# Casimir energy and radius stabilization

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Werkstattseminar SS 2011

## References:


- Itzykson, Zuber – "Quantum Field Theory" p. 138 – 141
- Pontón, Poppitz – "Casimir energy and radius stabilization in 5d & 6d Orbifolds" hep-ph/0105021v3
- Ghilencea, Hoover, Burgess, Quevedo – "Casimir energies for 6D supergravities compactified on $T_2/Z_N$ with Wilson lines" hep-th/0506164
- Buchmüller, Catena, Schmidt-Hoberg – "Enhanced symmetries of Orbifolds from moduli stabilization" 0902.4512v3 [hep-th]

  and "Small extra dimensions from the interplay of gauge and supersymmetry breaking" 0803.4501v2 [hep-ph]

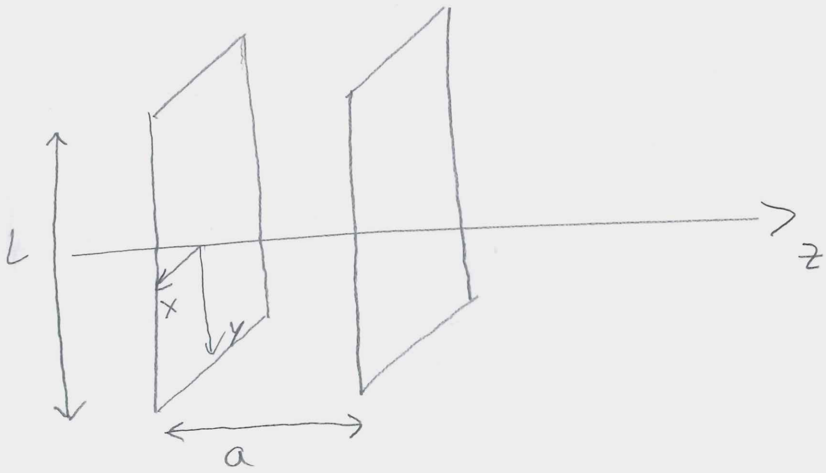
# I Casimir Effect in QED

Vacuum energy in 2nd quantization:

$$H = \sum_\alpha \frac{\hbar}{2} \omega_\alpha = \infty$$

is disregarded since it is unobservable in scattering processes. However, its variation can be measured.

Consider two large parallel perfectly conducting plates:




$a << L$

$E_{\parallel}|_R = B_{\perp}|_R = 0$ + Maxwellequation's for the vacuum $\Rightarrow \sin(k_z a) = 0 \Rightarrow k_z = \frac{n\pi}{a}, n \in \mathbb{N}_0$

$$\Rightarrow E = \sum_\alpha \frac{1}{2} \hbar \omega_\alpha = \frac{\hbar c}{2} \sum_\alpha |\vec{k}_\alpha| = \frac{\hbar c}{2} \int \frac{L^2 d^2k_\parallel}{(2\pi)^2} \, 2 \sum_{n=1}^{\infty} \left(k_\parallel^2 + \frac{n^2\pi^2}{a^2}\right)^{1/2}$$

Divergencies: $\int_0^\infty d\vec{k}_\parallel$ & $\sum_{n=1}^\infty$

$\Rightarrow$ Need to regularise & introduce counter terms

- Itzykson, Zuber: $E_0 = \frac{\hbar c}{2} \int \frac{L^2 dk_x dk_y a dk_z}{(2\pi)^3} 2\sqrt{k_x^2+k_y^2+k_z^2}$

  & cut-off $k \leq k_m$

- dimensional reg.: $\int d^2k_\parallel \to \int d^d k_\parallel$

  & regularise $\sum_{n=1}^\infty$

- zeta function reg.: $\omega_k \to \omega_k \, \omega_k^{-s}$ $\overset{\int d^2k_\parallel}{\vec{\sum}}$ $|_{s=0}$

  $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ (analytic continuation)

  for example $\zeta(-1) = \sum_{n=1}^\infty n = \frac{-1}{12}$

$\omega_k \to \omega_k \, \omega_k^{-s}$

$$E = \frac{L^2}{2} \int \frac{d^2 k_\parallel}{(2\pi)^2} 2 \sum_{n=1}^\infty \left(k_\parallel^2 + \frac{n^2\pi^2}{a^2}\right)^{\frac{1-s}{2}}$$

pol. coord. $d^2k_\parallel = dk\, k\, d\Omega_2$

$$= L^2 \frac{2\pi}{(2\pi)^2} \int_0^\infty dk \; k \sum_{n=1}^\infty \left(k^2 + \frac{n^2\pi^2}{a^2}\right)^{\frac{1-s}{2}}$$

This is finite for $s > 3$, then analytically continue to $s=0$ (Pole at $s=3$)

$$E = \frac{L^2}{2\pi} \sum_{n=1}^{\infty} \left[ \frac{1}{2} \frac{2}{3-s} \left( k^2 + \frac{n^2\pi^2}{a^2} \right)^{\frac{3-s}{2}} \right]_0^{\infty}$$

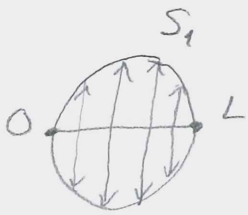
$$= - \frac{L^2 \left(\frac{\pi}{a}\right)^{3-s}}{2\pi(3-s)} \underbrace{\sum_{n=1}^{\infty} n^{3-s}}_{= \zeta(s-3)} \overset{s \to 0}{=} - \frac{L^2\pi^2}{6a^3} \underbrace{\zeta(-3)}_{1/120} \boxed{= - \frac{L^2\pi^2}{720\, a^3}}$$

=> Force per unit area:

$$\boxed{F = \frac{-dE}{da} \frac{1}{L^2} = - \frac{\pi^2}{240} \frac{\hbar c}{a^4} \simeq - \frac{1{,}3 \cdot 10^{-7}}{(a_{\mu m})^4} \frac{N}{cm^2}}$$

=> Has been measured!

## II Casimir energy on $S^1/\mathbb{Z}_2$




Choose the following parametrization of the interval $S_1$:

$$ds^2 = \phi^{-1/3} \left( g_{\mu\nu} + A_\mu A_\nu \phi \right) dx^\mu dx^\nu + 2\phi^{2/3} A_\mu dx^\mu dy + \phi^{2/3} dy^2 \qquad (*)$$

with $y \in [-L, L]$ before & $y \in [0, L)$ after orbifolding

The parametrization is chosen such that the 4d effective theory is in the Einstein frame.
The scale $L$ doesn't have any physical significance.
The physical size of the 5th dimension is $\phi^{1/3} L$.

$\mathbb{Z}_2$ symmetry $y \simeq -y$ => $ds^2$ has to be invariant

$$\Rightarrow \quad g_{\mu\nu}(y) = g_{\mu\nu}(-y), \quad \underbrace{A_\mu(y) = -A_\mu(-y)}_{\Downarrow}, \quad \phi(y) = \phi(-y)$$

no zero mode!

zero-mode effective theory (valid below energies $(\phi^{1/3}L)^{-1}$):

$$M_5^3 \int d^5x \sqrt{\mathfrak{g}}\, R(\mathfrak{g}) = \underset{\underset{L M_5^3}{\shortparallel}}{M_4^2} \int d^4x \sqrt{g}\left[R(g) + \frac{1}{6}\frac{\partial_\mu\phi\, \partial_\nu\phi\, g^{\mu\nu}}{\phi^2}\right]$$

<u>Kaluza-Klein Masses:</u> (see Seminar talk by J. Louis)

periodic boundary conditions: $\hat{\phi}(x^\mu, y+L) = \hat{\phi}(x^\mu, y)$

$$\Rightarrow \hat{\phi}(x^\mu, y) = \sum_{n=-\infty}^{\infty} \hat{\phi}^{(n)}(x^\mu)\, e^{-i\frac{\pi n y}{L}}$$

$$\mathfrak{g}_{MN} \overset{A_\mu = 0}{=} \begin{pmatrix} \phi^{-1/3} g_{\mu\nu} & 0 \\ 0 & \phi^{2/3} \end{pmatrix} \Rightarrow \mathfrak{g}^{MN} = \begin{pmatrix} \phi^{1/3} g^{\mu\nu} & 0 \\ 0 & \phi^{-2/3} \end{pmatrix}$$

$$\Box_5 = g^{MN}\partial_M \partial_N = \phi^{1/3}\Box_4 + \phi^{-2/3}\partial_y^2$$

$$\Box_5 \hat{\phi} = 0 \iff \sum_{n=-\infty}^{\infty}\left(\phi^{1/3}\Box_4 - \phi^{-2/3}\frac{n^2\pi^2}{L^2}\right)\hat{\phi}^{(n)}(x^\mu)\, e^{-\frac{i\pi n y}{L}} = 0$$

$$\Rightarrow \Box_4 \hat{\phi}^{(n)}(x^\mu) = \frac{\pi^2 n^2}{\phi L^2}\hat{\phi}^{(n)}(x^\mu) \Rightarrow \boxed{M_n^2 = \frac{\pi^2 n^2}{\phi L^2}}$$

anti-per. b.c. : $M_n^2 = \frac{\pi^2 (n+\frac{1}{2})^2}{\phi L^2}$

## Casimir energy

Consider background with vanishing 4d cosmological constant:

$$g_{\mu\nu} = \eta_{\mu\nu}, \quad A_\mu = 0, \quad \phi = \text{const}$$

The Casimir energy of a real scalar field is the 1-loop effective potential

$$V = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\log(k^2 + m^2)$$

In the case of periodic b.c. :

$\mu$ = Mass of canonically normalized field

periodic b.c.

$$V^{+,sc} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int \frac{d^4k}{(2\pi)^4}\left(k^2 + \underbrace{\frac{\pi^2 n^2}{\phi L^2}}_{\text{KK masses}} + \frac{\mu^2}{\phi^{1/3}}\right)$$

$$\equiv -\frac{d}{ds}\zeta^{+,sc}(s)\Big|_{s=0}$$ (Zeta function regularisation)

with $$\zeta^{+,sc} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^4k}{(2\pi)^4}\left(k^2+\frac{\pi^2n^2}{\phi L^2}+\frac{\mu^2}{\phi^{1/3}}\right)^{-s}$$

$\Downarrow$ Wick Rotation

$$\int\frac{dk_E\,k_E^3}{(2\pi)^4}\int d\Omega_4 = \int\frac{dk_E\,k_E^3}{8\pi^2}$$

$x=k_E^2$
$$= \frac{1}{32\pi^2}\sum_{n=-\infty}^{\infty}\int dx\; x\left(x+\frac{\pi^2n^2}{\phi L^2}+\frac{\mu^2}{\phi^{1/3}}\right)^{-s}$$ Converges for $s \geq 3$

$$= \frac{1}{32\pi^2}\frac{1}{(2-s)(1-s)}\sum_{n=-\infty}^{\infty}\left(\frac{\pi^2n^2}{\phi L^2}+\frac{\mu^2}{\phi^{1/3}}\right)^{2-s}$$

$$= \frac{1}{32\pi^2}\frac{1}{(2-s)(1-s)}\frac{\pi^{4-2s}}{L^{4-2s}\phi^{2-s}}F\left(s-2,\frac{\mu L\phi^{1/3}}{\pi}\right)$$

with $$F(s,c) = \sum_{n=-\infty}^{\infty}(n^2+c^2)^{-s}$$

In the massless case $\mu=0$: $F(s,0) = 2\zeta(2s)$

$$\Rightarrow \boxed{V^{+,sc} = -\frac{\pi^2}{16}\frac{\zeta'(-4)}{\phi^2L^4} = -\frac{3\zeta(5)}{64\pi^2}\frac{1}{\phi^2L^4}}$$

$\zeta'(-4) = \frac{3}{4\pi^4}\zeta(5)$

$\hookrightarrow$ attractive Force!

For anti-periodic b.c. $n \to n + \frac{1}{2}$

$$\Rightarrow \zeta(2s-4) \to (2^s - 1)\zeta(s)$$

$$\Rightarrow V^{-,sc} = -\frac{15}{16} V^{+,sc} \quad \to \text{repulsive Force!}$$

In the massive case (without derivation)

$$\boxed{V_\mu^{\pm,sc} = -\frac{3\mu^6 L^2}{\pi^2 x^6}\left[Li_5(\pm e^{-x}) + x\, Li_4(e^{-x}) + \frac{x^2}{3} Li_3(e^{-x})\right]}$$

with $x = 2\mu L \phi^{1/3}$ & polylogarithm $Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$

Finally, we have the 5d cosmological constant counter term:

$$\alpha \int d^5x \sqrt{g} = \alpha L \int d^4x\, \phi^{-1/3}$$

$$\Rightarrow \boxed{V_{c.c.} = \alpha L \phi^{-1/3} \sim x^{-6}}$$

For a consistent calculation (flat background) we have to demand

$$V(\phi_{min}) = V'(\phi_{min}) = 0$$

To obtain a minimum it would be sufficient to consider a 2-term Potential, for example a massless scalar (dilaton) and the c.c. counter term.

But then we can't fullfill $V(\phi_{min}) = 0$

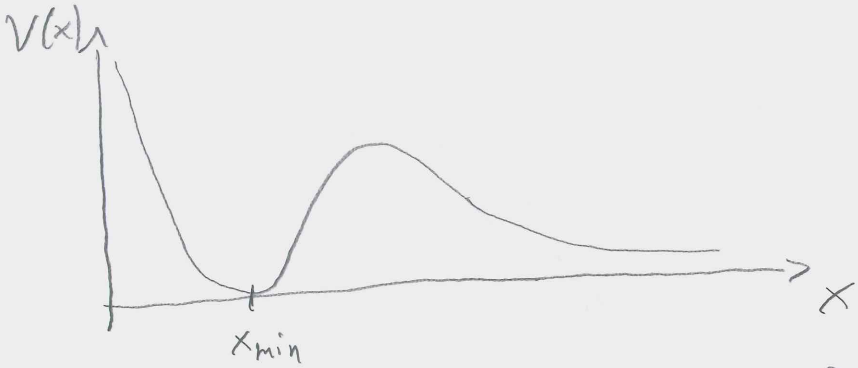
$\Rightarrow$ Need 3-term structure!

<u>Toy model</u>: Massless scalar + Massive scalar + c.c.

$$V(x) = V_{c.c.} + V_\mu^{+,sc} + V^{+,sc}$$

$$\boxed{V(x) = \frac{\text{const}}{x^6}\left[\alpha x^5 + \beta\left(Li_5(e^{-x}) + x\, Li_4(e^{-x}) + \frac{x^2}{3} Li_3(e^{-x})\right) + \gamma\right]}$$

We can have a meta-stable Minimum for $\alpha, \beta > 0$, $\gamma < 0$ with fine-tuned $\alpha$ so that $V(x_{min}) = V'(x_{min}) = 0$




$\Rightarrow$ Stabilized the radius $\langle \phi^{1/3} L \rangle$ of the $S_1$ using quantum effects of the vacuum <u>only</u>.

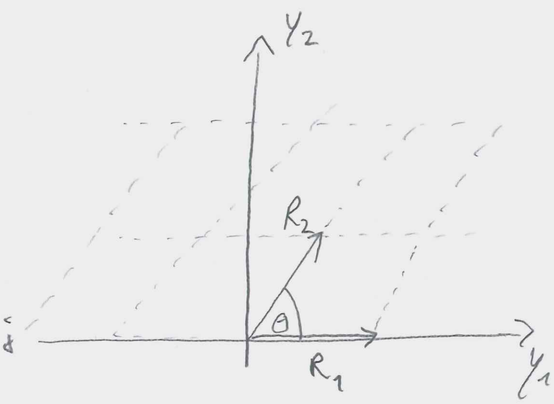
$\Rightarrow$ Typically $x_{min} = O(1)$ $\Rightarrow$ $\boxed{\langle \phi^{1/3} L \rangle = O(\mu^{-1})}$

In our mode expansion calculating $V^{+,sc}$ we have neglected that we are working in an orbifold. For fields even (odd) under the $\mathbb{Z}_2$ we have to throw away all odd (even) KK-excitations.

$$\Rightarrow \quad V^{+,sc}_{S_1/\mathbb{Z}_2} = \frac{1}{2} V^{+,sc}_{S_1}$$

Also in the $S_1/\mathbb{Z}_2$ case we get additional divergencies at the fixed points which can be renormalized by brane tension counter terms.

## III Casimir energy on $T^2/\mathbb{Z}_2$




Choose parametrization

$$ds^2 = A^{-1} g_{\mu\nu} dx^\mu dx^\nu + A \, \delta_{ij} dy^i dy^j$$

$$\delta_{ij} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad \tau = \tau_1 + i\tau_2 \qquad y^i \in [0, L]$$

Relation to radii $R_1, R_2$ and angle $\Theta$ on the lattice:

$$2\pi R_1 = L\sqrt{\frac{A}{\tau_2}}, \quad 2\pi R_2 = |\tau| L \sqrt{\frac{A}{\tau_2}}, \quad \Theta = \arccos \frac{\tau_1}{|\tau|}$$

Dimensional reduction of the 6d Einstein-Hilbert action: 

$$M_6^4 \int d^6x \sqrt{\mathfrak{g}}\, R(\mathfrak{g}) = M_6^4 L^2 \int d^4x \sqrt{g} \left( R(g) + \frac{g^{\mu\nu}\partial_\mu A \partial_\nu A}{A^2} + \frac{g^{\mu\nu}\partial_\mu \tau \partial_\nu \bar{\tau}}{2\tau_2^2} \right)$$

The group $SL(2,\mathbb{Z})$ $\quad \tau \to \frac{a\tau + b}{c\tau + d}$ ; $a,b,c,d \in \mathbb{Z}$, $ad - bc = 1$

relates modular parameters of diffeomorphic tori.

KK-mode expansion:

$$\phi(x,y) = \frac{1}{\sqrt{A}\,L} \sum_{m,n=-\infty}^{\infty} \phi_{n,m}(x) \exp\left[ \frac{2\pi i}{L\sqrt{A}\,\tau_2} \left( (m+\alpha)(\tau_2 y_1 - \tau_1 y_2) + (n+\beta) y_2 \right) \right]$$

$$M_{n,m}^2 = \frac{(2\pi)^2}{A L^2 \tau_2} \left| n + \beta - \tau(m+\alpha) \right|^2 \qquad \alpha, \beta \in \left\{0, \tfrac{1}{2}\right\}$$

## Stabilization of Shape Moduli $\tau_1, \tau_2$

→ Fixed points of the modular symmetry are extrema of $V_{eff}$. For example $\alpha = \beta = 0$ $(+,+)$ b.c. $(\tau_1, \tau_2) = (1/2, \sqrt{3}/2)$ is a Minimum.

## Stabilization of Volume Modulus $A$

→ $$V_{tot}(A, \tau_1, \tau_2) = V_{casimir}(A, \tau_1, \tau_2) + \underbrace{V_{ce}(A)}_{\text{tree-level contribution}} + \underbrace{V_{brane}(A)}_{\text{brane-tension counterterm}}$$

($V_{c.c.}$ does not depend on $A$ since $\sqrt{g} \sim A^{-1} A = 1$)

$$V_{casimir}(A, \tau_1, \tau_2) = \frac{1}{2} \Big[\sum_{m,n}\Big] \int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + \frac{M_{m,n}^2}{A} + \frac{M^2}{A}\right)$$

↳ some modes have to be modded out

$$= - \frac{d\,\zeta(s)}{ds}\Big|_{s=0} \rightarrow \text{extract finite part of Casimir Energy}$$

⇒ Brane tension can be tuned so that $V_{tot}(A_{min}) = 0$ at a local Minimum $A_{min}$ with $\boxed{\langle \sqrt{A_{min}}\, L \rangle \sim M^{-1}}$.