Sphalerons

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Literature

1. Christoph Horst – Instantons
2. Srednicki – Quantum Field Theory
3. Peskin, Schroeder – An Introduction to QFT
4. Smilga – Lectures on Quantum Chromodynamics
5. Rubakov – Classical Theory of gauge fields
6. Terning – Modern Supersymmetry
7. Healey – Gauge Symmetry and the Theta-Vacuum (Epsa Philosophical Issues in the Sciences, Vol 2)
8. Fort, Gambini – U(1) Puzzle and Strong CP Problem from a Holonomy Formulation Perspective (Internal Jour. of Theo. Phys. Vol 39, No 2. 2000)
9. Novikov, Shifman Vainstein, Zakharov – ABC of Instantons (Sov. Phys. Usp. 1982. vol 25, 553)
10. Manton – Topology in the Weinberg-Salam theory (Phys. Rev. D. 28 Number 8, 1983)
11. Callan, Dashen Gross – Towards a theory of the strong interactions (Phys. Rev. D. 17 Number 10, 1978)
12. 't Hooft – Symmetry breaking Through Bell-Jackiw Anomalies (Phys. Rev. lett. 37., p. 8-11, 1976)
13. 't Hooft – How instantons solve the U(1) problem (Physics Reports 142, No 6, 1986)
14. Anselm, Johansen – Can the electroweak Θ-term be observable? (Nucl. Phys. B 412 (1994) 553-573)
15. Hsu – Physical consequences of the QED theta angle (arXiv: 1012.2906 [hep-th])

1. Aspects of gauge symmetry

- What is a symmetry?
- What is an empirical symmetry?
- What is the status of gauge symmetry?
- Only working definitions open to discussion.

Symmetry of an abstract structure:

Automorphism - a transformation that maps the elements of an object back onto themselves so as to preserve the structure of that object.

A physical theory specifies a set of models - mathematical structures - that may be used to represent various different situations, actual as well as merely possible, and to make claims about them.

- Symmetries of a class of situations vs. symmetries of models.

Empirical symmetry:

A 1-1 mapping $\varphi : S \to S$ of a set of situations onto itself is an empirical symmetry if and only if any two situations related by φ are indistinguishable by means of measurements confined to each situation.

Theoretical symmetry:

A 1-1 mapping $f : \mathcal{M} \to \mathcal{M}$ of the set of models of a theory Θ onto itself is a theoretical symmetry of Θ if and only if the following condition obtains: For every model m of Θ that may be used to represent (a situation s in) a possible world ω, $f(m)$ may also be used to represent (s in) ω.

Some comments:

- Theoretical symmetries may be purely formal features of a theory, if all they do is to relate different but equivalent ways the theory has of one and the same empirical situation.

- A theoretical symmetry may entail a corresponding empirical symmetry, in which case it is not a purely formal feature of the theory.

- Example of empirical symmetry:

– Lorentz symmetry of SR:
  - Associate each model with inertial frame with respect to which a given situation is represented
  - Lorentz transform of any model is also a model.
  - Lorentz transform of a model may be used to represent a boosted duplicate of a situation (from the perspective of the original frame)
  - SR entails empirical symmetry associated with Lorentz invariance by implying that models connected by a Lorentz-transformation represent distinct but indistinguishable situations.

Question

Do gauge trafos reflect some empirical symmetry?

- Answer: Conventional wisdom is that successful employment of Yang-Mills theories warrants the conclusion that local gauge symmetry is a purely formal feature of these theories.
  ("Symmetry of mathematical model not of the world")

1.1. Large gauge transformations

Recall talk by Christoph on Instantons:

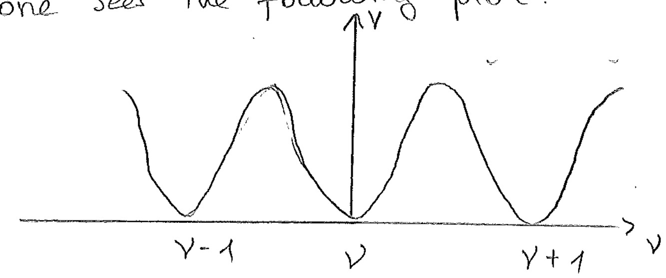
- Transformations which cannot be reduced to a continuous series of gauge transformations are called large gauge transformations. Furthermore these transformations are topologically non trivial:

Ex: $\Omega_*(x) = \exp\left\{\frac{i\pi x^a \tau^a}{\sqrt{x^2+\rho^2}}\right\}$ ρ finite and/arbitrary

$\Omega_*(0) = 1$, $|\vec{x}| \to \infty$, $\Omega_*(\vec{x}) \to -1$ irrespectively of $\frac{\vec{x}}{|\vec{x}|}$

(Note these are gauge trafos, not instantons)

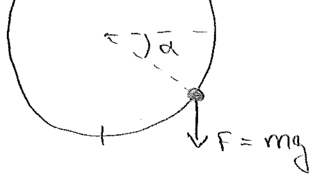
- If we consider pure gauge configurations: $A_i = -i\,\partial_i \Omega_* \Omega_*^+$ with some Ω_* (large gauge trafo) they correspond to vacua with different Chern-Simons number ν (winding number in Christoph's talk)
- Usually one sees the following plot:



- However note that the Chern-Simons current is not gauge invariant. All these vacua are connected by large gauge transformations.
- Maybe large gauge transformations play a special role?
- It is interesting to note that in principle one could eliminate the full gauge symmetry and consider the configuration space(*) as the space of gauge orbits. In this case large gauge trafos play no special role. Vacuum is a single point in configuration space.

(*) space of gauge fields

• Analogy: QM particle living on a vertically oriented circle subjected to a constant gravitational force

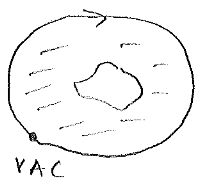


The particle not only oscillates near the minimum but can tunnel under the potential barrier, winding around the circle. However, the minimum is always the same point.

• Also in the configuration space the topology is non-trivial.
→ There is a <u>hole</u>.

• The points with different Chern-Simons numbers are physically one and the same point.

• Do we lose something if we look at the space of the gauge orbits?
• As always if a question is posed → the answer is no.
Physical consequences follow from <u>existence of noncontractible</u> loops in the configuration space.



• Instantons / Sphalerons are still with us.

• Comments: The topological charge of an instanton is gauge-invariant quantity.

$$n = \int d^4x \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} = N_{C.S}(\tau=\infty) - N_{C.S}(\tau=-\infty)$$

Instanton is not a large gauge transformation but a semi-classical tunneling trajectory connecting the vacuum to itself (non-contractible loop).

2. The Θ- Vacuum

- The picture of the vacuum-valley suggests to consider a superposition of vacuum states with different C.S numbers. (pre-vacua)
- This picture looks similar to the motion of an electron in a periodic potential. This problem was solved long ago, by Bloch.

$V(\vec{x}+\vec{a}) = V(\vec{x})$, $\psi_n(\vec{x}) = e^{i\vec{k}\vec{x}} u_{n\vec{k}}(\vec{x})$, $u_{n\vec{k}}(\vec{x}+\vec{a}) = u_{n\vec{k}}(\vec{x})$

$\vec{k}$: chrystal momentum or quasi-momentum.

- BUT : In a chrystal the points $\vec{x}+\vec{a}$ and $\vec{x}$ are physically different which is not the case for gauge theory. Remember this in the following.
- Gauge theory: Large gauge trafo operator Ω

(gauge dependent)

$$\Omega\, \psi[A_i(\vec{x})] = \psi[A_i^{\Omega *}(\vec{x})]$$

commutes with the Hamiltonian. $[H, \Omega] = 0$

$\to$ Diagonalize H & Ω simultaneously

- How do the physical (gauge-invariant) states behave under Ω? for small gauge-trafo this is specified by the Gauss's law constraint.

$$R\, \psi^{phys} = \psi^{phys}$$

- This is not the case for Ω. Usually one says that the requirement of gauge invariance is satisfied as long as ψ^{phys} are eigenstates of Ω

$$\underline{\Omega\, \psi^{phys} = e^{i\theta} \psi^{phys}}$$

- No physical principle determines what θ should be but since $[H, \Omega] = 0$ neither time-evolution nor local gauge-inv. perturbations will change θ.
- θ-labels superselection sectors of the theory and H must be block-diagonal in θ.

- Therefore $|\theta\rangle = \sum_{-\infty}^{\infty} e^{in\theta}|n\rangle$, where $\Omega_1 |n\rangle = |n+1\rangle$.
- $|\theta\rangle$ states are physically different from each other.
- If $\theta \neq 0$, $|\theta\rangle$ is not an eigenstate of parity and time reversal.

- Consider now following situation:

  If one looks from the beginning on gauge-inv. quantities, and by gauge-inv. we mean invariant under small and large gauge trafos, wouldn't one impose $\underline{\Omega \Psi^{phys} = \Psi^{phys}}$ for Ω now $\underline{\text{all}}$ gauge trafos?

  This is the claim of people working with holonomies instead of gauge fields.

- θ- Angle from cluster decomposition principle:

  Consider expectation value of an operator O in large Euclidean spacetime volume V where we look at fields with definite topological charge n.

  Suppose we assign weight $f(n)$ to each conf. with definite winding number. This sounds plausible but usually the only weight in the path integral is the exponential of the action. So why arbitrary $f(n)$ and not 1? ...

$$\langle O\rangle_V = \frac{\sum_n f(n)\int_n D\phi\, e^{-S_E^V(\phi)}\, O(\phi)}{\sum_n f(n)\int_n D\phi\, e^{-S_E^V(\phi)}}$$

- Divide V into very large volumes V_1 and V_2, O should be localized in V_1. Fields with top. charge n can be separated into fields in V_1 with n_1 and fields in V_2 with n_2. $n = n_1 + n_2$

$$\langle O\rangle_V = \frac{\sum_{n_1,n_2} f(n_1+n_2)\int_{n_1} D\phi\, e^{-S_E^{V_1}}\, O \int_{n_2} D\phi\, e^{-S_E^{V_2}}}{\sum_{n_1 n_2} f(n_1+n_2)\int_{n_1} D\phi\, e^{-S_E^{V_1}}\int_{n_2} D\phi\, e^{-S_E^{V_2}}}$$

- Cluster decomposition: $\langle 0|T\, O_1(x_1)\, O_2(x_2)|0\rangle \xrightarrow{|x_1-x_2|\to\infty} \langle O_1(x_1)\rangle\langle O_2(x_2)\rangle$
- Consistent if $f(n_1+n_2) = f(n_1)\cdot f(n_2) \Rightarrow f(n) = e^{i\theta n}$

- $f(n)\, e^{-S_E^V} = e^{i\theta n}\, e^{-S_E^V} = e^{-S_E^V + i\theta \xi \int d^4x\, F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}}$

- Weighting function is equivalent to adding θ-term to the Lagrangian.

2.1. Effects of the θ-Vacuum

- Suppose we add a θ-Term to the Lagrangian do we always see physical consequences?

2.1.1. QCD

- QCD with 1 flavour of massless quark:
- Symmetries: $SU(3)_C$ gauge symmetry
  $U(1)_V \times U(1)_A$ global symmetries of quark action

- $U(1)_A$ symmetry is anomalous!

Under $\left\{ \begin{array}{l} \psi \to e^{-i\alpha\gamma_5}\psi \\ \bar{\psi} \to \bar{\psi} e^{-i\alpha\gamma_5} \end{array} \right\}$ trafos the integration measure in

Path integral picks up a phase factor:

$$D\psi D\bar{\psi} \to \exp\left[-i \int d^4x \frac{g^2 \alpha}{16\pi^2} \tilde{F}^{a\mu\nu} F^a_{\mu\nu}\right] D\psi D\bar{\psi}$$

$\Rightarrow$ Effect of $U(1)_A$ is to change the value of θ from θ to $\theta + 2\alpha$. $\Rightarrow$ θ can be transformed to zero by $U(1)_A$ trafo. (equiv. to a change of the dummy integration variable in the path integral).

- Adding a massless quark has turned the θ-angle into a physically unobservable parameter.

- Now 1 massive quark

Write $\psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} \rightsquigarrow \mathcal{L}_{mass} = -m\chi\xi - m^*\xi^\dagger\chi^\dagger$

Since left & right handed fields are transformed differently (χ, ξ transformed in the same way) massterm breaks explicitly the $U(1)_A$ symmetry.
However it should be approximate, since the masses are small.

- Allow m to be complex: $m = |m|e^{i\phi}$

$\rightarrow \quad \mathcal{L}_{mass} = -|m|\bar{\psi}e^{-i\phi\gamma_5}\psi$

$\rightarrow$ $U(1)_A$ trafo changes ϕ to $\phi + 2\alpha$, Θ - changes to $\Theta + 2\alpha$

$\rightarrow$ $\Theta - \phi$ or $me^{-i\Theta}$ is unchanged.

Path integral depends on $me^{-i\Theta}$.

- To understand effects of Θ on hadronic physics one has to examine effective Lagrangian of QCD (Chiral QCD).
Θ-value appears in the quark mass-matrix and will lead to an electric dipol moment of the neutron. (Srednicki Chapter 94)

$d_n = 3.2 \cdot 10^{-16}\,\Theta\, e.cm.$

Experimental upper limit $|d_n| < 6.3 \cdot 10^{-26}\, e.cm$

$\rightsquigarrow \quad |\Theta| < 2 \cdot 10^{-10}$. Why is Θ so small? Strong CP Problem

2.2.2. SU(2) & U(1)

- Θ - terms added to $\mathcal{L}$ will always be unobservable, since it can be rotated away via the anomalous Baryon-number symmetry. (SU(2) part see Anselm 94)

- Main difference to QCD: No terms explicitly breaking this symmetry!

- U(1) part: no non-trivial gauge conf.

However HSU 2010 claims that $\frac{\Theta}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$ term added to $\mathcal{L}_{SM}$ can have observable consequences. ($F_{\mu\nu}$ corr. to U(1))

3. Level crossing and non conservation of fermion quantum numbers.

- Fermionic currents with anomalies in their divergence will lead in the background field of non-trivial topological configurations (instantons, sphalerons) to <u>non-concervation of fermion quantum numbers</u>. (Baryon number, Lepton number)

3.1. <u>Easy example</u> 2D QED (1 space + 1 time)

$$\mathcal{L} = \bar{\psi}(i\not{\partial})\psi - \frac{1}{4}(F_{\mu\nu})^2$$

- Currents $j^\mu = \bar{\psi}\gamma^\mu\psi$, $j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$ conserved without mass term.

- $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$, Free theory eq. for ψ_+
  $i(\partial_0 + \partial_1)\psi_+ = 0$; Solution: Waves that move to the right at the speed of light

- ψ_+: right-mover
  ψ_-: left-mover

- As in QCD the axial vector current is not conserved in the presence of electromagnetic fields, as the result of an anomalous behavior of its vacuum pol. diagram.

$$\partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}.$$

- <u>Global aspect</u>: In the free-fermion theory

$$\int d^2x\, \partial_\mu j^{\mu 5} = N_R - N_L = 0$$

together with the conservation of vector current N_R & N_L separately conserved.

- But $\partial_\mu j^{\mu 5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} = 2\partial_\mu(\epsilon^{\mu\nu}A_\nu)$ total derivative.

Consider now not trivial configuration:

A^1 constant in space and very slow time dependence

- Assume that system has finite length L with periodic boundary condition.
- Such field cannot be removed by a gauge trafo that satisfies the periodic boundary conditions.

- $H = \int dx\, \psi^{+} (-i\alpha^1 D_1)\psi$

$$= \int dx \left\{ -i\psi_+^{+} (\partial_1 - ieA^1)\psi_+ + i\psi_-^{+} (\partial_1 - ieA^1)\psi_- \right\}$$

- A^1 <u>constant</u> : easy to diagonalized:
  Eigenstates of covariant derivatives are wave-functions

$$e^{ik_n x} \quad , \quad k_n = \frac{2\pi n}{L} \quad , \quad n = -\infty, \dots, \infty$$

- Energies of single particle eigenstates:

$$\psi_+ \;:\; E_n = +(k_n - eA^1) \qquad (*)$$
$$\psi_- \;:\; E_n = -(k_n - eA^1)$$

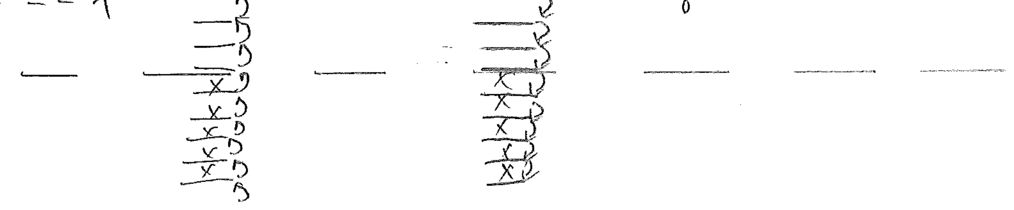
- Each type of fermion has an infinite tower of equally spaced levels.
- Find ground state of $\mathcal{H}$: Fill the negative-energy levels, interprete holes as antiparticles.

- Change <u>adiabatically</u> A^1 : Fermion energy levels slowly shift according to $(*)$

Let: $\Delta A^1 = \frac{2\pi}{eL}$ -> the spectrum of $\mathcal{H}$ returns to it original form.

- In this process each level of ψ_+ moves down to the next position and each level of ψ_- moves up to the next position.

$\gamma^5 = -1$ $\qquad\qquad$ $\gamma^5 = +1$



- The occupation numbers of levels should be maintained in this adiabatic process.

- $\Rightarrow$ One right-moving fermion disappears from the vacuum and one extra left-moving fermion appears.

- $\int d^2x \left(\frac{e}{\pi} \epsilon^{\mu\nu} F_{\mu\nu}\right) = \int dt\, dx \frac{e}{\pi} \partial_0 A_1 = \frac{e}{\pi} L(-\Delta A^1) = -2$

$$\Rightarrow \quad \underline{N_R - N_L = \int d^2x \left(\frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}\right)}$$

- The change in the fermion number corresponds exactly to the topological charge $\Rightarrow$ Consequence of Atiyah-Singer Index Theorem.

3.2 't Hooft determinant interaction

- The non-conservation of fermion quantum numbers can be deduced in 4D from Euclidean Path-Integral.
- The vacuum to vacuum transition amplitude vanishes due to the presence of zero-modes of the covariant derivative in the background instanton field. (Number of zero modes is connected to the topological charge: Atiyah-Singer Index Theorem)
- The amplitude vanishes because we kept the fermion numbers fixed. The initial and final chiral charges should be different.
- If one takes this into account the amplitude is final.
- 't Hooft asked then for an effective vertex that could mimic the same amplitude : <u>'t Hooft determinant interaction.</u>

3.2.1 Quick calculation

- Expand the gauge covariant derivative in eigenmodes (in instanton bdgr.)

$$i \not{D} f_i = \lambda_i g_i \qquad \text{(Spinors here)}$$

- Spinor in fundamental of SU(N), write eigenmodes ψ_i in terms of a Grassman variable and a complex spinor eigenfunction

$$\psi = a_0 f_0 + \sum_i a_i f_i \quad , \quad \bar{\psi} = \sum_i b_i^{+} g_i$$

f_0 corresponds to zero eigenvalue

- Path integration over this fermion

$$\int D\psi D\bar{\psi} = \int da_0 \int \prod_{i,j} da_i \, db_j$$

- $$\int D\psi D\bar{\psi} \exp\left(-\int \bar{\psi} \not{D} \psi\right) = \int da_0 \int \prod_{i,j} da_i \, db_j \exp\left(-\sum_n b_n \lambda_n a_n\right)$$

$$= \int da_0 \int \prod_{i,j} da_i \, db_j \prod_n (1 - \lambda_n b_n a_n)$$

use $\theta_i^2 = 0$ for θ_i Grassman number

$$= \int da_0 \prod_n \lambda_n = 0$$

since integrating a constant over a Grassmann variable vanishes.

- However if one inserts a fermion field into the path integral:

$$\int D\psi D\bar{\psi} \exp\left(-\int \bar{\psi} \not{D} \psi\right) \psi(x) = f_0(x) \prod_n \lambda_n$$

- Thus, for an instanton amplitude to be nonvanishing it must emit/absorb <u>one fermion for each zero mode</u>.

- At distances much larger than the instanton size 't Hooft showed that instantons produce effective interactions for a fermion Q_{nj} with SU(n) index n and flavour index $j = 1, \dots, F$

$$\mathcal{L}_{inst} = c \cdot \det \bar{Q}^{in} Q_{nj} + h.c. \qquad \text{dimension of } c \quad 4 - 3F$$

- $SU(2)_L$: non conservation of baryon and lepton number
- $SU(3)$: Breaking of $U(1)_A$: very heavy$_{(13)}$ η' - meson.

- Technical details
- Simple proof showed that one fermion in the eigenbasis should appear in the path integral.
- In general there is one zero-mode for each of the N flavors.
- Initial and final chiral charges in the Path integral should be different

$\Rightarrow$ product of $\underline{\bar{\psi}_R \psi_L}$ in the Eigenbasis

- In general basis one needs product of the eigenvalues and therefore $\det(\bar{\psi}_R \psi_L)$
- Then one takes into account the dilute instanton gas together with the weight

$$\leadsto A \propto \sum_{\nu_+=0}^{\infty} \sum_{\nu_-=0}^{\infty} \frac{\kappa^{\nu_+ + \nu_-}}{\nu_+! \nu_-!} e^{i\theta(\nu_+ - \nu_-)} \left(\int d^4x \det(\psi_L \bar{\psi}_R) \right)^{\nu_+}$$

$$\times \left(\int d^4x \det(\psi_R \bar{\psi}_L) \right)^{\nu_-} \qquad (*)$$

- summations can be carried out and exponentiate the integrals $\Rightarrow$ effective interaction

$$A \propto \exp \int d^4x \, \underbrace{\left[\kappa e^{i\theta} \det \psi_L \bar{\psi}_R + \kappa e^{-i\theta} \psi_R \bar{\psi}_L \right]}_{\mathcal{L}_{eff.}}$$

(*) comments: • Integral $\int d^4x$ over instanton locations

• Denominator $\frac{1}{\nu_+! \nu_-!}$ due to exchange symmetry

• Remark: Level crossing or determinant interaction are two ways to describe the same process.

3.3. Electroweak sector

• There are no true instantons in the Electroweak theory with Higgs-field. A scaling argument (see Rubakov) shows that in any finite-action solution of the 4D Euclidean Field Eq. the Higgs field cannot differ from its vacuum value in any finite region, so the only instantons are singular.

• However, there are smooth dynamical fields which represent a non-trivial loop and whose action is greater than $\frac{8\pi^2}{g^2}$ by an arbitrary small amount.

• Tunneling amplitude $\propto \exp(-8\pi^2/g^2)$, $g^2 \sim 10^{-2}$ ⟹ very small $\sim 10^{-160}$

3.3.1. Selection rules

• only left-handed fields interact with gauge fields.

• We have seen:

$$\Delta N_L = n(\Omega_f) - n(\Omega_i) = -\Delta n \qquad (**)$$

where ΔN_L: number of left-handed fermions

$n(\Omega_{f/i})$: Chern-Simons number of final/initial vacuum (in the sense of a loop ending in the same vacuum but with different N.C. number)

(**) is satisfied for every left-handed doublet!

• The number of right-handed fermions does not change:

(B-L conserved)

$$\Longrightarrow \quad \Delta L_e = \Delta L_\mu = \Delta L_\tau = \tfrac{1}{3}\Delta B$$

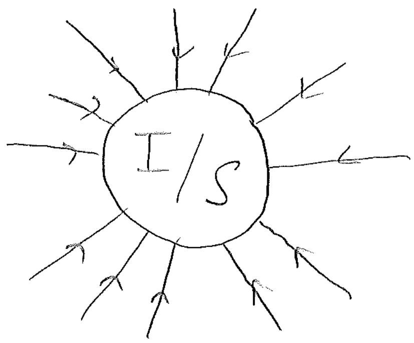
with $L_e = \#e + \#\nu_e - \#\bar{e} - \#\bar{\nu}_e = N_e + N_\nu$ and analogue for μ/τ.

• B: Baryon-number (# of left-handed doublets: $N_{gen} \cdot N_{colours} \cdot \frac{1}{3} = 3$)

$\frac{1}{3}$: Baryon-number of a quark.

⟹ Non conservation of Baryon and Lepton-number.

- In an instanton / sphaleron transition the total number of fermions decreases by 12 units.



" The blob "

$\underline{\Delta L = \Delta B = -3}$

4. Sphalerons

4.1 The hight of the barrier

- QCD Lagrangian at classical level contains no dimensional parameters.
- Instantons are solution of the classical (Euclidean) e.o.m. $\rightarrow$ their only dimensional constant is their variable size.
- The height of the barrier must have the dimension of mass $\rightarrow$ The smaller the size ρ the higher the barrier the instanton with the given ρ sees.
  The classical action stays always constant $\frac{8\pi^2}{g^2}$.
  This is only possible because the configuration space is infinite dimensional. (Calculation see Rubakov Chap. 13)
- Barrier can be arbitrarily small in classical theory
- The qu. QCD is not tractable quasiclassically in the infrared limit. It is impossible to determine the lowest possible height of the barrier under which the system tunnels. It should be of the order $\propto \Lambda_{QCD}$
- Situation changes in the Higgs regime. (SU(2) Part of SM)
  All gauge bosons have mass.
  If Higgs-VEV is larger than the corresponding strong scale (Λ_{QCD}) the coupling constant always stays small and the quasiclassical picture is applicable.
- What is then the minimal hight of the barrier?

4.2 Sphaleron solution

- If the system sits on top of the barrier, this is a solution of the static e.o.m. (Position on top is an equilibrium, but unstable one)
- This static solution is called sphaleron.
- The sphaleron energy is the height of the barrier separating the pre-vacua of the Yang-Mills theory in the Higgs regime.
- In gauge invariant terms sphaleron corresponds to a non-trivial loop in the configuration space which is a saddle-point of the energy functional.

Since it is a saddle-point its energy represent the height of the barrier between the lower energy regions in conf. space.

Therefore, both definitions are equivalent.

- The name comes from greek "σφάλλω" make to fall overthrow
- The sphaleron energy is

$$E_{sp} = \frac{2 m_V}{\alpha_V} B\left(\frac{m_\chi}{m_V}\right)$$

m_V : Vector boson mass

$\alpha_V = \frac{g^2}{4\pi}$

B varies within 1.56...2.27 as m_χ : Higgs-mass ranges from small to large values

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- Solution in $A_0 = 0$ gauge

$$A_i^a = \frac{1}{g} \varepsilon_{iak} \frac{x^k}{r} f(r) \quad , \quad X = \frac{\vec{\sigma}\vec{x}}{r} h(r) \quad ; r = \sqrt{x^2}$$

f, h : profile functions ; $f, h \xrightarrow[r\to 0]{} 0$ $\quad h \xrightarrow[r\to\infty]{} \eta$ $\quad f(r) \xrightarrow[r\to\infty]{} -\frac{2}{r}$

X is the Higgs-field as Matrix.

η : Higgs VeV

$$\mathcal{L}_H = \frac{1}{2} \mathrm{Tr}\, D_\mu X^\dagger D_\mu X - \frac{\lambda}{4}\left(\frac{1}{2} \mathrm{Tr}\, X^\dagger X - \eta^2\right)^2$$

$$X = \begin{pmatrix} \chi^1 & -\chi^{2\dagger} \\ \chi^2 & \chi^{1\dagger} \end{pmatrix}$$

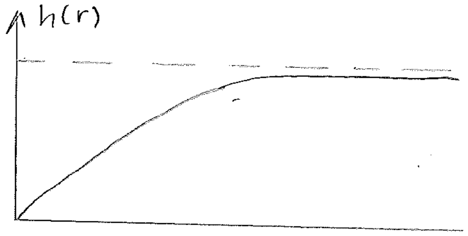
$m_V \sim g\eta$

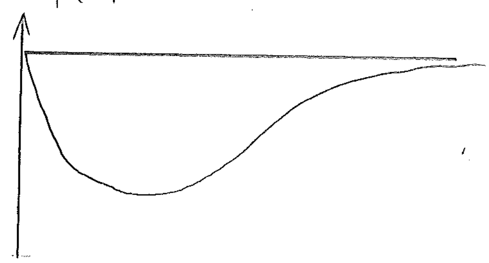
- Substitute Ansatz in the $\mathcal{L}_H$

$$\mathcal{H} = 4\pi \int_0^\infty r^2\, dr \left\{ \frac{1}{g^2}\left[f'^2 + \frac{2}{r^2} f^2 + \frac{2}{r} f^3 + \frac{1}{2} f^4 \right] + h'^2 + 2h^2 \left(\frac{1}{r} + \frac{f}{2} \right)^2 \right\}$$

- All terms positive definite $\to$ minimum exist
- It can be found numerically

- Profile functions:





- Rescale variables: $f = g\eta F$, $h = \eta H$, $r = R(g\eta)^{-1}$

$$H = 4\pi \underbrace{\frac{\eta}{g}}_{\text{The only parameter}} \int_0^\infty R^2\, dR \underbrace{\left\{ \left[F'^2 + \frac{2}{R^2} F^2 + \frac{2}{R} F^3 + \frac{1}{2} F^4 \right] + H'^2 + 2H^2 \left(\frac{1}{R} + \frac{F}{2} \right)^2 \right\}}_{\text{no parameters}}$$

$\to E_{sph} = \mathcal{H}_{min} = \text{const} \cdot \frac{\eta}{g} \propto \frac{m_V}{\alpha_V}$

- Chern-Simons number of sphaleron:

$(A_i)_{sph} \to i U \partial_i U^+$, $U = \frac{\vec{\sigma}\vec{x}}{r}$

- The matrix U takes different values as we approach infinity from different directions. The condition of comp. does not hold.

$\Rightarrow N_{C.S.}$ need not to be an integer.

For the given example: $N_{C.S} = \frac{1}{2}$

4.3. Sphalerons at high temperature

- At high temperature thermal fluctuations lead the system to emerge at a saddle point (sphaleron).

- Naive estimation: Rate of thermal jumps:

  - $\Gamma \propto \exp\left(-\frac{E_{sph}}{T}\right)$ Boltzmann exponent

  - $E_{sp}^{EW} = \frac{2m_W}{\alpha_W} B\left(\frac{m_H}{m_W}\right) \sim 10\,TeV$,

  - $T \gtrsim 1\,TeV$ the rate of thermal jumps ceases to be small.

- In fact the rate is underestimated.
The probability is controlled by the free energy F of the system.

- $\Rightarrow$ Extremum of free energy $F(\vec{A}, \varphi)$ important ("thermal sphaleron")

- $$\Gamma \sim T^4 \exp\left(-\frac{F_{sph}(T)}{T}\right)$$, T^4 on dimensional grounds,

  Γ: probability of process with non-conservation of the baryon and lepton numbers per unit time per unit spacial volume.

- $$F_{sph}(T) = \frac{2m_W(T)}{\alpha_W} B\left(\frac{m_H}{m_W}\right)$$

  decreases as the temperature grows, at $T > T_{crit}$ ($T > T_{crit}$ $SU(2)$ unbroken)
  $F_{sph} = 0$.

- $\Rightarrow$ For $T > T_{crit}$ processes with non-conservation of baryon and lepton number ceased to be exponentially suppressed.

- However, they can be then no longer described by semi-classical methods.

- Cross sections of such processes at colliders increase rapidly at energies in TeV-range.