Symmetry restoration at high temperatures 1.11.11

I QFT at finite T

II CW potential

III $\lambda\phi^4$

IV Abelian Higgs

I QFT at finite T

- Statistical ensemble $\{|n\rangle\}$, $0 \le p_n \le 1$

$$\langle \hat{O} \rangle = \sum_n p_n \langle n|\hat{O}|n\rangle = \mathrm{Tr}(\rho \hat{O})$$

- Density matrix

$$\rho = \sum_n p_n |n\rangle\langle n|, \quad \mathrm{Tr}(\rho) = 1$$

- Entropy $S = -\mathrm{tr}(\rho \ln \rho) = -\langle \ln \rho \rangle$
- Thermal equilibrium $[S = \max$ for $E = \langle \hat{H} \rangle$, $q_i = \langle \hat{Q}_i \rangle]$

$$\rho = \frac{1}{Z_G} e^{-\beta \hat{H} + \beta \mu_i \hat{Q}_i}, \quad \beta = \frac{1}{T}$$

- Kanonical ensemble $[\langle n|\hat{Q}_i|n\rangle = q_i]$

$$\rho = \frac{1}{Z} e^{-\beta \hat{H}}$$

- Free energy: $F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \ln \mathrm{tr}(e^{-\beta \hat{H}})$

$$\Rightarrow \quad E = \langle \hat{H} \rangle = \frac{\partial}{\partial \beta}(\beta F)$$

$$S = -\langle \log \rho \rangle = \ln Z + \beta \langle \hat{H} \rangle = -\beta F + \beta E \Rightarrow F = E - TS$$

- Time-evolution operator $\mathcal{U}(t,t') = e^{-i(t-t')\hat{H}}$

  $\rightarrow$ continue to complex half-plane $\mathrm{Im}(t-t') \leq 0$

  $\Rightarrow \quad Z = \mathrm{tr}\,(e^{-\beta \hat{H}}) = \mathrm{tr}\,(\mathcal{U}(-i\beta, 0))$

  Basis: $\hat{\Phi}(t=0,\vec{x})\,|\varphi\rangle = \varphi(\vec{x})\,|\varphi\rangle, \qquad |\varphi; t\rangle = \mathcal{U}(t,0)\,|\varphi\rangle$

  $$\Rightarrow \quad \mathbb{1} = \int \Big(\prod_x d\varphi(\vec{x})\Big)\, |\varphi; t\rangle\langle t; \varphi|$$

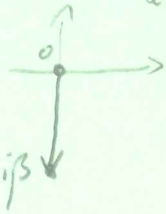
$$\Rightarrow \quad Z = \int \Big(\prod_x d\varphi(\vec{x})\Big)\, \langle \varphi; 0 |\, \mathcal{U}(-i\beta, 0)\, | \varphi; 0\rangle$$

$$= \int \prod_x d\varphi(\vec{x})\;\langle \varphi; -i\beta \,|\, \varphi; 0\rangle$$

Use $$\langle \varphi, t \,|\, \varphi', t' \rangle = \int_{\varphi(t',\vec{x}) = \varphi'(\vec{x})}^{\varphi(t,\vec{x}) = \varphi(\vec{x})} \mathcal{D}\varphi\; e^{\,i \int_{t'}^{t} dx^0 \int d^3x\, \mathcal{L}(x)}$$

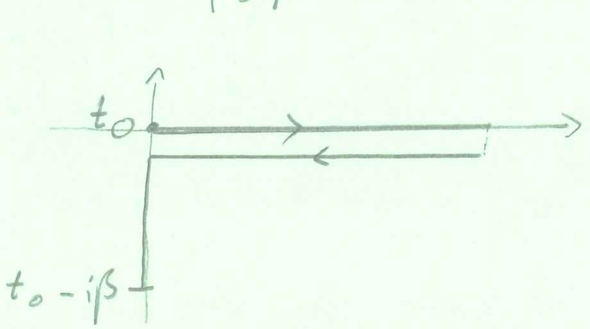
$$\Rightarrow \quad Z = \int_{\varphi(0,\vec{x}) = \varphi(-i\beta, \vec{x})} \mathcal{D}\varphi\; e^{\,i \int_0^{-i\beta} dx^0 \int d^3x\, \mathcal{L}(x)} \underset{\substack{x^0 = -i\tau \\ \mathcal{L}_E = -\mathcal{L}}}{=} \int \mathcal{D}\varphi\; e^{-\int_0^{\beta} d\tau \int d^3x\, \mathcal{L}_E(x)}$$

$\Rightarrow$ Imaginary time contour



- Generalization: $$Z = \int_{\varphi(0) = \varphi(-i\beta)} \mathcal{D}\varphi\; e^{\,i S_{\mathcal{C}}[\varphi]}, \qquad S_{\mathcal{C}}[\varphi] = \int_{\mathcal{C}} dx^0 \int d^3x\, \mathcal{L}(x)$$

$$\mathcal{C}:\; t(\tau) = \begin{cases} t_0 & \tau = 0 \\ t_0 - i\beta & \tau = \tau_{max} \end{cases}, \qquad \frac{d}{d\tau}\,\mathrm{Im}(t) \geq 0$$



Real time formalism: $t_0 \rightarrow -\infty$

- N-Point fctns

$$G(x_1,\dots,x_N) = \langle T_e \Phi(x_1)\cdots\Phi(x_N)\rangle = \frac{1}{Z}\int \mathcal{D}\varphi\, \varphi(x_1)\cdots\varphi(x_N)\, e^{iS_e}$$

$$\Rightarrow \quad G\big|_{x_i^0=0} = G\big|_{x_i^0=-i\beta}$$

periodic in imaginary time with period $\beta = \frac{1}{T}$

Fermions (N even)

$$G\big|_{x_i^0=0} = - G\big|_{x_i^0=-i\beta}$$

anti-periodic

- Fourier trf. along imaginary time $x^0 = -i\tau$

$$G(\omega_n,\vec{p}) = -i\int_0^\beta d\tau \int d^3x\, e^{i\omega_n\tau - i\vec{p}\vec{x}}\, G(-i\tau,\vec{x},0)$$

$$e^{i\omega_n\beta} = \pm 1 \quad \Rightarrow \quad \boxed{\omega_n = \frac{\pi}{\beta}\cdot\begin{cases} 2n & \text{bosons} \\ 2n+1 & \text{fermions}\end{cases}}$$

Matsubara frequencies

- Feynman rules

$- - - - \quad \frac{i}{p^2-m^2}, \quad p_0 = \frac{i\pi}{\beta}\cdot 2n$

$—— \quad \frac{i}{\not{p}-m} \quad p_0 = \frac{i\pi}{\beta}\cdot(2n+1)$

Loop: $\int\frac{dp^0}{2\pi} \longrightarrow \frac{1}{2\pi}\,\frac{2\pi i}{\beta}\sum_n = iT\sum_n$

Vertex: $(2\pi)^4\delta^{(4)}(\Sigma p_i) \longrightarrow \frac{1}{iT}\,\delta_{\Sigma\omega_i}\,(2\pi)^3\delta(\Sigma\vec{p}_i)$

II. Effective potential

- Generating functional $Z[J] = \int \mathcal{D}\varphi \, e^{-\int_0^\beta d\tau \int d^3x \{ \mathcal{L}_E(x) + \varphi(x) J(x) \}}$
  $= e^{-\beta F[J]}$

$\Rightarrow$ exp. value $\phi(x) = \langle \hat{\Phi}(x) \rangle = -\frac{1}{Z} \frac{\delta Z}{\delta J(x)} = \beta \frac{\delta F}{\delta J(x)}$

- Effective action

$$\Gamma[\phi] = \beta F - \int_0^\beta d\tau \int d^3x \, \phi(x) J(x)$$

$$\Rightarrow \frac{\delta \Gamma}{\delta \phi(x)} = -J(x) \rightarrow 0 \quad \text{for } J = 0$$

- Effective potential : $\phi(x) \rightarrow \phi = \text{constant}$

$$\Gamma[\phi] = \int_0^\beta d\tau \int d^3x \left[-V_{eff}(\phi) + \frac{1}{2} Z(\phi) (\partial\phi)^2 + \dots \right]$$

$\Rightarrow$ exp. value is determined by $\boxed{V_{eff}'(\phi) = 0}$

- Coleman-Weinberg pot. (1L)

  ↳ shift fields $\varphi_i(x) \rightarrow \phi_i + \varphi_i(x)$ such that $\langle \varphi_i \rangle = 0$

  ↳ expand $S[\phi + \varphi]$ to 2nd order in φ_i

$$S[\phi+\varphi] = \underbrace{S[\phi]}_{\int d^4x \, [-V_0(\phi)] \ \text{tree-level}} + \int d^4x \frac{\delta S}{\delta \phi_i} \varphi_i(x) + \frac{1}{2} \int d^4x \, d^4y \, \varphi_i(x) \underbrace{\frac{\delta^2 S}{\delta \phi_i \delta \phi_j}}_{\text{mass matrix } M_{ij}^2(\phi)} \varphi_j(y)$$

e.g. $M_F = Y_{ij} \phi$ from Yukawa-coupling

- Vacuum: $V_{eff} = V_0 + V_{1L} + \dots$

$$V_{1L}^{vac} = \frac{1}{2} \mathrm{Tr}_B \int \frac{d^4 p_E}{(2\pi)^4} \ln (p_E^2 + M_B^2(\phi)) - 2 \mathrm{Tr}_F \int \frac{d^4 p_E}{(2\pi)^4} \ln (p_E^2 + M_F^2(\phi))$$

- Finite-T: $\int \frac{dp^0}{2\pi} \rightarrow T \sum_n , \quad p_E^2 \rightarrow \omega_n^2 + \vec{p}^{\,2} = \begin{cases} \pi^2 T^2 (2n)^2 + \vec{p}^{\,2} & B \\ \pi^2 T^2 (2n+1)^2 + \vec{p}^{\,2} & F \end{cases}$

- Consider the contribution of one scalar d.o.f.

$$V_B(\phi) = \frac{1}{2} T \sum_{n=-\infty}^{+\infty} \int \frac{d^3p}{(2\pi)^3} \ln\Big((2\pi n T)^2 + \underbrace{\vec{p}^{\,2} + m_B^2(\phi)}_{\omega^2} \Big)$$

$$\hookrightarrow \quad V(\omega) = \sum_n \ln\left((2\pi n T)^2 + \omega^2 \right)$$

$$\frac{dV}{d\omega} = \sum_n \frac{2\omega}{(2\pi n T)^2 + \omega^2}$$

$\hookrightarrow$ Use $\displaystyle \sum_n \frac{y}{n^2+y^2} = -\frac{1}{2y} + \frac{\pi}{2} \coth(\pi y)$

$$\Rightarrow \quad \frac{dV}{d\omega} = 2\beta \left[\frac{1}{2} + \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \right]$$

$$V = 2\beta \left[\frac{\omega}{2} + \frac{1}{\beta} \ln\left(1-e^{-\beta\omega}\right) \right] + \text{const.}$$

$$\Rightarrow \quad V_B(\phi) = \int \frac{d^3p}{(2\pi)^3} \Big[\underbrace{\frac{\omega}{2}}_{\substack{V_B^{vac}(\phi) \\ + \text{const.}}} + \underbrace{\frac{1}{\beta} \ln\left(1-e^{-\beta\omega}\right)}_{V_B^T(\phi)} \Big]_{\omega = \sqrt{\vec{p}^{\,2} + m_B^2(\phi)}}$$

$$V_B^T(\phi) = \frac{T^4}{2\pi^2} J_B\left(\frac{m_B^2(\phi)}{T^2} \right), \quad J_B\left(\frac{m^2}{T^2}\right) = \int_0^\infty dx\, x^2 \ln\left(1-e^{-\sqrt{x^2+\frac{m^2}{T^2}}}\right)$$

$$= -\frac{\pi^4}{45} + \frac{\pi^2}{12}\frac{m^2}{T^2} - \frac{\pi}{6}\left(\frac{m^2}{T^2}\right)^{3/2} - \frac{1}{32}\frac{m^4}{T^4} \ln\frac{m^2}{a_b T^2} + \mathcal{O}\left(\frac{m^6}{T^6}\right)$$

with $a_b = 16\pi^2 e^{3/2 - 2\gamma_E}$

- One fermionic d.o.f.

$$V_F(\phi) = -\cancel{2}4 \int \frac{d^3p}{(2\pi)^3} \left[\frac{\omega}{2} + \frac{1}{\beta} \ln\left(1+e^{-\beta\omega}\right) \right]$$

$$V_F^T(\phi) = -\cancel{2}4 \frac{T^4}{2\pi^2} J_F\left(\frac{m_F^2(\phi)}{T^2} \right), \quad J_F\left(\frac{m^2}{T^2}\right) = \int_0^\infty dx\, x^2 \ln\left(1+e^{-\sqrt{x^2+\frac{m^2}{T^2}}}\right)$$

$$= \frac{7\pi^4}{360} - \frac{\pi^2}{24}\frac{m^2}{T^2} - \frac{m^4}{32T^4} \ln\frac{m^2}{a_f T^2} + \mathcal{O}\left(\frac{m^6}{T^6}\right)$$

$$a_f = \pi^2 e^{3/2 - 2\gamma_E}$$

III φ^4- theory

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V_0(\varphi) + \bar\psi(i\partial\!\!\!/ - m)\psi - g\varphi\bar\psi\psi \qquad \lambda \sim g^2 \ll 1$$

$$V_0(\varphi) = -\frac{1}{2}\mu^2\varphi^2 + \frac{1}{4}\varphi^4 = \frac{\lambda}{4}(\varphi^2 - v^2)^2 + \text{const}, \qquad v = \frac{\mu}{\sqrt{\lambda}}$$

$$\hat\varphi(x) \to \phi + \hat\varphi(x)$$

$$\mathcal{L}\big|_{\varphi\to\phi+\varphi} = \underbrace{\mathcal{L}\big|_{\substack{\varphi\to\phi \\ \psi\to 0}}}_{-V_0(\phi)} + (\text{lin. terms in } \varphi) + \underbrace{\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}V''(\phi)\varphi^2 + \bar\psi(i\partial\!\!\!/ - m)\psi - g\phi\bar\psi\psi}_{\substack{m_\varphi^2(\phi) = V''(\phi) = -\mu^2 + 3\lambda\phi^2 = \lambda(3\phi^2 - v^2) \\ m_\psi(\phi) = g\phi}} + \dots$$

$$V_{eff} = V_0(\phi) + \underbrace{V_{1L}^{vac}(\phi)}_{\text{Ren. cond.}} + \underbrace{V_{1L}^{T}(\phi)}_{\frac{T^4}{2\pi^2}\left\{J_B\left(\frac{m_\varphi(\phi)^2}{T^2}\right) - 4J_F\left(\frac{m_\psi(\phi)^2}{T^2}\right)\right\}}$$

Ren. cond.

$$\frac{dV_{1L}^{vac}}{d\phi}\Big|_{\phi=v} = 0$$

$$\frac{d^2V_{1L}^{vac}}{d\phi^2}\Big|_{\phi=v} = 0$$

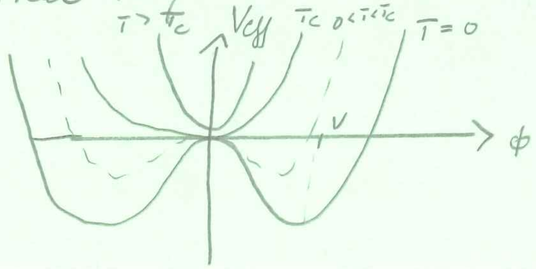
$$\Rightarrow V_{1L}^{vac} = \frac{1}{64\pi^2}\left\{m_\varphi^4(\phi)\left(\ln\frac{m_\varphi(\phi)^2}{m_\varphi(v)^2} - \frac{3}{2}\right) + 2m_\varphi^2(v)\,m_\varphi^2(\phi)\right\}$$

$$-\frac{1}{16\pi^2}\left\{ -\,\text{"}\,- \right\}_{m_\varphi\to m_\psi}$$

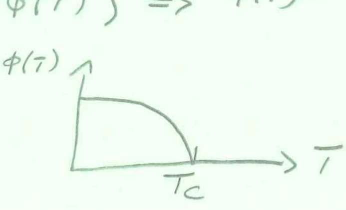
$$= -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{T^4}{2\pi^2}\left\{-\frac{\pi^4}{45} + \frac{\pi^2}{12}\frac{m_\varphi(\phi)^2}{T^2} - 4\frac{7\pi^4}{360} + 4\frac{\pi^2}{24}\frac{m_\psi(\phi)^2}{T^2} + \mathcal{O}(\lambda^{3/2}, \lambda^2\ln\lambda, g^4\ln g)\right\}$$

$$= -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{1}{2}T^2\left(\frac{3\lambda}{12} + \frac{4g^2}{24}\right)\phi^2 + f(T) + \mathcal{O}(\lambda^{3/2}, \dots)$$

$\Rightarrow$ critical temperature $T_c^2 = \dfrac{24\mu^2}{6\lambda + 4g^2} = \dfrac{24\lambda}{6\lambda + 4g^2}v^2 \sim v^2 \quad (\gg m_\varphi^2(v) = 2\lambda v^2)$



$$0 = V_{eff}'(\phi = \phi(T)) \Rightarrow \phi(T) = \begin{cases} 0 \\ \sim\sqrt{T_c^2 - T^2} & T < T_c \end{cases}$$



IV. Abelian Higgs Model

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^*(D^\mu\varphi) + \mu^2\varphi^*\varphi - \frac{\lambda}{2}(\varphi^*\varphi)^2$$

- Shifted field $\varphi \to \frac{1}{\sqrt{2}}(\phi + h + i\eta)$, choose ϕ = real

$$\Rightarrow \mathcal{L}\Big|_{\substack{\varphi=\phi/\sqrt{2} \\ A_\mu = 0}} = -V_0(\phi) \Rightarrow V_0(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4$$

- Quadratic part

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial h)^2 + \frac{1}{2}(\partial\eta)^2 + \frac{1}{2}e^2\phi^2 A_\mu A^\mu + \frac{1}{2}(\mu^2 - 3\lambda\phi^2)h^2$$
$$+ \frac{1}{2}(\mu^2 - \lambda\phi^2)\eta^2 - e\phi\,\eta\,\underbrace{\partial_\mu A^\mu}_{\to 0 \text{ in Lorentz gauge}}$$

$$\Rightarrow m_A^2(\phi) = e^2\phi^2$$
$$m_h^2(\phi) = 3\lambda\phi^2 - \mu^2 = \lambda(3\phi^2 - v^2)$$
$$m_\eta^2(\phi) = \lambda(\phi^2 - v^2)$$

- $$V_{1L}^T(\phi) = \frac{T^4}{2\pi^2}\left\{ J_B\left(\frac{m_h^2(\phi)}{T^2}\right) + J_B\left(\frac{m_\eta^2(\phi)}{T^2}\right) + 3\,J_B\left(\frac{m_A^2(\phi)}{T^2}\right)\right\}$$

(1) $\lambda \sim e^2 \Rightarrow m_A(v) \sim m_h(v) \ll T_C$

$$\Rightarrow V_{1L}^T \simeq \frac{1}{2}T^2\phi^2\frac{4\lambda + 3e^2}{12} \Rightarrow T_C^2 = \frac{12\mu^2}{4\lambda + 3e^2} = \frac{12\lambda}{4\lambda + 3e^2}v^2$$

(2) $\lambda \sim e^4 \Rightarrow m_h(v) \ll m_A(v) \lesssim T_C$

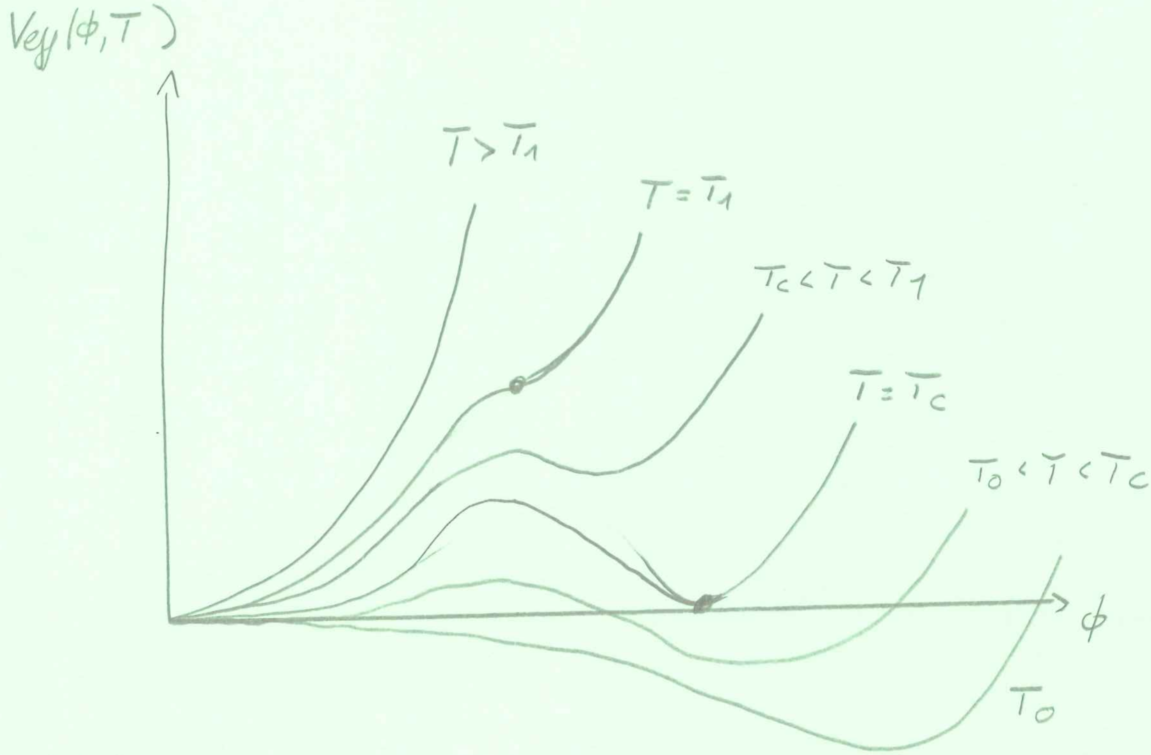
$$\Rightarrow V_{1L}^T(\phi) \simeq 3\frac{T^2}{2\pi^2}\left\{\frac{\pi^2}{12}\frac{m_A^2}{T^2} - \frac{\pi}{6}\frac{m_A^3}{T^3} - \frac{m_A^4}{32T^4}\ln\frac{m_A^2}{a_b T^2}\right\}$$

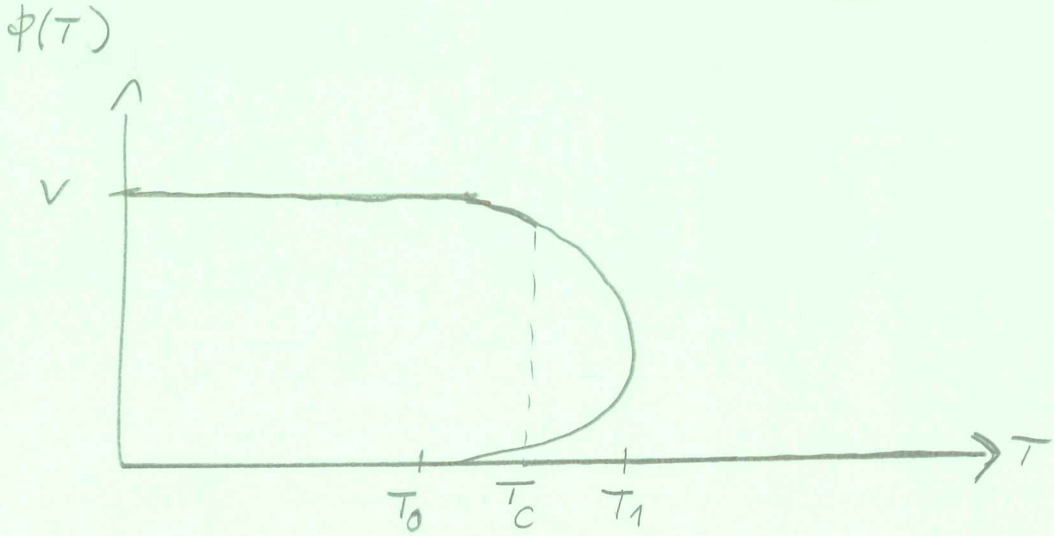
$$V_{1L}^{vac}(\phi) = 3\frac{1}{64\pi^2}\left\{m_A(\phi)^4\left(\ln\frac{m_A(\phi)^2}{m_A(v)^2} - \frac{3}{2}\right) + 2m_A(\phi)^2 m_A(v)^2\right\}$$

$$\Rightarrow \boxed{V_{eff}(\phi) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda(T)}{4}\phi^4}$$

where $D = \frac{e^2}{8}$, $DT_0^2 = \frac{\mu^2}{2} - \frac{3}{32\pi^2}e^4v^2 = \frac{1}{2}v^2\left(\lambda - \frac{3e^4}{16\pi^2}\right)$, $E = \frac{e^3}{4\pi}$,

$$\lambda(T) = \lambda - \frac{3e^4}{16\pi^2}\ln\frac{e^2v^2}{A_b T^2}, \quad A_b = 16\pi^2 e^{-2\gamma_E}$$





$$\frac{\phi(T_c)}{T_c} \simeq \frac{2E}{\lambda} = \frac{4Ev^2}{m_h^2}$$

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