Calabi-Yau compactification

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I) Introduction

Superstring theories are candidates for a quantum theory of gravity including fermions and without tachyons. However, dimension of spacetime is required to be $D = 10$ in order for negative norm states to be absent. Being aware of only 4 extended dimensions, we consider the possibility that the remaining 6 dimensions are compactified in order to arrive at a model that may connect to a supersymmetric standard model at lower energy. Thus, our spacetime is said to be a product manifold

$$M^4 \times Y^6$$

where M^4 is a maximally symmetric 4-dim. Lorentz manifold (de Sitter, Minkowski, anti de Sitter) and Y^6 is a compact 6-dim. Riemannian manifold.

In this talk want to discuss the following 2 questions:

- What is Y^6 ? $\rightsquigarrow$ II)
- How large is Y^6 ? $\rightsquigarrow$ III)

References:

- Candelas, Horowitz, Strominger, Witten: "Vacuum configurations for superstrings", Nucl. Phys. B258, 46-74 (1985)
- Green, Schwarz, Witten: "Superstring theory" (textbook, 1987)
- Louis: "Phenomenological aspects of string theory"
- Greene: "String theory on Calabi-Yau manifolds", hep-th/9702155v1

II) Low-energy SUSY in heterotic string theory

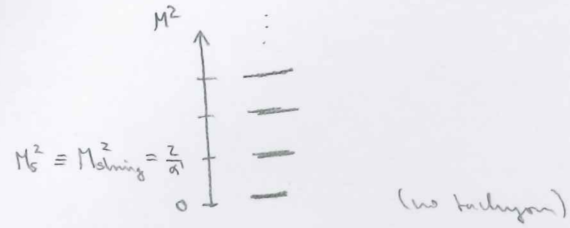
Heterotic string in 10-dim flat space

Among the 5 superstring theories (I, IIA, IIB, het SO(32), het $E_8 \times E_8$) we consider the heterotic ones because they have already internal gauge groups that contain the standard model gauge group:

$$SO(32) \supset SU(3) \times SU(2) \times U(1)$$

$$\text{or} \quad E_8 \times E_8 \supset SU(3) \times SU(2) \times U(1)$$

Their particle spectra (as in all superstring theories) consist of massless modes and an infinite tower of massive string excitations,



classified in terms of representations of their massless/massive little group of SO(1,9). Moreover, they form complete N=1 in D=10 SUSY multiplets, e.g. for the massless modes:

g_{MN}	metric / graviton	supergravity multiplet
ψ_M	spin-$\frac{3}{2}$ field / gravitino	
B_{MN}	2-form / Kalb-Ramond field	
λ	spin-$\frac{1}{2}$ field / dilatino	
ϕ	scalar field / dilaton	
A^a_M	vector / gauge boson	super Yang-Mills multiplet in the adjoint representation of the gauge group
χ^a	spin-$\frac{1}{2}$ field / gaugino	

Here $M, N = 1, \ldots, 10$; spinor indices are suppressed.

At M_S one would like to integrate out massive string excitations but this is technically difficult. Instead, one analyzes scattering amplitudes for massless particles with external momenta $\frac{p^2}{M_S^2} \ll 1$ and writes down a field theory action that in this approximation reproduces the results from string theory (S-matrix approach). Furthermore, this low-energy effective action shall be locally N=1 in D=10 supersymmetric. Recall that N=1 means that the

supercharges transform according to the smallest spinor representation of $SO(1,9)$ which we denote by its dimension: 16. (There is also another 16-dim. representation to be named $16'$.)
Hence, we have 16 supercharges.

In what follows, we will need the SUSY transformations of fermionic fields,

$$\delta_\eta \psi_M \sim D_M \eta + \mathcal{O}(H_{MNP}, \eta) + (\text{fermi})^2$$

$$\delta_\eta \lambda \sim \frac{1}{\sqrt{\phi}} (\Gamma \cdot \partial\phi)\eta + \mathcal{O}(H_{MNP}, \eta) + (\text{fermi})^2$$

$$\delta_\eta \chi^a \sim \frac{1}{\sqrt{\phi}} \Gamma^{MN} F^a_{MN} \eta + (\text{fermi})^2$$

where $\eta(X^M)$ is an infinitesimal SUSY parameter that depends on the spacetime coordinate X^M, and $D_M\eta = \partial_M \eta + \frac{1}{2}\omega_M{}^{ab}\Sigma_{ab}\eta$ is given in terms of the spin connection $\omega_M{}^{ab}$ and the Lorentz generators in the spinor representation. Moreover, $F = dA + A \wedge A$ $(A = A^a_M T^a dX^M)$ and $H = dB - \omega_{3Y} + \omega_{3L}$ with YM-Chern-Simons 3-form ω_{3Y} and Lorentz Chern-Simons 3-form ω_{3L} ...

Finally, Γ^M are γ-matrices in $D=10$ satisfying $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$, and $\Gamma^{MN} = \Gamma^{[M}\Gamma^{N]}$.

Heterotic string in $M^4 \times Y^6$ (rather heterotic supergravity in $M^4 \times Y^6$)

Consider heterotic supergravity from above but now on $M^4 \times Y^6$. It turns out that the demand that the 4-dim effective theory be invariant under N SUSY restricts the class of possible Y^6 manifolds, or put differently, Y^6 determines the amount of SUSY in 4D.

As $N=1$ in $D=4$ is of phenomenological interest (hierarchy problem), we start with this case:

The vacuum of a classical field theory is a background field configuration that is left invariant by the symmetry of the theory, in particular, supersymmetry:

" $(\text{bosonic field})^0 = \langle \text{bosonic field} \rangle$ "

" $(\text{fermionic field})^0 = \langle \text{fermionic field} \rangle$ "

Non-zero background values for fermionic fields would break 4D Lorentz symmetry because under the branching $SO(1,9) > SO(1,3) \times SO(6)$, $16 = (2,4) \oplus (\bar{2},\bar{4})$, etc. which does not contain a subspace that is left invariant by $SO(1,3)$.

$$\Rightarrow \delta_\eta (\text{bosonic field})^0 \sim (\text{fermion})^0 \eta = 0$$

which is why only variations of fermions were listed above.

On the other hand, (bosonic field)$^0 \neq 0$ ok as long as 4D Lorentz symmetry is preserved,

e.g. $\phi^0 \neq 0$, $B^0_{ij} \neq 0$, $H^0_{ijk} \neq 0$, $F^0_{ij} \neq 0, \ldots$, but e.g. $B^0_{\mu\nu} = 0$ where $\mu,\nu = 1,\ldots,4$ and $i,j,k = 5,\ldots,10$.

Choosing $H^0_{ijk} = 0$, η parametrizes a supersymmetry of the vacuum if and only if

$$\left.\begin{aligned} 0 &\stackrel{!}{=} \delta_\eta \psi^0_M \sim D_M \eta \\ 0 &\stackrel{!}{=} \delta_\eta \lambda^0 \sim \frac{1}{\sqrt{\phi}} (\Gamma \cdot \partial \phi^0) \eta \\ 0 &\stackrel{!}{=} \delta_\eta \chi^{a0} \sim \frac{1}{\sqrt{\phi}} \Gamma^{MN} F^a_{MN} \eta \end{aligned}\right\} \quad (*)$$

Now drop superscript ^{0}!

Solve $(*)$: $\boxed{\delta_\eta \lambda = 0}$ $\Rightarrow$ $0 = \Gamma^M \partial_M \phi \, \eta = \Gamma^i \eta \, \partial_i \phi$ because $\partial_\mu \phi = 0$

$$\begin{aligned} \Rightarrow \quad 0 &= (\Gamma^j \partial_j \phi)(\Gamma^i \partial_i \phi) \eta \\ &= \Gamma^j \Gamma^i \eta \, \partial_j \phi \, \partial_i \phi \\ &= \underbrace{\Gamma^{(j} \Gamma^{i)}}_{g^{ij}} \eta \, \partial_j \phi \, \partial_i \phi = \eta \sum_i |\partial_i \phi|^2 \end{aligned}$$

$\Rightarrow$ $\partial_i \phi = 0 \;\; \forall i$, i.e. $\boxed{\phi = \text{const on } M^4 \times Y^6}$

$\boxed{\delta_\eta \psi_\mu = 0}$ $\Rightarrow$ $D_\mu \eta = 0$

$\Rightarrow$ $0 = [D_\mu, D_\nu] \eta = \frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \eta$ (see e.g. (6.73) in "Supergravity" by Freedman & van Proeyen)

$= \frac{1}{4} \mathcal{R} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \gamma^{\rho\sigma} \eta$ (M^4 max. symmetric)

$= \frac{1}{4} \mathcal{R} (\gamma_{\mu\nu} - \gamma_{\nu\mu}) \eta = \frac{\mathcal{R}}{2} \gamma_{\mu\nu} \eta$

$\equiv \frac{\mathcal{R}}{2} (\gamma_{\mu\nu} \otimes 1) \eta$

For $\mu \neq \nu$, $\gamma_{\mu\nu}$ is invertible $\Rightarrow$ $\boxed{\mathcal{R} = 0}$

i.e. M^4 is Minkowski space! ($D_\mu = \partial_\mu$)

(Note that generically supergravity in AdS space is also possible.)

$\Rightarrow$ $\partial_\mu \eta = 0$, i.e. η independent of $M^4 = M^{1,3}$

Now $(*)$ reads:

$$\left.\begin{aligned} 0 &\stackrel{!}{=} \delta_\eta \psi_i \sim D_i \eta \\ 0 &\stackrel{!}{=} \delta_\eta \chi^a \sim \Gamma^{ij} F^a_{ij} \eta \end{aligned}\right\} \quad (*')$$

Solving $\delta_\eta \chi^a = 0$ is beyond the scope of this talk (see e.g. Green, Schwarz, Witten) but we will now embark on a discussion of $\boxed{D_i \eta = 0}$ $\forall i$. Such an η is referred to as covariantly constant spinor field on Y^6:

Recall (from general relativity) the definition of parallel transport:

Let $\gamma : (0,1) \to Y^6$ be a smooth curve with tangent vector $t^i \frac{\partial}{\partial x^i}$ at $\gamma(t)$. Then a spinor η given at each $\gamma(t)$ is said to be parallelly transported along γ if

$$t^i D_i \eta = 0$$

at each point of the curve $\gamma(t)$. We write $P_\gamma \eta|_{\gamma(t_1)} = \eta|_{\gamma(t_2)}$ for $0 < t_1 \leq t_2 < 1$.

Our covariantly constant spinor η is defined on Y^6 and $D_i \eta = 0$ on Y^6.

Hence, η is parallelly transported along all curves in Y^6

In particular, η is parallelly transported along contractible closed curves and returns to its original value!

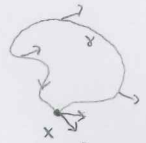
$$\boxed{P_\gamma \eta = \eta}$$

This, however, restricts the <u>holonomy</u> of Y^6:

The (restricted) holonomy group of a Riemannian manifold M with dimension n, (at $x \in M$):

$$\mathrm{Hol}_x(M) = \{ P_\gamma : T_x M \to T_x M \mid \forall \text{ closed (contractible) loops } \gamma \text{ at } x \} \subset SO(n)$$

if using Levi-Civita connection, Y^6 orientable (length conserved under parallel transport)



differs by $SO(n)$ rotation

In our case, $\dim Y^6 = 6$ and, thus, the maximal holonomy is $SO(6)$. Being a spinor of $SO(6)$, η transforms as a 4 of the universal cover group $SU(4)$.

$P_\gamma \eta = \eta$ $\forall$ closed contractible loops requires an invariant subspace in the 4 which is only possible if the holonomy is $SU(3)$ or smaller; $SU(3) \subset SU(4)$: $4 = 3 \oplus 1$

invariant under $SU(3)$

$SU(2) \subset SU(4)$: $4 = 2 \oplus 1 \oplus 1$

$SU(1) = 1 \subset SU(4)$: $4 = 1 \oplus 1 \oplus 1 \oplus 1$

e.g. $\eta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}$, $SU(3) = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & 1 \end{array} \right) \in SU(4) \right\}$

Note that for the discussion of holonomy we ignored that η also transforms under $SO(1,3)$. So one rather has:

$$SO(1,9) \supset SO(1,3) \times SO(6) \supset SO(1,3) \times SU(3)$$

$$\underbrace{16}_{\substack{\text{supercharges} \\ \text{of } N=1 \text{ in } D=10}} = (2,4) \oplus (\bar{2},\bar{4}) = (2,3) \oplus \underline{(2,1)} \oplus (\bar{2},\bar{3}) \oplus \underline{(\bar{2},1)}$$

covariantly constant on Y^6: $(2+\bar{2},1)$

$(2+\bar{2},1)$ is a real representation which, being covariantly constant, gives rise to 1 four-component Majorana spinor of $SO(1,3)$ of supercharges that leave the vacuum invariant, i.e. we have $N=1$ in $D=4$ SUSY.

We finally arrive at the following result:

holonomy / Sugra		SU(4)	SU(3)	SU(2)	1 = SU(1)
het./I	16	–	N = 1	N = 2	N = 4
IIA	16, $\overline{16}$	–	N = 2	N = 4	N = 8
IIB	16, 16	–	N = 2	N = 4	N = 8
		e.g. S^6	CY^3	$\mathbb{C}/\Gamma \times K3$	$\mathbb{C}^3/\Gamma \sim T^6$

SUSY in D = 4

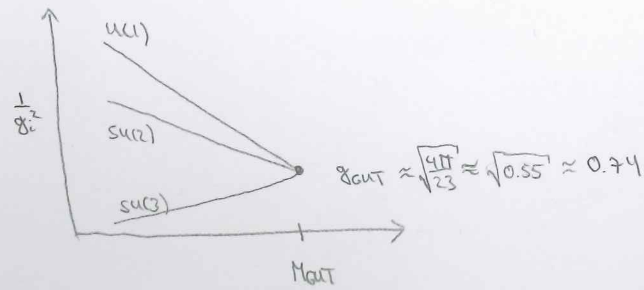
CY^3 are called Calabi-Yau threefolds.

Note that the case N=1 in D=4 is phenomenologically the most interesting one because for N ≥ 2 in D=4, massless fermions are non-chiral in that they transform in real representations of the gauge group.

III A bit of string phenomenology

heterotic string

Experimental data and extrapolation based on 1-loop beta functions point to a unification of the standard model gauge couplings at a scale $M_{GUT} \approx 3 \cdot 10^{16}$ GeV if the spectrum is that of the (next-to-)minimal supersymmetric standard model ((N)MSSM), i.e. N=1 in D=4.



In view of the result of II) one may consider heterotic string theory as a more fundamental theory of the (N)MSSM. A first Ansatz could be to set

$$M_s = M_{comp.} = M_{GUT} \approx 3 \cdot 10^{16}\ \text{GeV} \qquad V_{CP} = V = (M_{comp})^{-6}$$

which is tantamount to saying that at this scale massive string excitations are "integrated out" and, at the same scale, the resulting field theory of massless string modes is compactified on $M^{1,3} \times CY^3$. While $N=1$ in $D=4$ is guaranteed by the holonomy, we also assume that all but the (N)MSSM particles become massive and the resulting gauge group is $SU(3) \times SU(2) \times U(1)$ as otherwise gauge coupling unification is jeopardized. Moreover, it would be natural to have the matching condition

$$g_{GUT} = g_s \approx 0.74$$

at the GUT-scale. Here, g_s is the unique string coupling (constant).

However, as a theory of gravity, heterotic string theory should contain Einstein gravity. Comparing scattering amplitudes for gravitons one finds the following relation:

$$\begin{aligned} M_s &\approx 0.05\, g_s\, M_{Pl} \\ &\approx g_s \cdot 5 \cdot 10^{17}\,\text{GeV} \end{aligned}$$

For $g_s \approx 0.74$,

$$M_s \approx 3.7 \cdot 10^{17}\,\text{GeV} \approx 10\, M_{GUT}$$

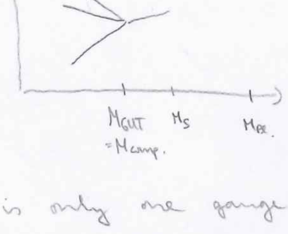
which shows that M_s and M_{GUT} coincide up to a factor of 10.

Interpretations of this mismatch range from "qualitatively the right picture" to "excluded by experimental data".

One proposal to deal with the mismatch would be to consider the following scenario

$$3 \cdot 10^{16}\,\text{GeV} \approx M_{GUT} = M_{comp} < M_s$$

where at a scale M_s string excitations are "integrated out" which yields a supergravity field theory in $D=10$ flat spacetime to be compactified at a lower scale $M_{GUT} = M_{comp}$.



Here we assume that there is only one gauge coupling in the $D=10$ field theory.

Compactification yields an effective 4-dim gauge coupling

$$\left(g^{(4)}\right)^{-2} = \left(g^{(10)}\right)^{-2} \frac{V}{l_s^6} \qquad (**)$$

$\left(l_s = \frac{1}{M_s}\right)$

that matches the unified gauge couplings

$$g_{GUT} = g^{(4)}$$

at $M_{GUT} = M_{comp}$.

However, for a perturbative D=10 coupling $g^{(10)} < 1$ one now finds the following inequality:

$$0.55 \approx g_{GUT}^2 = (g^{(4)})^2 = (g^{(10)})^2 \frac{l_s^6}{V} < \frac{l_s^6}{V}$$

Thus,

$$\frac{V}{l_s^6} < 1.82 \qquad \text{i.e. } V = \mathcal{O}(l_s^6) \text{ or smaller}$$

$$\Rightarrow \frac{M_s}{M_{comp}} < \sqrt[6]{1.82} = 1.10$$

Hence, the largest M_s would be $M_s \approx 1.10\, M_{comp}$ which is close to M_{comp}.

But for $M_s \approx M_{comp}$ we are back in the first scenario and find $M_s \approx 10\, M_{GUT}$ which is still a contradiction in terms of a factor of 10.

Note that the case $M_s < M_{comp}$ is technically much harder. Moreover, (**) no longer holds in this case.

Type I string

While compactification of type I supergravity on CY^3 gives also $N=1$ in $D=4$ SUSY, there is no generic internal gauge group. (probably D-branes needed...). Still, let us again consider the Ansatz:

$$3 \cdot 10^{16}\,\text{GeV} \approx M_{GUT} = M_{comp} < M_s$$

Here, the effective 4-dim. gauge coupling reads

$$(g^{(4)})^{-2} \sim (g^{(10)})^{-1} \frac{V}{l_s^6}$$

and consistency with Einstein gravity requires

$$M_{Pl}^2 \sim l_s^{-2} (g^{(4)})^{-4} \frac{l_s^6}{V} = (g^{(4)})^{-4} \frac{l_s^4}{V} \qquad (***)$$

Demanding $M_{GUT} = V^{\frac{1}{6}}$ and the matching condition $g^{(4)} = g_{GUT} \approx 0.74$ one can now determine l_s (M_s, respectively) s.t. (***) gives the right Planck mass.

$$l_s = \sqrt[4]{V M_{Pl}^2 (g^{(4)})^4} = g^{(4)} \sqrt{M_{comp}^{-3} M_{Pl}}$$

or

$$M_s = (g^{(4)})^{-1}\sqrt{M_{comp}^3\, M_{Pl}^{-1}}$$
$$= (0.74)^{-1}\sqrt{(3\cdot 10^{16})^3\,(1.2\cdot 10^{19})^{-1}}\ \mathrm{GeV}$$
$$\approx 2.0\cdot 10^{15}\ \mathrm{GeV}$$

which is smaller than $M_{GUT} = M_{comp}$ by a factor of 15. Of course, this is also not quite what we want...

IV A glance at Calabi-Yau moduli

Compact 2n-dim manifolds with SU(n) holonomy ≡ Calabi-Yau manifolds have a rich mathematical structure:

- complex manifold (notion of holomorphic functions on manifold exist)
- Kähler (holonomy contained in U(n))
- ∃ Ricci-flat metric on manifold (⇒ CY manifolds are vacuum solutions of Einstein's equations)

...

They come in families in that many of them are related by deformations of the following kind:

Given a CY-threefold M with Ricci-flat metric g, i.e. $Ric(g) = 0$ one can analyze slightly perturbed metrics $g + \delta g$ such that still $Ric(g+\delta g) = 0$. Here,

$$\delta g = \delta g_{ij}\, dz^i dz^j + \delta g_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}} + c.c.$$

One finds that $\delta g_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}}$ can be identified with elements in the cohomology group

$$H^{1,1}_{\bar\partial}(M) = \frac{\{(1,1)\text{ forms } \omega^{1,1} \mid \bar\partial\omega^{1,1} = 0\}}{\{(1,1)\text{ forms } \alpha^{1,1} \mid \alpha^{1,1} = \bar\partial\beta^{1,2}\}}$$

while $\Omega_{ijk}\, g^{k\bar{k}}\, \delta g_{\bar{k}\bar{l}}\, dz^i dz^j d\bar{z}^{\bar{l}}$ uniquely corresponds to elements of the cohomology group

$$H^{2,1}_{\bar\partial}(M) = \frac{\{(2,1)\text{ forms } \omega^{2,1} \mid \bar\partial\omega^{2,1} = 0\}}{\{(2,1)\text{ forms } \alpha^{2,1} \mid \alpha^{2,1} = \bar\partial\beta^{2,1}\}}$$

One finds that $\delta g_{ij} \neq 0$ change the complex structure ⇒ $H^{2,1}_{\bar\partial}(M)$ space of complex structure deformations

while $\delta g_{i\bar{j}} \neq 0$ change the Kähler class in $H^{1,1}_{\bar\partial}(M)$ ⇒ $H^{1,1}_{\bar\partial}(M)$ space of Kähler deformations

The dimensions of all cohomology groups $H^{p,q}_{\bar\partial}(M)$ is always finite and denoted by the so-called Hodge numbers

$$h^{p,q}_{(M)} = \dim H^{p,q}_{\bar\partial}(M) < \infty$$

Displaying them people usually draw what is called a Hodge diamond

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$h^{1,1}$ and $h^{2,1}$ are the dimensions of the aforementioned deformation spaces.

⇝ give rise to moduli which need to be stabilized for the predictiveness of the theory

↳ next talk by Valerie ...