisms in extra-dimensions (e.g. explanations of fermion masses, EW symmetry breaking) could work equally well if $\Lambda_{XD} \sim \Lambda_{GUT}$. In such a case another mechanism would be required to stabilise the TeV scale, for which SUSY is an obvious option.

- N=1 SUSY in d=5,6 after dimensional reduction looks like N=2 SUSY in d=4, as showed in detail later.

- SU(5) in 5d and SO(10) in 6d can be broken by orbifold compactification on a circle or torus to give the SM gauge group in 4d, see SS12.

- The breaking of such GUT groups in 4d is much more involved e.g. requires mass splitting between the doublet and triplet Higgs as on integrating out the Higgs triplet one gets effective dim. 5 operators which are highly constrained as they give rise to proton decay. In higher dimensions such operators are naturally absent as the Higgs triplets live in the bulk and do not couple to SM matter, see SS12.

- In order to study SUSY in higher dimensions we require rules to write down a supersymmetric Lagrangian, so we can couple fields localised on 3-branes in a SUSY preserving way

N=2 SUSY: Basic formalism

- As stated, the motivation for SUSY is not only the cancellation of quadratic divergences, but also the possibility to unify forces, only possible in extended SUSY

- Unification is not possible in N=1, as for unification require gauge transformations which do not commute with SUSY, which are related to co-ordinate transforms, i.e. the generators of gauge symmetries ~~shall~~ not commute with SUSY generators (largest group U(N)), so N>1 is needed to have a non-Abelian group in a gauged supergravity theory

- A first step towards this is N=2, which resembles N=1 SUSY in d=5,6 after dimensional reduction, and therefore it is essential to understand N=2 SUSY in order to understand SUSY in d=5,6.

- Helicities in a supermultiplet have a range N/2 ⇒ for

N=2 SUSY gauge multiplet	helicity	-1	-1/2	0	1/2	1
	states	1	2	1+1	2	1

Therefore it can be thought of as a superposition of the N=1 gauge and chiral multiplets. The resulting Lagrangian is also approximately a superposition of the N=1 for the gauge and chiral multiplets, however in order to ensure SU(2) invariance we define a symplectic Majorana spinor $\lambda^i \equiv \begin{pmatrix} -i\,\varepsilon^{ij}\lambda_{\alpha j} \\ \bar\lambda^{\dot\alpha i} \end{pmatrix}$; $\bar\lambda_i = (\lambda_i^{\alpha}, i\,\varepsilon_{ij}\bar\lambda^j_{\dot\alpha})$

which satisfies the symplectic reality condition $\lambda^i = \varepsilon^{ij}\gamma_5 C \bar\lambda_j^T$
(↳ refers to ε_{ij} used to raise and lower SU(2) indices)

- $$\mathcal{L} = \mathrm{tr}\Big\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{i}{2}\bar\lambda_i\gamma^\mu\nabla_\mu\lambda^i + \tfrac{1}{2}\nabla_\mu M\nabla^\mu M + \tfrac{1}{2}\nabla_\mu N\nabla^\mu N$$
$$-\tfrac{i}{2}\bar\lambda_i[\lambda^i, M] - \tfrac{i}{2}\bar\lambda_i\gamma_5[\lambda^i, N] + \tfrac{1}{2}[M,N]^2\Big\}$$

which is invariant under SU(2) but also chiral U(1) transforms $M+iN \to e^{-2i\alpha}(M+iN)$, $\lambda^i \to e^{\alpha\gamma_5}\lambda^i$, $A_\mu \to A_\mu$.

- SUSY transforms for which $\mathcal{L}$ is a density are given by

$\delta A_\mu = i\bar\xi_i\gamma_\mu\lambda^i$ $\quad \delta M = i\bar\xi_i\lambda^i$ $\quad$ (require $\xi \to \xi_i$)

$\delta N = i\bar\xi_i\gamma_5\lambda^i$ $\quad \delta\lambda^i = -\tfrac{i}{2}\sigma^{\mu\nu}\xi^i F_{\mu\nu} - \gamma^\mu\nabla_\mu(M+\gamma_5 N)\xi^i - i\gamma_5\xi^i[M,N]$

then $[\delta^{(1)}, \delta^{(2)}] = 2i\bar\xi_i^{(1)}\gamma^\mu\xi^{i(2)}\partial_\mu + \delta_{gauge}$, $\quad \{Q_i, \bar Q_j\} = 2\delta_{ij}\gamma^\mu P_\mu$

as in WZ gauge

matter multiplet: $s_0=0$, $z=m$, $N=2$ spectrum is double of $N=1$
(hypermultiplet) (chiral multiplet)

- In order to see $N=2$ on a single chiral $N=1$ multiplet, we require a second SUSY generator $Q_{\alpha 2}$, different from $Q_{\alpha 1}$, which sees the multiplet as anti-chiral
  $[A,\bar{Q}_1]=0$, $[A,Q_2]=0$, $[A,Q_1]=2i\psi$, $[A,\bar{Q}_2]=\alpha\bar{\psi}$

- For $N=2$: $\{Q^i_\alpha,\bar{Q}_{\dot{\beta}j}\}=2m\,\delta^i{}_j\,\delta_{\alpha\dot{\beta}}$, $\{Q^i_\alpha,Q^j_\beta\}=2\dot{z}\,g^{ij}\varepsilon_{\alpha\beta}$,
  $\{\bar{Q}_{\dot{\alpha}i},\bar{Q}_{\dot{\beta}j}\}=-2\dot{z}\,g_{ij}\,\varepsilon_{\dot{\alpha}\dot{\beta}}$.

  $\dot{z}$ is the quantum no. of Z: CENTRAL CHARGE

  (The simplest algebra with a central charge is for $N=2$, one Z)
  Diagonalising above for $Q_{(1)}\equiv Q_1{}^1-\bar{Q}_{\dot{z}2}$, $Q_{(2)}=Q_2{}^1+\bar{Q}_{i2}$,
  (+ hermitian conjs) $Q_{(3)}=Q_1{}^1+\bar{Q}_{\dot{z}2}$, $Q_{(4)}=Q_2{}^1-\bar{Q}_{i2}$

  $\{Q_{(1)},Q_{(2)}\}=\{Q^\dagger_{(1)},Q^\dagger_{(2)}\}=0$, $\{Q_{(r)},Q^\dagger_{(s)}\}=4(m\pm\dot{z})\delta_{rs}$ $\begin{matrix}+ \text{ for } r=1,2\\ - \text{ for } r=3,4\end{matrix}$

$\Rightarrow$ for $\dot{z}=0$ have 4 raising operators $\Rightarrow$ get non-renormalisable theory
  for $m=|\dot{z}|$ only 2 raising operators as in massless case.
  operator equivalent: $P^2=Z^2$: Multiplet shortening condition.

- $\{\psi,\bar{Q}_{\dot{\alpha}1}\}=\not{\partial}A$, $\{Q_2,\bar{\psi}\}=2i\alpha^{-1}\not{\partial}A$, $\{Q_1,\bar{\psi}\}=\not{\partial}A^\dagger$, $\{\bar{\psi},Q_2\}=-2i\alpha^{*}\not{\partial}A^\dagger$

  $\Rightarrow[A,\{Q,\bar{S}\}]=\left(\frac{4}{\alpha^*}+\alpha\right)\not{\partial}A^\dagger$, we require $\{Q_1,\bar{Q}_2\}=0$ so $A=$ const.

- Therefore we require 2 complex fields A_i, on which the 2 SUSY generators Q_i act: $[A_i,\bar{Q}_j]+[A_j,\bar{Q}_i]=0\Rightarrow D_{\alpha(i}\varphi_{j)}=\bar{D}_{\dot{\alpha}(i}\varphi_{j)}=0$
  defines the hypermultiplet φ in analogy to the χSF definition.
  $\varphi_i=(A_i;\psi;f_i)$ chiral superfield

- $\delta A_i=2\bar{\xi}_i\psi$ $\quad\delta\psi=-i\zeta^i f_i-i\not{\partial}\zeta^i A_i$, $\delta f_i=2\bar{\xi}_i\not{\partial}\psi$

  $[\delta^{(1)},\delta^{(2)}]=2i\,\bar{\xi}_i{}^{(1)}\gamma^\mu\zeta^{i(2)}\partial_\mu+2i\zeta_i^{(1)}\zeta^{i(2)}\delta_Z$

  $\delta_Z A_i=F_i$, $\delta_Z\psi=\not{\partial}\psi$, $\delta_Z F_i=\Box A_i$, $[\delta,\delta_Z]=0$, $\delta_Z^2=\Box\Leftrightarrow P^2=Z^2$

- Construct a Lagrangian from φ_i, $\delta_Z\varphi_i$, eq. of motion $\delta_Z\varphi_i=im\varphi_i$

$$\mathcal{L}=\frac{i}{2}(\bar{\varphi}^i\cdot\delta_Z\varphi_i)+\frac{m}{2}(\bar{\varphi}^i\cdot\varphi_i)$$

(Can easily be extended to include gauge interactions)

SUSY in $d=5,6$

- Spinors in higher dimensions: $\{\Gamma_a,\Gamma_b\}=2\eta_{ab}\mathbb{1}$, $\Sigma_{ab}=i\Gamma_{[a}\Gamma_{b]}$

$$\Gamma_\mu=\begin{pmatrix}0&\gamma_\mu\\\gamma_\mu&0\end{pmatrix},\ \Gamma_5=\begin{pmatrix}0&\gamma_5\\\gamma_5&0\end{pmatrix},\ \Gamma_6=\begin{bmatrix}0&-\mathbb{1}\\\mathbb{1}&0\end{bmatrix},\ \Gamma_7=\begin{bmatrix}-\mathbb{1}&0\\0&\mathbb{1}\end{bmatrix}$$

Complex matrix dim of spinor reps: $n=\begin{cases}2^{d/2} & d \text{ even}\\ 2^{(d-1)/2} & d \text{ odd}\end{cases}$

Smallest possible dim of spinor reps is $2n$, Large. This can be reduced by reality or chirality conditions. Note there are 8 spinorial charges in $d=6$, like $N=2$ in $d=4$.
Chirality conditions only possible in even dimensions where Γ_{d+1} is non-trivial. The extension of $(1\pm\gamma_5)/2$ in higher d is $\frac{1}{2}(1\pm\sqrt{\beta}\,\Gamma_{d+1})$. Reality conditions are Majorana conditions
i.e. $\psi=X\psi^*$ (not poss in $d=5,6$)

- Trivial dimensional reduction of $d=6$ SUSY.
  for
  $$\mathcal{L}=\mathrm{tr}\left\{-\frac{1}{4}F_{ab}F^{ab}+\frac{i}{2}\bar{\lambda}\Gamma^a\overleftrightarrow{\nabla}_a\lambda\right\}$$ for a gauge field A_a

  where $a=0..3,5,6$ and chiral spinor $\lambda=\frac{1}{2}(1-\Gamma_7)\lambda$. Let $\lambda=\begin{pmatrix}\chi\\0\end{pmatrix}$ for a complex 4-spinor χ:
  $$\frac{i}{2}\bar{\lambda}\Gamma^a\overleftrightarrow{\nabla}\lambda=\frac{i}{2}\bar{\chi}\gamma_\mu\overleftrightarrow{\nabla}\chi-\frac{i}{2}\bar{\chi}\gamma_5\overleftrightarrow{\nabla}_5\chi-\frac{i}{2}\bar{\chi}\overleftrightarrow{\nabla}_6\chi$$

  $\partial_5=\partial_6=0$: $\nabla_{5,6}\chi=i[A_{5,6},\chi]$, $F_{\mu5}=\nabla_\mu A_5$, $F_{\mu6}=\nabla_\mu A_6$, $F_{56}=i[A_5,A_6]$

$$\mathcal{L}=\mathrm{tr}\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\nabla_\mu A_5\nabla^\mu A_5+\frac{1}{2}\nabla_\mu A_6\nabla^\mu A_6+\frac{i}{2}\bar{\chi}\gamma^\mu\overleftrightarrow{\nabla}_\mu\chi$$
$$-\bar{\chi}[\chi,A_6]-\bar{\chi}\gamma_5[\chi,A_5]+\frac{1}{2}[A_5,A_6]^2\Big\}$$

  which is identical to the $N=2$ Lagrangian with $\chi=\frac{1}{\sqrt{2}}(\lambda_1-i\lambda_2)$, $A_5=N$, $A_6=M$
  and obey the SUSY transforms in $d=6$ with $\zeta=\frac{1}{\sqrt{2}}\begin{pmatrix}\zeta_1-i\zeta_2\\0\end{pmatrix}$
  $i\bar{\zeta}\Gamma_\mu\lambda=\frac{i}{2}\bar{\zeta}_i\gamma_\mu\lambda_i+\frac{\varepsilon_{ij}}{2}\bar{\zeta}_i\gamma_\mu\lambda_j$,
  $i\bar{\zeta}\Gamma_5\lambda=\quad\gamma_\mu\to\gamma_5$
  $i\bar{\zeta}\Gamma_6\lambda=\quad\gamma_\mu\to\mathbb{1}$

  $\delta A_a=i\bar{\zeta}\Gamma_a\lambda-i\bar{\lambda}\Gamma_a\zeta$
  $\delta\lambda=-\frac{i}{2}\Sigma^{ab}\zeta F_{ab}$

  generalise to case $\partial_5\neq0\neq\partial_6$

and we have

$$[\delta^{(1)},\delta^{(2)}] = 2i(\bar\xi^{(1)}\Gamma^a\xi^{(2)} - \bar\xi^{(2)}\Gamma^a\xi^{(1)})\partial_a + \delta_{gauge} + \text{eq. of motion}$$

This is all consistent with $N=2$ SUSY with 2 central charges in $d=4$:

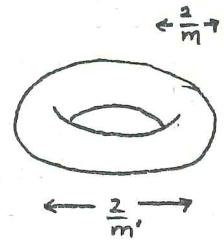
$$\{Q_i,\bar Q_j\} = 2\,\delta_{ij}\gamma^\mu P_\mu + 2i\,\varepsilon_{ij} Z + 2i\,\varepsilon_{ij}\gamma_5 Z' , \quad [Q_i,P_\mu] = [Q_i,Z] = [Q_i,Z'] = 0$$

$$[P_\mu,P_\nu] = [P_\mu,Z] = [P_\mu,Z'] = [Z,Z'] = 0$$

$$\{Q,\bar Q\} = (1+\Gamma_7)\Gamma^a P_a , \quad \{Q,Q\} = 0 , \quad [Q,P_a] = 0 , \quad \{P_a,P_b\} = 0$$

with $Q = \frac{1}{\sqrt2}\begin{pmatrix} 0 \\ Q_1 - iQ_2 \end{pmatrix}$, $P_5 = -Z'$, $P_6 = -Z$

- For a hypermultiplet consisting of 2 complex scalars A, B and an anti-chiral spinor $\Psi = \frac12(1+\Gamma_7)\Psi$



$$\mathcal{L} = \partial_a A^\dagger \partial^a A + \partial_a B^\dagger \partial^a B + \frac{i}{4}\bar\Psi\Gamma^a \overleftrightarrow{\partial_a}\Psi$$

which is invariant under the transformations

$\delta A = \bar\xi\Psi$, $\delta B = \bar\xi^c\Psi$, $\delta\Psi = -i\Gamma^a\partial_a(A\xi + B\xi^c)$

$$[\delta^{(1)},\delta^{(2)}] = 2i(\bar\xi^{(1)}\Gamma^a\xi^{(2)} - \bar\xi^{(2)}\Gamma^a\xi^{(1)})\partial_a + \text{eq. of motion.}$$

- Assume fields periodic in x^5, x^6 with periods $1/m$, $1/m'$, Fourier decompose:

eg. $A(x^\mu,x^5,x^6) = \sum_{nn'} e^{(-in'm'x^5 - inmx^6)} A_{nn'}(x^\mu)$

$$\mathcal{L}_{nn'} = \partial_\mu A^\dagger_{nn'}\partial^\mu A^{nn'} + \partial_\mu B^\dagger_{nn'}\partial^\mu B_{nn'} + \frac{i}{4}\bar\Psi_{nn'}\overleftrightarrow{\partial}\Psi^{nn'}$$

$$- (n'^2m'^2 + n^2m^2)(A^\dagger_{nn'}A_{nn'} + B^\dagger_{nn'}B_{nn'})$$

$$- \frac{n'm'}{2}\bar\Psi_{nn'}\gamma_5\Psi_{nn'} + \frac{nm}{2}\bar\Psi_{nn'}\Psi_{nn'}$$

where $\Psi_{nn'}$ are 4d Dirac spinors, bottom half of 6d anti-chiral spinors, $M_{nn'} = \sqrt{n^2m^2 + n'^2m'^2}$

$Z_i^2 = P_\mu P^\mu \Leftrightarrow \partial_a\partial^a = 0$, $\quad \delta_2\varphi = im\varphi \Leftrightarrow \frac{\partial}{\partial x^6}\varphi = -im\varphi$

References :

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