# Supergravity in $d = 11$ and $d = 10$

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## 1 $p$-forms

These are geometric animals which will frequently appear, so we start by giving a (somewhat sloppy) overview of $p$-form notation. A $p$-form in $d \geq p$ dimensions is an antisymmetric tensor field with $p$ indices. In particular, 0-forms are scalars and 1-forms are vectors. All operations on a $p$-form $\Omega$ can be defined in terms of its components $\Omega_{\mu_1 \ldots \mu_p}$ in some local coordinates. What we need is

- Addition: Two $p$-forms are added by adding their components.
- Exterior differentiation: The d operator turns a $p$-form into a $(p+1)$-form. It acts as

$$(\mathrm{d}\,\Omega)_{\nu\mu_1\ldots\mu_p} = (p+1)\,\partial_{[\nu}\Omega_{\mu_1\ldots\mu_p]}\,. \tag{1}$$

  Here (as always) [ ] denotes weighted antisymmetrization. Note that

$$\mathrm{d}\,\mathrm{d}\,(\text{anything}) = 0 \qquad \text{by symmetry}\,. \tag{2}$$

- The wedge product: Given a $p$-form $\Omega$ and a $q$-form $\Omega'$, one may form a $(p+q)$-form $\Omega \wedge \Omega'$ whose components are

$$(\Omega \wedge \Omega')_{\mu_1\ldots\mu_p\,\nu_1\ldots\nu_q} = \frac{(p+q)!}{p!\,q!}\Omega_{[\mu_1\ldots\mu_p}\Omega'_{\nu_1\ldots\nu_q]}\,. \tag{3}$$

- Hodge dualization: The $*$ operator turns a $p$-form into a $(d-p)$-form in $d$-dimensional space:

$$(*\Omega)_{\mu_1\ldots\mu_{d-p}} = \frac{\sqrt{-g}}{p!}\epsilon_{\mu_1\ldots\mu_{d-p}}{}^{\nu_1\ldots\nu_p}\Omega_{\nu_1\ldots\nu_p}\,. \tag{4}$$

  Here $g = \det(g_{\mu\nu})$, and the $\epsilon$ symbol with all lower indices is defined to be $\pm 1$ or 0 as usual, with indices raised by the inverse metric $g^{\mu\nu}$.
- Integration: A $p$-form can be integrated over a $p$-dimensional manifold. In local coordinates,

$$\begin{aligned}\int_{\mathcal{M}} \Omega &= \int \mathrm{d}^p x\; \Omega_{0\,1\,\cdots\,(p-1)} \\ &= \int \mathrm{d}^p x\; \sqrt{-g}\;(*\Omega)\,.\end{aligned} \tag{5}$$

  The first equality shows that the integral does not depend on the metric. The second one keeps general covariance manifest (here the metric and Hodge $*$ are those of $\mathcal{M}$, so $*\Omega$ is a scalar).

A familiar example where $p$-form notation can sometimes be useful is electromagnetism in $d = 4$. There is a 1-form $A$ with components $A_\mu$ (the gauge potential), which is locally determined only up to a U(1) gauge transformation

$$A \to A + \mathrm{d}\,\Lambda \tag{6}$$

with $\Lambda$ any 0-form. It gives rise to a 2-form $F = \mathrm{d}\,A$ with components $F_{\mu\nu}$ (the field strength). $F$ is invariant under gauge transformations of Eq. (6) by Eq. (2). The kinetic action in some 4d space-time $\mathcal{S}$ is

$$S = -\frac{1}{2}\int_{\mathcal{S}} F \wedge *F = -\frac{1}{4}\int \mathrm{d}^4x\,\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu}\,. \tag{7}$$

One may couple an external particle of charge $q$ (an "electron") to the gauge field. This is done by integrating the 1-form $A$ over the particle's 1-dimensional worldline $\mathcal{C} \subset \mathcal{S}$:

$$S = -\frac{1}{2}\int_{\mathcal{S}} F \wedge *F + q\int_{\mathcal{C}} A + \ldots \tag{8}$$

Alternatively one could group all charged particles into a current density 1-form $J$:

$$S = -\frac{1}{2}\int_{\mathcal{S}} F \wedge *F + \int_{\mathcal{S}} A \wedge *J + \ldots \tag{9}$$

In this formulation Maxwell's equations are very short:

$$\mathrm{d}\,F = 0, \qquad \mathrm{d}\,*F = *J\,. \tag{10}$$

The homogeneous equation, or "Bianchi identity" is as usual a direct consequence of working with a gauge potential. In $p$-form language it follows from $F = \mathrm{d}\,A$ and Eq. (2).

## 2 Spinors in $d$ dimensions

The $d$-dimensional Clifford algebra can be defined in analogy to the four-dimensional case. It is represented by $d$ Dirac matrices $\Gamma^0, \ldots, \Gamma^{d-1}$ such that

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{1}\,. \tag{11}$$

A representation of the Lorentz algebra $\mathfrak{so}(d-1,1)$ is then furnished by

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu]\,. \tag{12}$$

In $d = 2$, a possible choice of Dirac matrices is

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

From this one can iteratively construct the $\Gamma^\mu$ for even $d$:

$$\begin{aligned} \Gamma^\mu &= \widetilde{\Gamma}^\mu \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\mu = 0, \ldots, d-3), \\ \Gamma^{d-2} &= \mathbb{1} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^{d-1} = \mathbb{1} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{14}$$

where $\widetilde{\Gamma}^\mu$ are the Dirac matrices in $d-2$ dimensions, and $\mathbb{1}$ is the $2^{d/2-1} \times 2^{d/2-1}$ unit matrix. The dimension of the Dirac representation is thus $2^{d/2}$, and there are $2^{d/2+1}$ real degrees of freedom contained in a $2^{d/2}$-dimensional complex Dirac spinor.

The analogue of $\gamma^5$ in $d = 4$ is

$$\Gamma = i^{d/2-1}\Gamma^0\Gamma^1 \cdots \Gamma^{d-1}\,. \tag{15}$$

$\Gamma$ has the properties

$$(\Gamma)^2 = \mathbb{1}, \quad \{\Gamma, \Gamma^\mu\} = 0, \quad [\Gamma, \Sigma^{\mu\nu}] = 0\,. \tag{16}$$

The eigenvalues of $\Gamma$ are $\pm 1$. Dirac spinors in even $d$ are reducible since they admit a decomposition into two Weyl spinors of opposite $\Gamma$ eigenvalues, or chiralities: The Dirac representation splits into two Weyl representations of dimension $2^{d/2-1}$.

If $d$ is odd, one obtains the Clifford algebra by taking the Dirac matrices of the $(d-1)$-dimensional Clifford algebra and adding $\Gamma^d = \Gamma$. The Dirac representation of the Lorentz algebra is then $2^{(d-1)/2}$-dimensional and irreducible.

In certain dimensions it is possible to further halve the independent degrees of freedom of a spinor by imposing the Majorana condition

$$\zeta^* = \mathcal{B}\zeta \tag{17}$$

where $\mathcal{B}$ is some product of Dirac matrices satisfying

$$\mathcal{B}\,\Sigma^{\mu\nu}\,\mathcal{B}^{-1} = -\Sigma^{\mu\nu *}\,. \tag{18}$$

It turns out that one may consistently impose this condition on Dirac spinors if $d \equiv 0, 1, 2, 3, 4 \mod 8$. One may impose it on Weyl spinors only if $d \equiv 2 \mod 8$.

From these considerations we obtain the number of real degrees of freedom $f$ in the minimal spinor representation in $d$ dimensions:

| d | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | 1 | 2 | 4 | 8 | 8 | 16 | 16 | 16 | 16 | 32 | 64 |

## 3 Why $d = 11$ is special

Using the above table we can deduce that the maximal dimension for supersymmetry is $d = 11$. The argument goes as follows: For arbitrary $d$ we can always compactify on a torus $T^{d-4}$ to $d = 4$, which preserves all supersymmetries. Supersymmetry transformations are generated by spinorial supercharges. A $d = 4$ supercharge has $f = 4$ degrees of freedom; a $d = 11$ supercharge has $f = 8 \cdot 4$ degrees of freedom, i.e. it corresponds to 8 distinct supercharges $\{Q^A\}$, or $\mathcal{N} = 8$ SUSY, after torus compactification to $d = 4$. A hypothetical $d \geq 12$ supercharge would correspond to at least $\mathcal{N} = 16$ SUSY in $d = 4$. We will show that such a large number of $d = 4$ supercharges would lead to 4d fields with spin $> 2$, and comment on why this is unacceptable.

In $d = 4$ the supercharges are written as two-component complex Weyl spinors which satisfy the algebra (in the absence of central charges)

$$\begin{aligned} &\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu\, \delta^A_B, \\ &\{Q, Q\} = \{\overline{Q}, \overline{Q}\} = [P, Q] = [P, \overline{Q}] = [P, P] = 0 \end{aligned} \tag{19}$$

Here $P_\mu$ is the four-momentum, $\alpha$ and $\dot\beta$ are Weyl spinor indices, $A$ and $B$ label the supercharges, and we have suppressed all indices in the second line.

Consider now a massless supermultiplet, $P^2 = 0$. In a reference frame where $P = (-E, 0, 0, E)$, the first line of Eq. (19) becomes

$$\{Q^A_\alpha, \overline{Q}_{\dot\beta B}\} = \begin{pmatrix} 4\,E & 0 \\ 0 & 0 \end{pmatrix} \delta^A_B\,. \tag{20}$$

The operators $Q^A_2$ and $\overline{Q}_{\dot 2 A}$ are represented by zero in this frame since they anticommute with everything. The operators $Q^A_1$ lower the helicity of any

state by $\frac{1}{2}$. A $\mathcal{N}=8$ massless multiplet contains a spin-2 field (a graviton); its two polarisation states $|h_{+2}\rangle$ and $|h_{-2}\rangle$ with helicities $\pm 2$ are related by repeatedly acting with the $Q_1^A$:

$$|h_{-2}\rangle \sim Q_1^1 \cdots Q_1^8 \, |h_{+2}\rangle \,. \tag{21}$$

A ficticious massless multiplet of $\mathcal{N}=16$ SUSY in $d=4$ would have to contain a spin-4 field. It is often stated that interacting theories with fields of spin $>2$ cannot be consistently coupled to gravity, although recently some exceptions may have been discovered (involving an infinite number of fields, and on curved backgrounds). Barring such exotic constructions, the maximal supersymmetry in $d=4$ would be $\mathcal{N}=8$, and the maximal dimension for supersymmetry $d=11$.

The above arguments can also be applied, with slight modifications, to massive multiplets, and generalized to the case non-vanishing central charges.

## 4 $d=11$ supergravity

In $d=11$ there is a unique supergravity theory; it contains 32 real supercharges, or equivalently a $d=11$ Majorana spinor supercharge. The gravitational multiplet is the unique supersymmetry multiplet. It contains

- the graviton,
- a gravitino $\Psi$,
- and a 3-form $A_3$.

There is a U(1) gauge symmetry under which the gravitino is charged and for which $A_3$ is the gauge field. It transforms as

$$A_3 \;\to\; A_3 + \mathrm{d}\,\Lambda_2 \tag{22}$$

with $\Lambda_2$ any 2-form, analogous to Eq. (6).

The bosonic action up to two derivatives contains the usual Einstein-Hilbert term along with a kinetic term and a Chern-Simons term for $A_3$,

$$\begin{aligned} S \supset & \frac{1}{2\,\kappa^2}\int \mathrm{d}^{11}x\,\sqrt{-g}\,R \\ & + \frac{1}{2\,\kappa^2}\int F_4 \wedge *F_4 - \frac{1}{12\,\kappa^2}\int A_3 \wedge F_4 \wedge F_4 \,. \end{aligned} \tag{23}$$

Here $F_4 = \mathrm{d}\,A_3$ is the U(1) field strength 4-form. The Chern-Simons term is gauge invariant up to a total derivative, since under a gauge transformation Eq. (22) it transforms as

$$\begin{aligned} & \int A_3 \wedge F_4 \wedge F_4 \\ & \to \int A_3 \wedge F_4 \wedge F_4 + \int \mathrm{d}\,\Lambda_2 \wedge F_4 \wedge F_4 \\ & = \int A_3 \wedge F_4 \wedge F_4 + \int \mathrm{d}\,(\Lambda_2 \wedge F_4 \wedge F_4) \,. \end{aligned} \tag{24}$$

The last equality follows from the product rule and from $\mathrm{d}\,F_4 = 0$

The terms in the action involving $\Psi$ are somewhat more complicated. They are however also fully fixed by local supersymmetry and gauge invariance.

## 5 Type IIA in $d=10$

When compactifying $d=11$ SUGRA on $S^1$ and truncating to the massless spectrum, one obtains type IIA SUGRA in $d=10$. The number of real supercharges is still 32, contained in two Majorana-Weyl spinors of opposite chiralities. The field content is

- the graviton and its gravitino superpartner,
- a real scalar dilaton $\phi$ and its dilatino superpartner $\psi$,
- a 2-form gauge potential $B_2$ with field strength $H_3 = \mathrm{d}\,B_2$,
- a 1-form gauge potential $C_1$ and a 3-form gauge potential $C_3$ with field strengths $F_2 = \mathrm{d}\,C_1$ and $F_4 = \mathrm{d}\,C_3$. Alternatively, one may dualize the field strengths to obtain $F_6 = *F_4$ and $F_8 = *F_2$, with "magnetic" gauge potentials $C_5$ and $C_7$, such that $\mathrm{d}\,C_5 = F_6$ and $\mathrm{d}\,C_7 = F_8$. The formulations in terms of either $C_1$ and $C_7$ are equivalent, and similarly for $C_3$ and $C_5$.
- There is a version, so-called *massive type IIA*, in which also a 0-form field strength $G_0$ is allowed. $G_0$ does not descend from any gauge potential and does not appear in dimensional reduction (but it does appear in string theory). We will not discuss this theory any further.

The origin of these fields can be understood from torus compactification (cf. the very first seminar on Kaluza-Klein theory). $C_1$ originates from the gravitational multiplet in $d = 11$, just as the original KK theory had a massless photon 1-form. The dilaton we are also familiar with; essentially, it parameterizes the radius of the circle. $B_2$ and $C_3$ descend from the $A_3$ of the $d = 11$ SUGRA: $B_2$ is the zero mode of $A_3$ with one leg along the $S^1$ direction, while $C_3$ is the zero mode of $A_3$ with all legs in the uncompactified dimensions.

The gauge variation for $C_3$ takes a slightly non-standard form:

$$\begin{aligned} C_1 &\to C_1 + d\Lambda_0, \\ B_2 &\to B_2 + d\Lambda_1, \\ C_3 &\to C_3 + d\Lambda_2 + \Lambda_0 \wedge H_3 \end{aligned} \tag{25}$$

with $\Lambda_{0,1,2}$ arbitrary. It is convenient to define the invariant field strength

$$\widetilde{F}_4 = F_4 - C_1 \wedge H_3 \,. \tag{26}$$

$\widetilde{F}_4$ is not a closed form, but satisfies a non-standard Bianchi identity:

$$d\widetilde{F}_4 = -F_2 \wedge H_3 \,. \tag{27}$$

With this definition the bosonic action is

$$\begin{aligned} S = &\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \, \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2\right) \\ &- \frac{1}{4\kappa_{10}^2} \int \left(e^{-2\phi} H_3 \wedge *H_3 + F_2 \wedge *F_2 + \widetilde{F}_4 \wedge *\widetilde{F}_4\right) \\ &- \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 \,. \end{aligned} \tag{28}$$

Again the Chern-Simons term in the last line is gauge invariant up to a total derivative.

Type IIA SUGRA is the low-energy limit of type IIA string theory. Type IIA string theory contains several higher-dimensional objects which couple to, and provide sources for, the higher $p$-form gauge fields. Classically, this is analogous to electrons coupling to the electromagnetic gauge field, which we described by integrating the gauge potential over the electron worldline. Type IIA string theory contains $D$-branes with $D = 0, 2, 4, 6$. These are objects with $D$ spatial dimensions and $D + 1$-dimensional world-volumes, coupling to $C_1$, $C_3$, $C_5$, and $C_7$ respectively. For instance, the action in the presence of a 2-brane of tension $T$ and with world-volume $\mathcal{W}$ contains a term

$$S = T \int_{\mathcal{W}} C_3 + \ldots \,, \tag{29}$$

the analogue of Eq. (8).

## 6 Type IIB in $d = 10$

There is a second maximal supergravity in ten dimensions which cannot be (directly) obtained from $d = 11$. It contains 32 real supercharges in two Majorana-Weyl spinors of the same chirality. The field content of the gravitational supermultiplet is

- the graviton and the gravitino,
- a real scalar dilaton $\phi$ and its dilatino superpartner $\psi$,
- a 2-form $B_2$ with field strength $H_3 = \mathrm{d}\, B_2$,
- a 0-form $C_0$, a 2-form $C_2$ and a 4-form $C_4$ with field strengths $F_1 = \mathrm{d}\, C_0$, $F_3 = \mathrm{d}\, C_2$ and $F_5 = \mathrm{d}\, C_4$.

Motivated again by non-standard gauge transformation laws, it is convenient to define the combinations

$$\begin{aligned} \widetilde{F}_3 &= F_3 - C_0 \wedge H_3, \\ \widetilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \,. \end{aligned} \tag{30}$$

The field strength combination $\widetilde{F}_5$ is restricted to be self-dual:

$$\widetilde{F}_5 = *\widetilde{F}_5 \tag{31}$$

(a restriction which cannot be derived from an action, but must be imposed on the equations of motion as a further constraint). The bosonic action reads

$$\begin{aligned} S = &\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \, \sqrt{-g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2\right) \\ &- \frac{1}{4\kappa_{10}^2} \int \Big(e^{-2\phi} H_3 \wedge *H_3 + F_1 \wedge *F_1 \\ &\qquad + \widetilde{F}_3 \wedge *\widetilde{F}_3 + \widetilde{F}_5 \wedge *\widetilde{F}_5\Big) \\ &- \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \,. \end{aligned} \tag{32}$$

Type IIB supergravity is the low-energy limit of type IIB string theory, which contains a fundamental string and various $D$-branes of odd $D$ coupling to the gauge fields, similar to the even $D$-branes of type IIA .

The action exhibits an interesting symmetry under modular transformations. We perform a Weyl rescaling to the Einstein frame metric $G_{\mu\nu} = e^{-\phi/2} g_{\mu\nu}$ and define

$$\tau = C_0 + i e^{-\phi} (\text{the “axio-dilaton”}) \tag{33}$$

and

$$G_3 = F_3 - \tau H_3 \,. \tag{34}$$

The action Eq. (32) becomes (with the Einstein frame metric used everywhere)

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G} \left( R - \frac{\partial\overline{\tau}\partial\tau}{2(\operatorname{Im}\tau)^2} \right) \\ & - \frac{1}{2\kappa_{10}^2} \int \left( \frac{1}{6} \frac{G_3 \wedge *\overline{G}_3}{\operatorname{Im}\tau} + \frac{1}{2} \widetilde{F}_5 \wedge *\widetilde{F}_5 \right) \\ & - \frac{1}{8\, i\, \kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \overline{G}_3}{\operatorname{Im}\tau} \,. \end{aligned} \tag{35}$$

Recall that $\mathrm{SL}\,(2,\mathbb{R})$ is the group of real $2\times 2$ matrices with determinant one. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}\,(2,\mathbb{R})$ acts on the fields as

$$\begin{aligned} \tau &\to \frac{a\tau + b}{c\tau + d}, \\ G_3 &\to \frac{G_3}{c\tau + d}, \\ \widetilde{F}_5 &\to \widetilde{F}_5, \\ G_{\mu\nu} &\to G_{\mu\nu} \,. \end{aligned} \tag{36}$$

The action Eq. (35) is invariant under this transformation. This is because

$$\begin{aligned} \partial\tau &\to \partial \left( \frac{a\tau + b}{c\tau + d} \right) = \frac{\partial\tau}{(c\tau + d)^2}, \\ \operatorname{Im}\tau &\to \frac{\operatorname{Im}\tau}{|c\tau + d|^2} \,. \end{aligned} \tag{37}$$

The invariance under the subgroup $\mathrm{SL}(2,\mathbb{Z})$, where $a, b, c, d$ are integers, lifts to “S-duality” in type IIB string theory, which might play some role in the upcoming talks.

# 7 Type I in $d = 10$

Type I is not a maximal supergravity since it only contains 16 supercharges in a single Majorana-Weyl spinor. It is obtained by compactifying $d = 11$ supergravity on $S^1/\mathbb{Z}_2$ and adding gauge degrees of freedom. There are two multiplets. The gravitational multiplet contains

- the graviton and the gravitino,
- the dilaton and the dilatino,
- a 2-form $B_2$ with field strength $H_3$.

The vector multiplet contains

- a 1-form non-abelian gauge potential $A_1$ in the adjoint representation of some gauge group
- and its gaugino superpartner.

While it would seem at first sight that this leaves much freedom to engineer theories with various vector multiplets and large gauge symmetries, it turns out that the theory is in fact tightly restricted by anomaly cancellation. The classic result of Green and Schwarz in the 1980s shows that for gravitational anomalies to cancel, the dimension of the gauge group should be 496. Cancellation of gauge anomalies then restricts the possible gauge groups to either SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ (the cases $\mathrm{E}_8 \times \mathrm{U}(1)^{248}$ and $\mathrm{U}(1)^{496}$, while not covered by this argument, are believed not to admit any UV completion).

Both SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$ gauged supergravities have UV completions in terms of string theories. The SO(32) can arise as the low-energy limit of either type I string theory or SO(32) heterotic string theory. The $\mathrm{E}_8 \times \mathrm{E}_8$ case is the low-energy limit of $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory.

# 8 Compactification

A trivial exact solutions to the field equations is always given by flat $d = 10$ or $d = 11$ Minkowski space with all other fields vanishing.

For various reasons (among others, we know only of four non-compact dimensions), it is interesting to look at solutions of the form $\mathcal{M} \times \mathcal{C}$, where

$\mathcal{M}$ is an $m$-dimensional non-compact manifold and $\mathcal{C}$ is a $(10-m)$- or $(11-m)$-dimensional compact manifold. We recapitulate some of the statements made in talk on Kaluza-Klein theory:

- One obtains a field theory on $\mathcal{M}$.
- The degrees of freedom are the KK modes, i.e. (for bosonic fields) the eigenfunctions of the covariant Laplacians $\nabla^2$ on $\mathcal{C}$.
- Even if the metric on $\mathcal{C}$ (and thus the Laplace and Dirac operators) is not known explicitly, information about the massless spectrum can be gained from topology.
- If $\mathcal{C}$ has isometry group $G$, there will be a gauge symmetry with gauge group $G$.

The last point motivates a search for a solution with some $\mathcal{C}$ whose isometry group is large enough to contain the Standard Model. Note however that, upon compactification on a product space, the curvature scalar splits as follows:

$$R = R_{\mathcal{M}} + R_{\mathcal{C}} + \dots \tag{38}$$

so the Einstein-Hilbert term in $d$ dimensions becomes, after carrying out the integral over $\mathcal{C}$,

$$\begin{aligned} &\frac{1}{2\kappa_d^2}\int \mathrm{d}^d x\sqrt{-g}\, R \\ &= \frac{1}{2\kappa_m^2}\int \mathrm{d}^m x\sqrt{-g_{\mathcal{M}}}\,(R_{\mathcal{M}} + \Lambda) + \dots \end{aligned} \tag{39}$$

where $\Lambda \sim \int_{\mathcal{C}} R_{\mathcal{C}}$ is a cosmological constant. One faces two options:

- Calabi-Yau compactification on manifolds $\mathcal{C}$ with $R_{\mathcal{C}} = 0$, but whose Riemann tensor is still non-vanishing – otherwise one has just a torus compactification with too much SUSY. These manifolds have no isometries and therefore no gauge symmetry.
- Compactifications with large isometry groups, e.g. on homogeneous spaces $\mathcal{C} = G/H$ with $G$ and $H$ compact Lie groups. $\mathcal{C}$ then has the isometry group $G$. However, for small compactification radii the cosmological constant will be unacceptably large.

Together with the difficulties of obtaining chiral fermions which was also mentioned in the KK theory lecture, it seems that the KK idea cannot be straightforwardly applied to 10-dimensional or 11-dimensional supergravity to produce a realistic model of the real world. String theory does however provide ways to avoid these obstacles.

Because they are of some interest for the upcoming talks (especially on moduli stabilization and on AdS/CFT), we close with an interesting class of exact solutions of type IIB SUGRA on homogeneous spaces. Take $\mathcal{M}$ and $\mathcal{C}$ to be solutions to the $d = 5$ vacuum Einstein equations in Lorentzian and Riemannian signatures respectively; $\mathcal{M}$ negatively and $\mathcal{C}$ positively curved. The most important examples have $\mathcal{M}$ =AdS$_5$, the unique $d = 5$ Lorentzian manifold of topology $\mathbb{R}^5$ and constant negative curvature. One obtains a solution to the equations of motion provided that there is also some 5-form flux in the background. Explicitly, with $m, n\dots$ indices on $\mathcal{M}$ and $\alpha, \beta\dots$ indices on $\mathcal{C}$:

$$\begin{aligned} \widetilde{F}_{mnopq} &= -r\sqrt{-g_{\mathcal{M}}}\,\epsilon_{mnopq}, \\ \widetilde{F}_{\alpha\beta\gamma\delta\varepsilon} &= r\sqrt{-g_{\mathcal{C}}}\,\epsilon_{\alpha\beta\gamma\delta\varepsilon}, \\ R_{mn} &= 4\,r^2\,g_{mn}, \\ R_{\alpha\beta} &= -4\,r^2 g_{\alpha\beta}, \\ \text{all other fields} &= 0\,. \end{aligned} \tag{40}$$

In the simplest example, $\mathcal{C}$ is $\mathcal{C} = S^5 = \mathrm{SO}(6)/\mathrm{SO}(5)$ a 5-sphere of radius $r$, and $\mathcal{M}$ =AdS$_5$ with scale $1/r$. This is the near-horizon geometry for a stack of $D3$ branes (recall that these indeed source the 5-form gauge field). Another important example has $\mathcal{C} = T^{1,1} = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{U}(1)$. While these solutions have no immediate relevance for 4d physics, they are extremely important in the AdS/CFT correspondence.

## References


- Differential forms: Nakahara, *Geometry, Topology and Physics*
- SUSY in $d = 4$: Wess & Bagger, *Supersymmetry and Supergravity*, Chapter 2
- Everything and more: Polchinski, *String Theory Vol. II*, Appendix B

## Disclaimer

If you think you've found a sign error, a missing prefactor, or a gaping hole in the reasoning, you're probably correct.