AdS/CFT

(Martin Schasny)

> AdS/CFT is a conjectured duality between two apparently different theories:
>
> i) $\mathcal{N}=4$ Super Yang-Mills theory (SYM) on flat 4D Minkowski space.
>
> ii) Type IIB string theory on $AdS_5 \times S^5$

Remarks: i) is no (Kaluza-Klein-like) truncation of ii).
AdS/CFT is a special case of a conjectured more general concept, the Holographic Principle.

Outline:

1. Basic setting: D3 branes in type IIB string theory

   Two alternative descriptions of the region near to the brane

2. The AdS/CFT conjecture

   Possibility to compute strong coupling dynamics of one theory in the weak coupling regime of the other

3. Properties of AdS-space

4. Derivation of the AdS/CFT dictionary

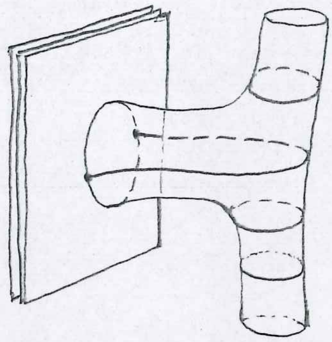
   Operator correspondence, symmetries, conformal weights

5. Example: Scalar 2-point function

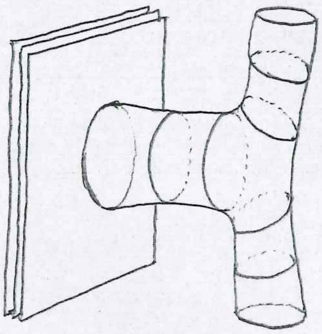
1. Basic setting: D3 branes in type IIB string theory

- $\exists$ two alternative descriptions of string dynamics in the neighbourhood of N branes:

a) Open + closed string perspective:



b) Only closed string perspective:



Additionally, there are closed bulk-strings in both perspectives.
There are two corresponding descriptions:

a) $S_{open} + S_{closed} + S_{I}$ (in flat space) Perturbative regime: $g_s < 1$, $g_s N < 1$	b) S_{closed} (in a black brane background) Perturbative/classical geometry regime: $g_s < 1$, $R \gg l_s$

The two regimes are complementary:
Consider the low energy/classical description of the black brane, $S_{closed} \approx S_{SUGRA}$. For a particular extremal BPS ($Q=M$) solution we have:

$$ds^2 = H(r)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H(r)^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2 \right),$$

$$C_4 = \left(H(r)^{-1} - 1 \right) g_s^{-1} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

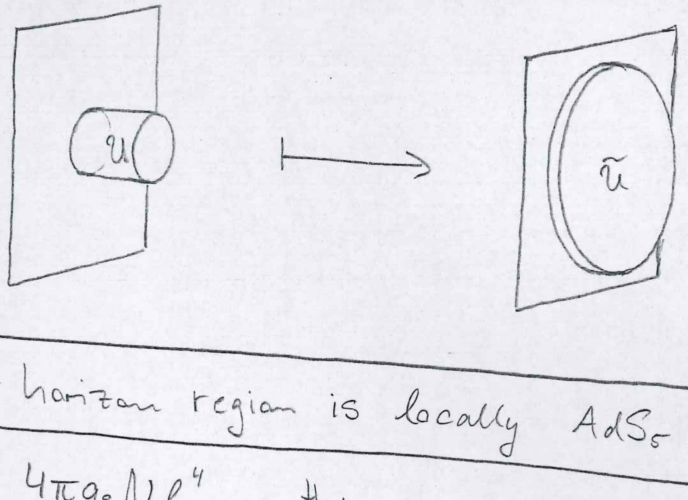
$$H(r) = 1 + \frac{4\pi g_s N l_s^4}{r^4}$$

$$\boxed{\text{Background curvature depends on } g_s, N, l_s}$$

- Decoupling limit:

To get a more precise statement about the two complementary regimes, zoom into the near horizon region $(r \to 0)$:

$$\mathcal{U} = [0,r] \times B_4 \,, \quad r \mapsto \frac{r}{\xi}\,, \; x^\mu \mapsto \xi x^\mu \,, \quad \xi > 1 .$$



The near horizon region is locally $AdS_5 \times S_5$:

$$L^4 := 4\pi g_s N \ell_s^4 \,, \quad \text{then:}$$

$$d\tilde{s}^2 = \left(1 + \frac{L^4 \xi^4}{r^4}\right)^{-\frac{1}{2}} \xi^2 \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{L^4 \xi^4}{r^4}\right)^{\frac{1}{2}} \left(\frac{dr^2}{\xi^2} + \frac{r^2}{\xi^2} d\Omega_5^2 \right)$$

$$= \frac{\xi^2}{\xi^2 \sqrt{\frac{1}{\xi^4} + \frac{L^4}{r^4}}} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{\xi^2 \sqrt{\frac{1}{\xi^4} + \frac{L^4}{r^4}}}{\xi^2} \left(dr^2 + r^2 d\Omega_5^2 \right)$$

$$\overset{r\to 0}{\approx} \underbrace{\frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2}_{AdS_5 \text{ with radius } L} + \underbrace{L^2 d\Omega_5^2}_{S^5 \text{ with radius } L}.$$

$\Rightarrow$ Classical geometry regime: $\frac{L^4}{\ell_s^4} = 4\pi g_s N \gg 1$.

Dynamics in the near horizon region is decoupled from the bulk by the gravitational red-shift:

Compare proper time measures:

$$d\tau(r) = \sqrt{g_{00}(r)}\, dt$$
$$d\tau(\infty) = \sqrt{g_{00}(\infty)}\, dt = dt$$

$$\Longrightarrow \quad \frac{d\tau(\infty)}{d\tau(r)} = \frac{1}{\sqrt{g_{00}(r)}} = H(r)^{\frac{1}{4}} = \left(1 + \frac{4\pi g_s N \ell_s^4}{r^4}\right)^{\frac{1}{4}}$$

$$\overset{r\to 0}{=} \frac{(4\pi g_s N \ell_s^4)^{\frac{1}{4}}}{r} \longrightarrow \infty .$$

Frequencies/energies are red-shifted by the inverse function, $\frac{E(\infty)}{E(r)} \longrightarrow 0$.

$\Rightarrow$ Decoupling in b) $\quad \boxed{S_{\text{closed}} \longrightarrow S^{\text{near}}_{\text{closed}} + S^{\text{far}}_{\text{closed}}}$

- From the a)-perspective we have a similar decoupling.

$$\boxed{S_{\text{open}} + S_{\text{closed}} + S_{I} \longrightarrow S_{\text{open}} + S^{\text{bulk}}_{\text{closed}}}$$

("... the closed strings have irrelevant interactions and decouple.")

Additionally, massive open string effects are suppressed as

$$\frac{\alpha'}{x^2} \longmapsto \frac{\alpha'}{\xi^2 x^2} \longrightarrow 0$$

Reduction to the massless sector : $\quad \boxed{S_{\text{open}} \longrightarrow S_{\text{SYM}}}$

- Summary:

$$\begin{tikzcd}
 & \text{IIB string theory} \arrow[dl, "g_s N \ll 1"'] \arrow[dr, "g_s N \gg 1"] & \\
S_{\text{SYM}} + S^{\text{bulk}}_{\text{closed}} & & S^{\text{near}}_{\text{closed}} + S^{\text{far}}_{\text{closed}}
\end{tikzcd}$$

Until now: One system is described in two different parameter regimes by two different theories.

2. AdS/CFT conjecture

The (adiabatic) continuation between small and large $g_s N$ commutes with the decoupling limit:

$$\begin{tikzcd}
 & \text{IIB string theory} \arrow[dl] \arrow[dr] & \\
S_{SYM} + S_{closed}^{bulk} \arrow[rr, leftrightarrow] & \circlearrowright & S_{closed}^{near} + S_{closed}^{far}
\end{tikzcd}$$

This means, the two limiting theories are equivalent for all values of $g_s N$. Identify $S_{closed}^{bulk} = S_{closed}^{far}$ to get:

$$\boxed{S_{SYM} \;\hat{=}\; S_{closed}^{near}}$$

on 4D Minkowski (world volume of the D3-branes)   on $AdS_5 \times S^5$.

Equivalence:
- 1-1 mapping of the spectra at all energies and couplings
- equality of all observables, i.e. correlation functions of operators.

Evidence: Symmetries on both sides match
- $AdS_5 \times S^5$ has isometry group $SO(2,4) \times SO(6)$
- SYM ($\mathcal{N}=4$) has conformal group $SO(2,4)$ and R-symmetry $SU(4) \cong SO(6)$.

(Local symmetries like $SU(N)$ do not match, the duality only acts on gauge invariant quantities.)

• Application: Calculate strong coupling dynamics of one theory in the weak coupling regime of the other theory.

Example: t'Hooft limit for non-perturbative effects in SYM:

SYM with $G = SU(N)$ have the effective coupling

$$\lambda = g_{YM}^2 N \quad , \quad g_{YM}^2 = g_s$$

For classical SUGRA on the AdS side we need

$$g_s \ll 1 \, , \quad g_s N \gg 1 \quad \Longrightarrow \quad N \gg 1 , \; \lambda \gg 1 .$$

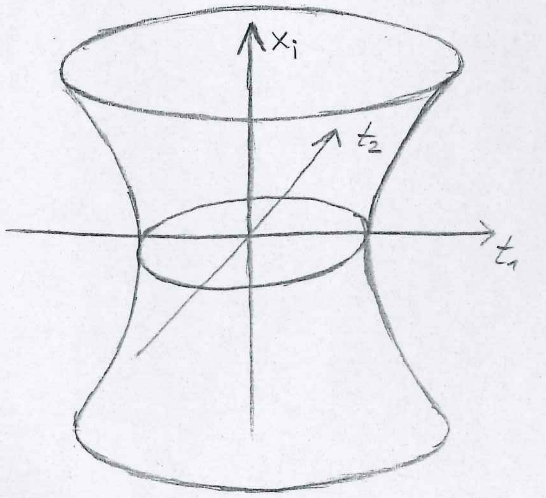
3. Properties of AdS-space

Motivation: AdS/CFT is a bulk-boundary duality, i.e. a realization of the holographic principle.

• AdS is the maximally symmetric space(time) with negative scalar curvature.

• AdS_5 as a submanifold of $\mathbb{M}^{2,4}$ (Minkowski space):

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - t_1^2 - t_2^2 = -R^2$$



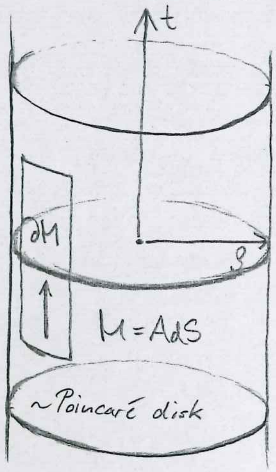
This contains closed timelike curves

$\Longrightarrow$ physical AdS is the universal cover.

- Boundary of AdS: Take conformal compactification

Sausage coordinates: $AdS_5 \cong S^4 \times \mathbb{R}$ topologically

$$\boxed{ds^2 = -dt^2 + \frac{4}{(1+\varrho^2)^2}\left(d\varrho^2 + \varrho^2 d\Omega_4^2\right), \quad \varrho \in [0,1].}$$



Conformal boundary is timelike, and identical to spatial infinity.

$$\partial M \cong S^3 \times \mathbb{R} \quad \text{topologically.}$$

Moreover, $S^3 \times \mathbb{R}$ is a conformal compactification of $M^{1,3}$ (where the SYM theory lives).

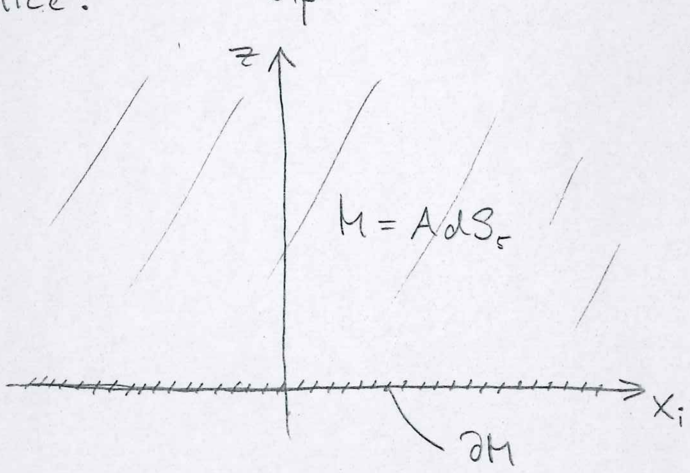
- Boundary $\varrho \to 1$ is an infinite proper distance away from $\varrho = 0$, but lightrays can reach and return in a finite observer time.

  $\Rightarrow$ AdS is not globally hyperbolic. Problems with Cauchy surfaces.

- Alternative coordinate patch: Poincaré half plane

$$\boxed{ds^2 = \frac{1}{z^2}\left(dt^2 - dz^2 - dx^i dx^i\right), \quad x^i \in \mathbb{R}, \ z \in \mathbb{R}^+.}$$

Spatial slice:



Boundary:

$$\partial M \cong \mathbb{R}^4 \cup p \quad \text{(topologically)}$$

4. Derivation of the AdS/CFT dictionary

Recall: $\boxed{\mathcal{N}=4 \text{ SYM in 4D} \;\hat{=}\; \text{IIB closed strings on } AdS_5\times S^5}$

t'Hooft limit: Reduce to classical IIB SUGRA on $AdS_5 \times S^5$

KK-reduce the S^5: $\mathcal{N}=8$ gauged SUGRA on AdS_5.

Basic idea:

$$\boxed{\left\langle \exp \int_{\partial M} \phi_0 \mathcal{O} \right\rangle_{SYM} = e^{iS_{SUGRA}[\Phi_{cl.}]}}$$

here: ϕ_0 = boundary value of the SUGRA-field ϕ,

$\mathcal{O}$ = some operator in the SYM (CFT),

$e^{iS_{SUGRA}[\Phi_{cl.}]} = \lim_{\hbar\to 0} Z_{SUGRA}$ (saddle point approximation),

$\left\langle \exp \int_{\partial M} \phi_0 \mathcal{O} \right\rangle_{SYM} = \frac{1}{Z_0}\int \mathcal{D}\mathcal{O}\, e^{iS_{SYM}[\mathcal{O}] + \int \phi_0 \mathcal{O}} = Z_{SYM}[\phi_0]$.

Application: Compute non-perturbative SYM correlation functions in the dual SUGRA picture:

$$\langle T\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle = \frac{\delta}{\delta\phi_0^1}\frac{\delta}{\delta\phi_0^2}\frac{\delta}{\delta\phi_0^3} Z_{SYM}[\phi_0] = \frac{\delta}{\delta\phi_0^1}\frac{\delta}{\delta\phi_0^2}\frac{\delta}{\delta\phi_0^3} e^{iS_{SUGRA}[\phi]}$$

Problem: Which SYM-operators $\mathcal{O}$ couple to which SUGRA-fields ϕ?

$\longrightarrow$ Identify pairs by their symmetry group representation and their conformal weight!

- Conformal weights:

Let $ds^2 \longmapsto e^{2\omega} ds^2$ be a conformal change of the metric. A field/operator has conformal weight Δ, if it transforms as:

$$\mathcal{O}(x) \longmapsto e^{-\omega\Delta}\, \mathcal{O}(x)$$

Example: The "tensor density" $\sqrt{g}$ has conformal weight $\Delta = -d$.

$$g_{\mu\nu} \longmapsto e^{2\omega} g_{\mu\nu} \implies \det g \longmapsto e^{2d\omega} \det g,$$

$$\implies \sqrt{g} \longmapsto e^{d\omega} \sqrt{g}, \quad \Delta = -d.$$

- Condition:

$$\boxed{\int_{\partial M} \phi_0\, \mathcal{O} = \int_{\partial M} \phi_0(x)\, \mathcal{O}(x) \sqrt{g}\, d^d x \quad \text{must be conformally invariant!}}$$

$$\longmapsto \int_{\partial M} e^{\lambda\omega} \phi_0(x)\, e^{-\omega\Delta}\, \mathcal{O}(x)\, e^{d\omega} \sqrt{g}\, d^d x,$$

$$\implies \lambda - \Delta + d \overset{!}{=} 0,$$

$$\Delta = \lambda + d$$

Problem: How/why should the classical boundary field ϕ_0 transform under conformal maps?

Only massive SUGRA-fields ϕ have non-zero conformal weight on the boundary of AdS:

Sketch of proof:
Consider the massive Laplace equation on Euclidean AdS_d

polar coord.: $ds^2 = dr + \sinh^2(r)\, d\Omega^2_{d-1}$, $r \in [0, \infty]$.

$$(\Delta + m^2)\phi = \left(-\frac{1}{\sinh(r)^d} \frac{d}{dr} \sinh(r)^d \frac{d}{dr} + \frac{L^2}{\sinh^2(r)} + m^2 \right) \phi = 0$$

Ansatz: $\phi = \sum_a c_a \phi_a(r) \Psi_a(\Omega)$

Study asymptotic behaviour $\phi_a(r \to \infty)$:
Effective equation:

$$\left(-\frac{d}{dr} e^{dr} \frac{d}{dr} + e^{dr} m^2 \right) \phi_a(r) = 0$$

Solution: $\phi_a(r) \propto e^{\lambda r}$

$$\curvearrowright \quad -\frac{d}{dr} e^{dr} \frac{d}{dr} e^{\lambda r} + e^{dr} m^2 e^{\lambda r} = 0$$

$$-\frac{d}{dr}\left(\lambda e^{(d+\lambda)r}\right) + m^2 e^{(d+\lambda)r} = 0$$

$$-\lambda(d+\lambda)\, e^{(d+\lambda)r} + m^2 e^{(d+\lambda)r} = 0$$

$$\Rightarrow \lambda(\lambda + d) = m^2,$$

$$\boxed{\lambda_\pm = -\frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}}$$

i) $m = 0$: $\quad \lambda_+ = 0 \; , \; \lambda_- = -d$

The solutions $\phi_a(r) = a e^0 + b e^{-dr}$ are bounded and have the finite boundary value $\phi_a(r \to \infty) = a$.

ii) $m \neq 0$: $\quad \lambda_+ > 0 \; , \; \lambda_- < 0$

The solutions are asymptotically:

$$\phi_a(r) = a e^{\lambda_+ r} + b e^{\lambda_- r} \longrightarrow a e^{\lambda_+ r} \longrightarrow \infty .$$

$\Rightarrow$ No finite boundary values.

Regularization: $\quad \phi_a(r) = \phi_0 \cdot f^{-\lambda_+} \; , \quad f(r) = e^{-r}$

f has a simple zero at the boundary and defines an induced metric on ∂M, up to rescalings

$$f \mapsto e^{\omega} f \;\Rightarrow\; ds^2\big|_{\partial M} \mapsto e^{2\omega} ds^2\big|_{\partial M}$$

Therefore, under conformal transformation on ∂M ϕ_0 transforms nontrivially:

$$\phi_0 \cdot f^{-\lambda_+} \longmapsto \tilde{\phi}_0 \cdot (e^{\omega} f)^{-\lambda_+} = \tilde{\phi}_0 \cdot e^{-\lambda_+ \omega} f^{-\lambda_+}$$

$$\Rightarrow \boxed{\tilde{\phi}_0 = e^{\lambda_+ \omega} \phi_0}$$

"The regularization breaks the conformal symmetry."

• Examples:

SUGRA-scalar	m^2	SYM-operator	Δ(SYM)
ϕ	0	$\mathrm{Tr}(F \wedge * F)$	4
C_0	0	$\mathrm{Tr}(F \wedge F)$	4
φ	-4	$\mathrm{Tr}[\phi_{(i}\phi_{j)}] - \frac{1}{6}\delta_{ij}\mathrm{Tr}[\phi_k \phi_k]$	2

φ: KK-scalar from the S^5-reduction. $m^2 < 0$ but in the allowed range of the Breitenlohner–Freedman bound.

5. Example: Scalar 2-point-function

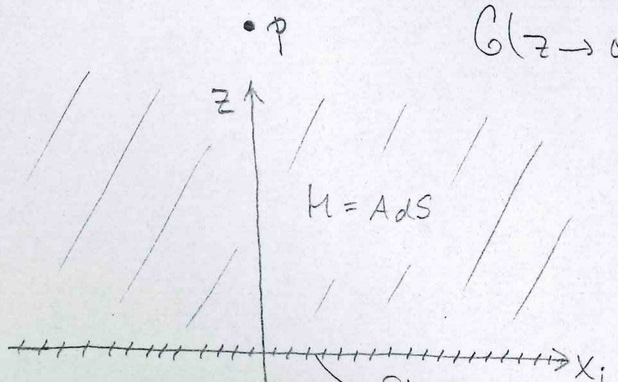
$$\langle T\, \mathcal{O}(x)\, \mathcal{O}(x') \rangle_{SYM} = \frac{\delta}{\delta\phi_0(x)} \frac{\delta}{\delta\phi_0(x')} Z_{SYM}[\phi_0]$$

$$\stackrel{AdS/CFT}{=} \frac{\delta}{\delta\phi_0(x)} \frac{\delta}{\delta\phi_0(x')} e^{iS[\phi]}$$

→ Need an explicit formula $\phi = \phi(\phi_0)$:
Green's function for the homogeneous Laplace equation with given boundary value ϕ_0:

$$\phi(z, x_i) = \int_{\partial M} G(z, x_i, x_i')\, \phi_0(x_i'), \qquad \text{(Poincaré coordinates)}$$

$$G(z \to \infty, x_i, x_i') \propto \delta^{(d)}(x_i - x_i').$$



For the right asymptotics, $G(z \to 0) = 0$, it contains just the λ_+ exponent.

• Result:

$$\boxed{\phi(z,x) = \int_{\partial M} d^4x' \, \frac{z^{4+\lambda_+}}{(z^2+|x_i - x_i'|^2)^{4+\lambda_+}} \phi_0(x_i')}$$

• Insert into action:

$$S[\phi] = \frac{1}{2}\int_M \langle d\phi, d\phi\rangle + \frac{1}{2}\int_M m^2\phi^2$$

$$= \frac{1}{2}\int_M \Big(d\langle d\phi, \phi\rangle - \underbrace{\langle\Delta\phi, \phi\rangle\Big) + \frac{1}{2}\int m^2\phi^2}_{=0}$$

$$= \frac{1}{2}\int_M d\langle d\phi, \phi\rangle$$

$$= \frac{1}{2}\int_{\partial M} \langle d\phi, \phi\rangle$$

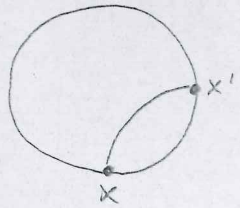
$$= \frac{1}{2}\int_{\partial M} \sqrt{g}\, d^4x \; \phi\,(\vec{n}\cdot\vec{\nabla}\phi)$$

Inserting the Green's function:

$$\Longrightarrow \quad S[\phi] = \frac{1}{2}\int_{\partial M} d^4x \,(d+\lambda_+)\, \frac{\phi_0(x)\,\phi_0(x')}{|x-x'|^{d+\lambda_+}}$$

$$\Longrightarrow \boxed{\frac{\delta}{\delta\phi_0(x)}\frac{\delta}{\delta\phi_0(x')} S[\phi] = \frac{1}{2}(d+\lambda_+)\frac{1}{|x-x'|^{d+\lambda_+}}}$$

This is a correct 2-point function $\langle T\, O(x)\, O(x')\rangle$ of a SYM-operator with conformal weight $\Delta = d+\lambda_+$.



Higher order correlation functions using Witten diagrams: