Convex and nonconvex potentials

① Analogy between the effective potential and free energy [1,2].

At the classical level $\langle \Phi \rangle$ is given by a minimum of the potential energy. But at the quantum level $\langle \Phi \rangle$ can be shifted by loop corrections. So, we need some replacement of the potential, which will define $\langle \Phi \rangle_{\text{corrected}}$ and provide correct masses and couplings, which can be obtained from the Feynman diagrams
$\Rightarrow$ idea of the effective potential.

Let's start from an <u>example 1</u>: in magnetics the thermo. ground state is defined via the minimum of the Gibbs free energy G. Partition function:

$$Z(H) = e^{\overbrace{-\beta}^{\beta = 1/kT} \overbrace{F(H)}^{\text{Helmholz free energy}}} = \int \underbrace{Ds}_{\text{spin configurations}} \exp\Big\{ -\beta \int dx \big(\underbrace{\mathcal{H}(s)}_{\text{spin energy density}} - \underbrace{H}_{\text{external magn. field}} \cdot \underbrace{S(x)}_{\text{spin configurations}} \big) \Big\}$$

then the magnetization is

$$M = -\frac{\partial F}{\partial H}\Big|_{\beta = \text{fix}} = \frac{1}{\beta}\frac{\partial}{\partial H} \log Z = \int dx \langle S(x) \rangle$$

Gibbs free energy is defined by means of the Legendre transformation:

$$G = F + M \cdot H$$

then

$$\frac{\partial G}{\partial M} = +\frac{\partial F}{\partial M} + M\frac{\partial H}{\partial M} + H = \frac{\partial H}{\partial M}\cdot\frac{\partial F}{\partial H} + M\frac{\partial H}{\partial M} + H = H$$

$\Rightarrow$ $H=0$ defines an extremum of G.

Example 2 Let's repeat this in a quantum field theory

$$Z[J] = e^{-iE[J]} = \int D\varphi \cdot \exp\left\{ i\int d^4x \left(\mathcal{L}[\varphi] + J\varphi \right) \right\}$$

vacuum energy — $E[J]$; Lagrangian — $\mathcal{L}$; field configuration — φ; external source — J

$$\varphi_{cl} \equiv \langle \Omega | \varphi(x) | \Omega \rangle_J = -\frac{\delta}{\delta J(x)} E[J] = \frac{\delta}{i\delta J(x)} \operatorname{Log} Z$$

Legendre transform:

$$\Gamma[\varphi_{cl}] = -E[J] - \int d^4y\, J(y)\, \varphi_{cl}(y) \leftarrow \text{effective action.}$$

$$\frac{\delta}{\delta \varphi_{cl}(x)} \Gamma[\varphi_{cl}] = \dots = -J(x) \Rightarrow \left.\frac{\delta \Gamma[\varphi_{cl}]}{\delta \varphi_{cl}(x)}\right|_{J\equiv 0} = 0$$

We can also define the effective potential by

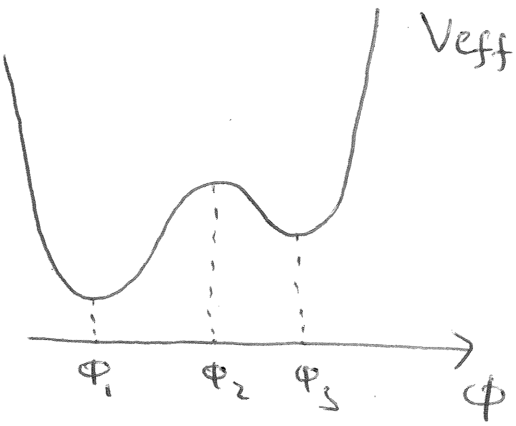
$$\Gamma[\varphi_{cl}] = -V_4 \cdot V_{eff}(\varphi_{cl}),$$ therefore

$$\frac{\partial}{\partial \varphi_{cl}} V_{eff}(\varphi_{cl}) = 0. \quad \text{— an equation on } \varphi_{cl}. \qquad (*)$$

to make the analogy more transparent let's summarize the correspondence between the thermo. quantities and ones from QFT:

Magnetic system	QFT
$\bar{x}$	$x = (t, \bar{x})$
$S(\bar{x})$	$\varphi(x)$
H	$J(x)$
$\mathcal{H}(s)$	$\mathcal{L}(\varphi)$
$Z(H)$	$Z[J]$
$F(H)$	$E[J]$
M	$\varphi_{cl}(x)$
$G(M)$	$-\Gamma(\varphi_{cl})$.

Let's suppose we computed the effective potential and it looks line one depicted below:



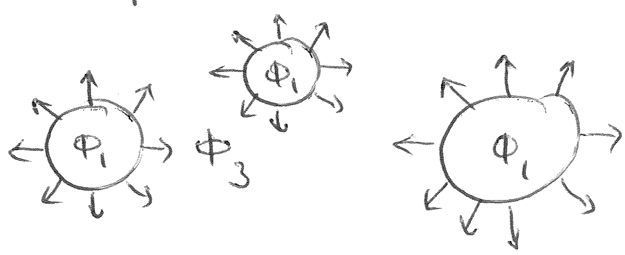
Here $\varphi_i(x) = \varphi_i$ are the solutions of (*) then

φ_1 – true vacuum (stable)

φ_3 – false vacuum (metastable)

φ_2 – unstable

Here we consider a homogeneous situation. But constant background field does not provide the minimum of the energy. Instead we consider an inhomogeneous one

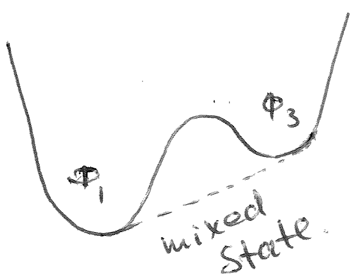


$$\varphi_{cl} = \lambda \varphi_1 + (1-\lambda)\varphi_3$$

then

$$V_{eff}(\varphi_{cl}) = \lambda V_{eff}(\varphi_1) + (1-\lambda) V_{eff}(\varphi_3)$$

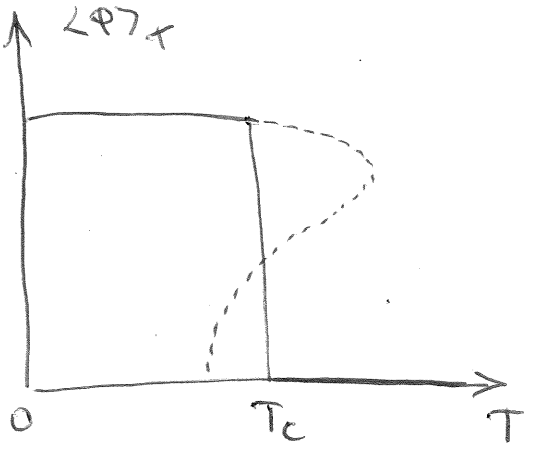
$\Rightarrow$ Maxwell construction

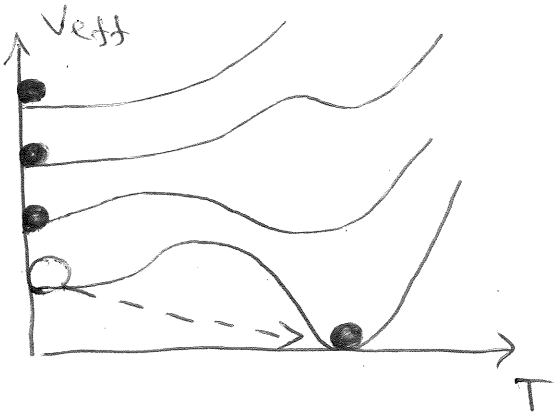


Phase transitions

$\langle \Phi \rangle_T$ – order parameter

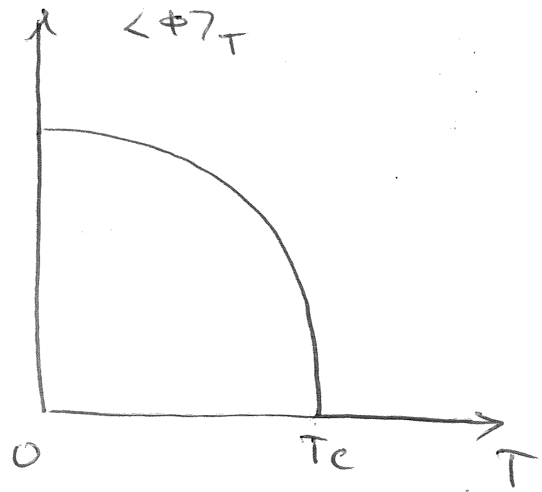
I order

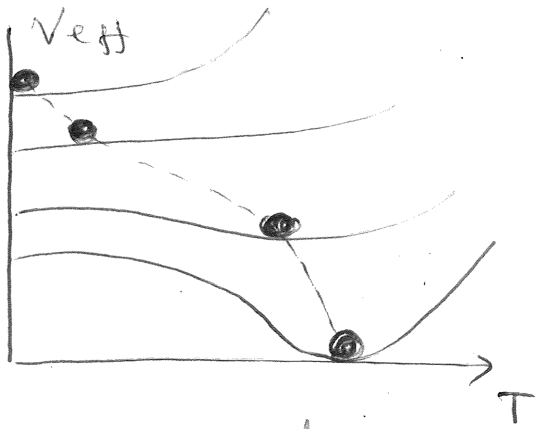




boiling water
melting ice
e.w. with $m_H < m_W$

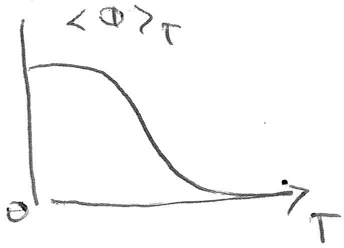
II order





ferromagnetics
superconductivity
superfluidity

Notice also existing of a crossover:



Clausius – Clapeyron equation:

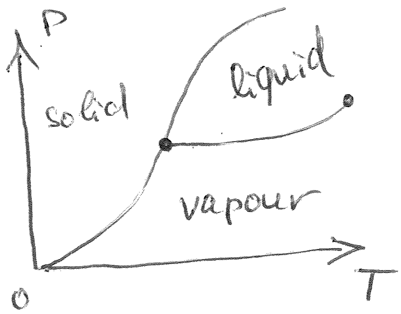
pressure

latent heat

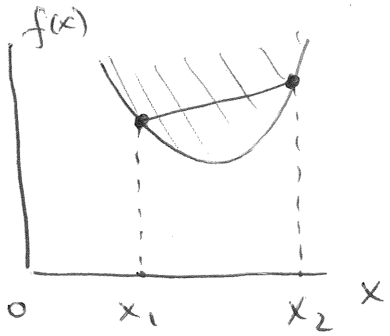
$$\frac{dP}{dT} = \frac{L}{T \Delta V}$$

temperature of the transition

volume change



② Convexity [3]



$$f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

$\lambda \in [0,1]$ by def. convex

replacing "$\le$" by "$\ge$" we get concave function.

Useful inequality (Hölder): for two positive functions f and g and any measure $d\mu$

$$\int d\mu\, f^\alpha g^{(1-\alpha)} \le \left(\int d\mu\, f\right)^\alpha \left(\int d\mu\, g\right)^{1-\alpha}, \quad \alpha \in [0,1] \qquad (**)$$

Let's consider a field theory at finite T (in Euclidean time)

$$e^{-E} = \langle \exp\left(-\int d^4x\, J(x)\, \Phi(x)\right) \rangle$$

Using Hölder inequality (**) we see that

$$e^{-E(\lambda J_1 + (1-\lambda) J_2)} = \langle \left(e^{-\int d^4x\, J_1(x)\Phi(x)}\right)^\lambda \rangle \times$$
$$\times \langle \left(e^{-\int d^4x\, J_2(x)\Phi(x)}\right)^{1-\lambda} \rangle \le e^{-\lambda E[J_1]} e^{-(1-\lambda)E[J_2]}$$

$\Rightarrow$ $E = E[J]$ is a concave function.

Next step: Using the definition of Φ_{cl} we get

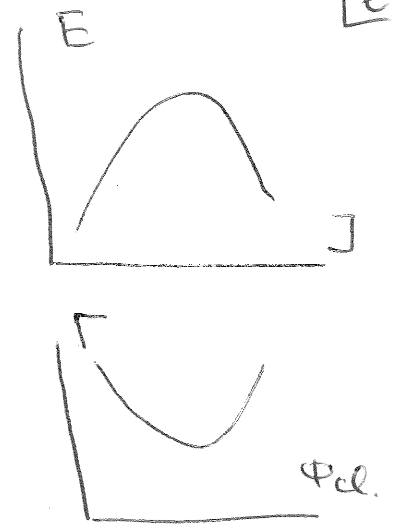
$$\Gamma[\Phi_{cl}] = \max_J \left[E[J] - \int d^4y\, J(y)\, \Phi_{cl}(y)\right]$$ and then

$$\Gamma(\lambda \Phi_{cl_1} + (1-\lambda)\Phi_{cl_2}) = \max_J \left[\lambda\left(E[J] - \int d^4y\, J(y)\Phi_{cl_1}(y)\right) + \right.$$
$$\left. + (1-\lambda)\left(E[J] - \int d^4y\, J(y)\Phi_{cl_2}(y)\right)\right] \le \lambda \max_J [\ldots] +$$
$$+ (1-\lambda)\max_J[\ldots] = \lambda\Gamma[\Phi_{cl_1}] + (1-\lambda)\Gamma[\Phi_{cl_2}]$$

$\Rightarrow$ $\Gamma[\Phi_{cl}]$ is convex by definition!

NOTE: Sometimes $E[J]$ is not differentiable at the maximum.

(3) But sometimes it is not convex (as a result of our calculation) [1,2,4,5], which means that what we've found is not the true effective potential (for example, the initial potential was not convex and perturbative calculations just deformed it a bit).



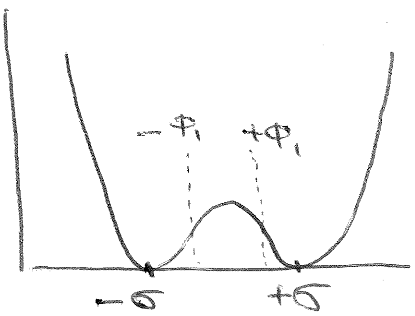
<u>Example</u>: a scalar field theory

$$L = \int d^3x \left[\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \right]$$

$$V_{1\text{-loop}} = V(\phi) + \frac{1}{64\pi^2}\left\{ (V''(\phi))^2 \ln V''(\phi) \right\} + P(\phi)$$

$P(\phi)$ is a polynomial.

1) if



then for $\phi \in (-\phi_1, \phi_1)$ we have $V'' < 0$ and $V_{1\text{-loop}}$ is complex! Keep this in mind.

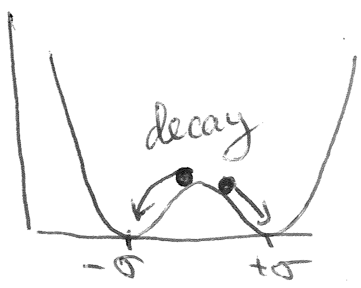
2) second, the real part of $V_{1\text{-loop}}$ is not convex. (a trouble of the perturbative calculation). We already know, how to resolve it <u>classically</u> – by the Maxwell construction (mixed state). In <u>quantum</u> theory we get a quantum superposition of two vacuum states.

Now, resolution of the problem (1) is the following

$|\phi| \le \phi_1$ a perturbative decay to $\pm\sigma$

$\phi_1 \le |\phi| \le \sigma$ a non-perturbative decay to $\pm\sigma$

$\text{Im}(V_{1\text{-loop}}) \propto (\text{decay rate})$



$\varphi(x) = \varphi_0 + \tilde{\varphi}(\bar{x})$, where $\varphi_0 = \frac{1}{\Omega} \int d^3x \, \varphi(\bar{x})$

↑ 3-volume

$\varphi_{cl} = \langle \varphi_0 \rangle$

then in a finite volume with the periodic bound. conditions

$$L = \Omega \left(\frac{\dot{\varphi}^2}{2} - V(\varphi_0) + \int d^3x \left(\frac{1}{2} (\partial_\mu \tilde{\varphi})^2 - \frac{V''(\varphi_0)}{2} \tilde{\varphi}^2 + \dots \right) \right).$$

$$H_{quad} = \Omega V(\varphi_0) + \int d^3x \left(\frac{1}{2} \dot{\tilde{\varphi}}^2 + \frac{1}{2} (\bar{\nabla} \tilde{\varphi})^2 + \frac{V''(\varphi_0)}{2} \tilde{\varphi}^2 \right)$$

$$\tilde{\varphi}(\bar{x}) = \frac{1}{\sqrt{2\Omega}} \sum_{\bar{k}} \left(\varphi_{\bar{k},1} \sin \bar{k} \cdot \bar{x} + \varphi_{\bar{k},2} \cos \bar{k} \cdot \bar{x} \right)$$

$$H_{quad} \to \Omega V(\varphi_0) + \frac{1}{2} \sum_{\bar{k}} \sum_{j=1}^{2} \frac{1}{2} \left(\dot{\varphi}_{\bar{k},j}^2 + \varphi_{\bar{k},j}^2 \cdot \omega_{\bar{k}}^2 \right),$$

where $\omega_{\bar{k}}^2 = \bar{k}^2 + V''(\varphi_c)$

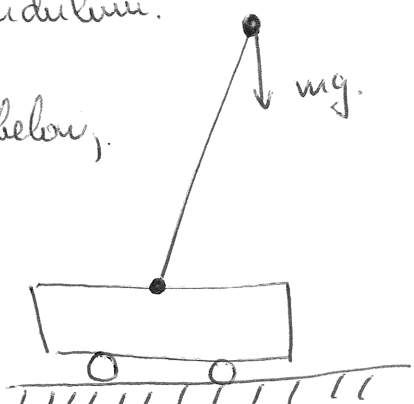
1) $V''(\varphi_c) > 0 \longrightarrow$ harmonic oscillator states

$$\frac{E_{quad}}{\Omega} = V(\varphi_c) + \frac{1}{\Omega} \sum_{k} \frac{1}{2} \omega_k .$$

2) $V''(\varphi_c) < 0$: $\bar{k}^2 > |V''|$ still harm. oscillator.

$\bar{k}^2 < |V''|$ upside-down oscillator (imaginary frequency).

Let's consider a quantum inverted pendulum.



$\hat{H} = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \eta^2 q^2$ is not bounded from below,

but for small q (perturbative $\tilde{\varphi}$) we can

consider $\langle \psi | q^2 | \psi \rangle \leq a \, \eta^{-1}$

the solution is given by

$$\psi(q,t) = \left(\frac{\eta \sin 2\theta}{2\pi} \right)^{1/4} \frac{1}{(\cos(\theta - i\eta t))^{1/2}} \exp\left\{ -\frac{\eta q^2}{2} \tan(\theta - i\eta t) \right\},$$

where $\theta \equiv \operatorname{arctg}(1/2a)$.

Decay probability:

$$\hat{P}(t) = |\langle \varphi(0) | \varphi(t) \rangle|^2 = \left(1 + \frac{\sinh^2(\gamma t)}{\sin^2(2\theta)}\right)^{-\frac{1}{2}} \underset{t\to\infty}{\sim} 2\sin 2\theta \, e^{-\gamma t}$$

So, the decay rate $\hat{\Gamma} \equiv \gamma$ doesn't depend on a-constraint.

<u>NOTE</u>: decay rate for $V(\phi) = -\lambda \frac{\phi^4}{4!}$ also doesn't depend on the final state.

Now, for the scalar field:

$$\frac{1}{2}\frac{\Gamma}{\Omega} = \frac{1}{(2\pi)^3}\int d^3K \cdot \frac{1}{2}\left[|V''(\varphi_{cl})| - \bar{K}^2\right]^{1/2}\Big|_{V'' > \bar{K}^2}$$

$$= \frac{1}{64\pi}\left[V''(\varphi_{cl})\right]^2 = \operatorname{Im}(V_{1\text{-loop}})$$

<u>NOTE</u> $\varphi_1 < |\varphi| < \sigma$ are unstable too.

(4) Electroweak phase transition [5].

$m_H < m_W$ — I^{st} order transition

$m_H > m_W$ — Crossover.

$$S_E = \int dx \, \mathrm{Tr}\left[\frac{1}{2} W_{\mu\nu} W^{\mu\nu} + (D_\mu \Phi)^+ (D^\mu \Phi) + \mu \Phi^+ \Phi + 2\lambda (\Phi^+ \Phi)^2\right]$$

where

$$\int_\beta dx \equiv \int_0^\beta d\tau \int_\Omega d^3x \qquad D_\mu = \partial_\mu - ig W_\mu \qquad \Phi = \frac{1}{2}(\sigma + i\vec{\pi}\vec{\tau})$$

$$e^{-E(T,J)} = \int D\Phi D\Phi^+ DW_\mu \exp\left\{-\int_\beta dx (L + J\Phi^+\Phi)\right\}$$

<u>important</u>: Source J couples to the gauge invariant composite field $\Phi^+\Phi$, which makes the free energy E gauge-independent. (***)

if $W(T,J) \equiv \frac{E(T,J)}{\Omega\beta}$, then the effective potential

$$V(T,\rho) = W(T,J) - \frac{1}{2}\rho J \equiv$$

$$\equiv W(T,J) - \frac{1}{\Omega}\int_\Omega d^3x \langle \Phi^+(x)\Phi(x)\rangle \cdot J.$$

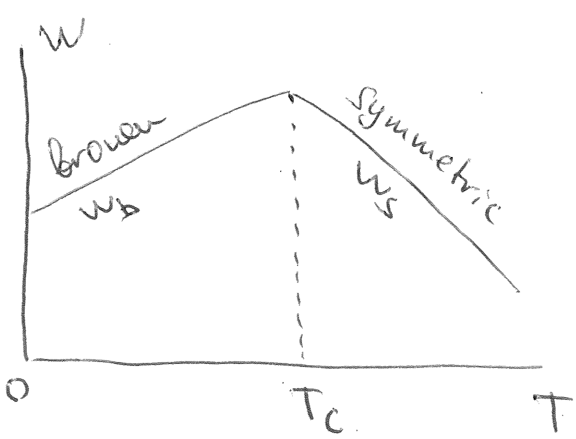
Here ρ plays a role of an "order parameter". It is well defined and suitable for the lattice studies because of (***). Otherwise, for instance, for $\langle \Phi \rangle$ it would be resolution-dependent (dependence on lattice spacing) and can even vanish identically in the continuum limit.

NOW, back to the <u>Clausius-Clapeyron</u> equation:

$$-\frac{1}{T}\Delta Q \equiv \frac{\partial}{\partial T}(W_s - W_b) =$$

$$= -\frac{\partial}{\partial J}(W_s - W_b)\frac{dJ(T)}{dT} =$$

$$= -\frac{\Delta\rho}{2}\frac{dJ}{dT}$$



$$\Rightarrow \quad \Delta Q = \frac{1}{2}\Delta\rho \cdot T \cdot \frac{dJ}{dT}$$

latent heat

"molar volume" change $\langle \Phi^+\Phi \rangle$

from the dimensional analysis

$$\mu + J(T) = C(g^2, \lambda)\, T^2$$

$$\Rightarrow T \frac{dJ(T)}{dT}\Big|_{J\to 0} = 2\mu$$

from 1-loop effective potential $C(g^2; \lambda) \simeq -\left(\frac{3}{16} g^2 + \frac{\lambda}{2}\right)$

$$\longrightarrow \quad \boxed{\Delta Q = -\frac{1}{2} m_H^2 \Delta\rho \left(1 + O(g^2; \lambda)\right)}$$

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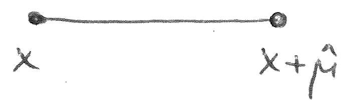
(5) Basics of the lattice EW transition [6].

1) Lattice itself is usually of the size $(N_s \cdot a)^3 \times (N_\tau \cdot a)$, gauge fields are periodic and thus $T = \frac{1}{N_\tau \cdot a}$ $\Rightarrow$ thin lattice means high T.

2) Gauge fields are defined at lines via Wilson lines $U_\mu(x) = e^{-ia W_\mu(x + \frac{\hat\mu}{2})}$

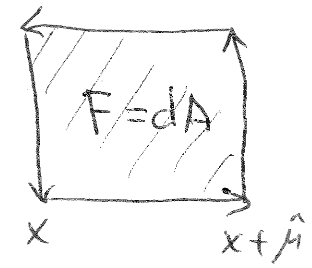


the scalars pick up this phase.

Scalars are defined at sites $\Phi = \Phi(x)$.

Field strength $W_{\mu\nu}$ is defined on a plaquette

$$P_{\mu\nu} = U_\mu(x) U_\nu(x+\hat\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x)$$



3) Action:

$$S = \beta_G \sum_x \sum_{i<j} \left(1 - \frac{1}{2} \mathrm{Tr}\, P_{ij}\right) +$$

$$+ \beta_H \sum_x \sum_i \frac{1}{2} \mathrm{Tr}\, \Phi^\dagger(x) U_i(x) \Phi(x+\hat i) +$$

$$+ \sum_x \frac{1}{2} \mathrm{Tr}\, \Phi^\dagger(x)\Phi(x) + \beta_R \sum_x \left(\frac{1}{2} \mathrm{Tr}\, \Phi^\dagger(x)\Phi(x) - 1\right)^2$$

Here (I) kinetic term for W_μ
(II) kinetic term for $\Phi(x)$ "hopping term"
(III) potential for $\Phi(x)$

4) "order parameters" (not true o.p., but still useful ones).

$\langle \Phi^+ \Phi \rangle$, $\langle L \rangle$ (L is the hopping term).

m_H (GeV)	T_c (GeV)
35	~94
60	~138
70	~154

(translatable to T_c).

5) Monte-Carlo:
Changing values of the fields randomly in order to minimize the action (long story, let me drop it in the notes).

Literature:

[1] Peskin - Schröder "Introduction to Quantum Field Theory".
[2] Rubakov "Introduction to the theory of the early Universe".
[3] Iliopoulos, Itzykson, Martin, Rev. Mod. Phys 47 (1975) 165
[4] Weinberg, Wu, Phys. Rev. D36 (1987) 2474.
[5] Buchmüller, Fodor, Hebecker, Nucl. Phys. B 447 (1995) 317
[6] Kajantie et al. hep-lat/9510020

By. T. Kalagodzhyan 2011.