Instantons

Christoph Horst, 13/12/11

<u>Def.:</u> An <u>instanton</u> is a topologically nontrivial stationary point of a Euclidean finite action.

I) a) Instantons in quantum mechanics (D=2 dimensional spacetime)
 b) Relation between instantons and tunneling

II) a) Yang-Mills instantons

 b) Emergence of the Θ-angle

I) a) Instantons in QM

The nature of the problem: Given a classical theory, investigate the vacuum of the quantum theory, i.e. want to compute the vacuum energy and try to obtain some information regarding the overlap of the quantum vacuum with classical vacua.

Note, could solve this problem using Schrödinger's equation, but opt for another method that can later also be applied in quantum field theory.

We consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$$

and the action of a classical trajectory $x(t)$ with $x(-t_0/2) = x_i$, $x(t_0/2) = x_f$

$$S = \int_{-t_0/2}^{t_0/2} dt\, \mathcal{L} .$$

The Hamiltonian density reads

$$\mathcal{H} = -\mathcal{L} + p\cdot\frac{dx}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)$$

and the energy is

$$E = \int_{-t_0/2}^{t_0/2} dt\, \mathcal{H} = \int_{-t_0/2}^{t_0/2} dt \left(\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \right).$$

In stable theories where $V(x)$ is bounded from below, classical vacua are absolute minima of the energy (functional) E, i.e. $x(t) = \eta$ and η is an absolute minimum of the potential V. We arrange V in such a way that

$$V(\text{absolute minima}) = 0$$

and restrict ourselves to the case where classical vacua are isolated points in $\mathbb{R}$.

In the quantum theory the transition amplitude between position eigenstates $|x_i\rangle$ and $|x_f\rangle$ is given as a path integral:

$$\langle x_f | e^{-iHt_0} | x_i \rangle = N \int Dx\, e^{iS}$$

where H is the Hamiltonian operator and N is a normalization constant. Dx denotes integration over all classical trajectories $x(t)$ with $x(\pm t_0/2) = x_{f/i}$, see Patrick's talk for the precise definition of this integral measure. As in Kai's talk we insert a $\mathbb{1}$ in terms of a complete basis of energy eigenstates $|n\rangle$ and <u>Wick rotate the time axis</u>

$$t \to -i\tau$$

because now, in the limit $\tau_0 \to \infty$, we obtain

$$e^{-E_0 \tau_0} \underbrace{\langle x_f | 0 \rangle}_{\psi_0(x_f)} \underbrace{\langle 0 | x_i \rangle}_{\psi_0(x_i)^*} = \lim_{\tau_0 \to \infty} N \int Dx\, e^{-S_E}$$

where

$$-S_E = -\int_{-\tau_0/2}^{\tau_0/2} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right] = i \int_{-\tau_0/2}^{\tau_0/2} d(-i\tau) \left[\frac{1}{2}\left(\frac{dx}{d(-i\tau)}\right)^2 - V(x) \right] \xleftarrow{\text{Wick}} iS$$

and the boundary condition is now

$$x(\pm\tau_0/2) = x_{f/i}\,.$$

As a result, as soon as we can compute / approximate the path integral (in the limit $\tau_0 \to \infty$), we learn something about the vacuum energy E_0 and about the overlap of the vacuum $|0\rangle$ with position eigenstates $|x_{i/f}\rangle$.

If $x_{i/f} = \eta_{i/f}$ are classical vacua ($V(\eta_{i/f}) = 0$) there exist field configurations $x(\tau)$ with $\lim_{\tau\to\pm\infty} x(\tau) = \eta_{f/i}$ such that

$$\lim_{\tau_0\to\infty} S_E[x(\tau)] < \infty$$

is <u>finite</u>. Among those the <u>stationary</u> configurations (satisfying the Euler-Lagrange equations of the Euclidean Lagrangian) give the zeroth order contribution to the path integral.

On the other hand, if x_i or x_f was not a classical vacuum, then the action $\lim_{\tau_0\to\infty} S_E[x(\tau)]$ would be infinite for all such configurations $x(\tau)$. In those cases the path integral (in the limit $\tau_0\to\infty$) would vanish and, hence, $E_0 > 0$ or $\Psi_0(x_f)\Psi_0(x_i)^* = 0$. Continuity of the modulus of the wavefunction and $\Psi_0(\eta) \neq 0$ will finally imply $E_0 > 0$.

Finite action configurations $x(\tau)_{\eta_j\to\eta_j}$ between the same classical vacuum η_j are understood as topologically trivial, while the ones relating different vacua, $x(\tau)_{\eta_j\to\eta_k}$ with $\eta_j\neq\eta_k$, are called <u>topologically non-trivial</u>. On the set of finite action configurations $x(\tau)_{\eta_i\to\eta_f}$ one has the following equivalence relation:

$$x(\tau)_{\eta_i\to\eta_f} \sim x(\tau)_{\eta_i'\to\eta_f'} \;:\Leftrightarrow\; \eta_i = \eta_i' \text{ and } \eta_f = \eta_f'$$

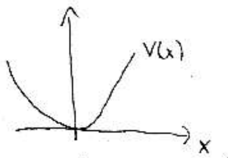
Inequivalent finite action configurations are separated by configurations with infinite action, and, obviously, cannot be continuously deformed into each other. In contrast, equivalent finite action configurations can be deformed into each other and, in particular, to the stationary configurations in this class. For fixed $x_{i/f} = \eta_{i/f}$, such deformations of stationary configurations will contribute to the path integral as quantum fluctuations (see again Patrick's talk).

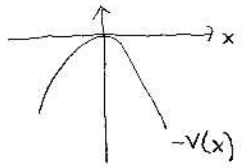
$$S[X(\tau) + \delta x(\tau)] = S_0 + \int_{-\tau_0/2}^{\tau_0/2} d\tau\, \delta x \left[-\frac{1}{2}\frac{d^2}{d\tau^2}\delta x + \frac{1}{2} V''(X)\, \delta x \right]$$

There is, however, one caveat: The computation/approximation of the path integral requires physical intuition due to the occurrence of so-called collective coordinates / zero modes.

Examples:

1) $V(x) = \frac{\omega^2 x^2}{2}$ (harmonic oscillator)





Only one classical vacuum $\eta = 0$

Finite action configurations are always topologically trivial. (no instanton)

Compute $\langle \eta=0 | e^{-H\tau_0} | \eta = 0 \rangle = \mathcal{N} \int \mathcal{D}x\, e^{-S_E}$

Euclidean action has only one stationary point: $x(\tau) = \eta = 0$

(Recall from previous talks that Euclidean e.o.m is $\frac{d^2 x}{d\tau^2} = -\left(-\frac{dV}{dx}\right)$)

For $x(\tau) = \eta$, $S_E = \int_{-\tau_0/2}^{\tau_0/2} d\tau \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right] = 0$ $\Rightarrow$ no zeroth order contribution

Hence,

$$\langle \eta=0 | e^{-H\tau_0} | \eta=0 \rangle = \mathcal{N} \left[\det\left(-\frac{d^2}{d\tau^2} + \omega^2 \right) \right]^{-\frac{1}{2}} (1 + \text{subleading terms})$$

$$= \quad \dots \qquad \text{(see: ABC of instantons pp 212)}$$

$$= \sqrt{\frac{\omega}{\pi}}\, (2 \sinh \omega\tau_0)^{-1/2}$$

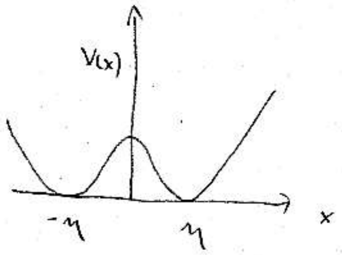
$$\xrightarrow{\tau_0 \to \infty} \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2} \left(1 + \frac{1}{2} e^{-2\omega\tau_0} + \dots \right)$$

$$\stackrel{!}{=} e^{-E_0 \tau_0} |\Psi_0(0)|^2$$

$\Rightarrow E_0 = \frac{\omega}{2}$ and $|\Psi_0(0)|^2 = \sqrt{\frac{\omega}{\pi}}$

which is in accordance with the standard results for the harmonic oscillator.

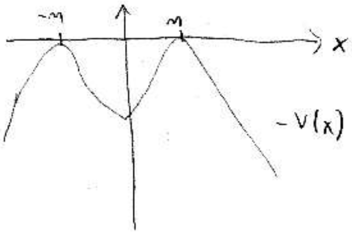
2) $V(x) = \lambda(x^2-\eta^2)^2$ (double-well potential)



Have two classical vacua: $X(\tau) = \pm\eta$

$\omega^2 := V''(\pm\eta) = 8\lambda\eta^2$

In addition to the topologically trivial classes $x(\tau)_{\pm\eta\to\pm\eta}$ have non-trivial classes $x(\tau)_{-\eta\to\eta}$ and $x(\tau)_{\eta\to-\eta}$.
Stationary configurations therein are referred to as instantons/anti-instantons, respectively.
In the limit $\tau_0\to\infty$ one finds (anti)instantons

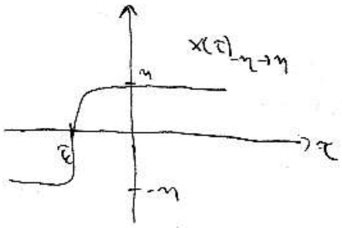


$$X(\tau)_{\mp\eta\to\pm\eta} = \eta\tanh\frac{\omega(\pm\tau-\tau_c)}{2}$$

for any $\tau_c\in\mathbb{R}$.

[$0=E=\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x) \Rightarrow \frac{dx}{d\tau} = -\sqrt{2\lambda}\,(x^2-\eta^2)$ which can easily be solved.]

e.g.



Euclidean action: $S_E[x(\tau)_{\mp\eta\to\pm\eta}] = \int_{-\infty}^{\infty} d\tau \left(\frac{dx}{d\tau}\right)^2 = \int_{-\eta}^{\eta} dx\,(-\sqrt{2\lambda})(x^2-\eta^2) = \frac{\omega^3}{12\lambda} =: S_0$

The occurrence of collective coordinates (here: τ_c) is a general property of instantons. Such coordinates cause the path integral to become infinite which can already be seen at the zeroth order contribution to the path integral

$$\lim_{\tau_0\to\infty}\int_{x(\tau)_{-\eta\to\eta}} \mathcal{D}x\, e^{-S_E}\Big|_{\text{zeroth order}} = \int_{\{x(\tau)_{-\eta\to\eta}=\eta\tanh\frac{\omega(\tau-\tau_c)}{2}\,|\,\forall\tau_c\in\mathbb{R}\}} \mathcal{D}x\, e^{-S_E} = e^{-\frac{\omega^3}{12\lambda}}\int \mathcal{D}x\cdot 1 = \infty$$

This divergence signals that there is a point where the validity of the semi-classical approximation of the path integrals starts to fail. In fact, in picking an arbitrary instanton with $\tau_c\in\mathbb{R}$ and freezing the direction in the space of fluctuations that corresponds to a shift in τ_c, one can compute
[see ABC of instantons, pp 217-223]

$$\therefore \langle -\eta | e^{-H\tau_0} | \eta \rangle_{\substack{1\text{ instanton}\\ \text{at } \tau_c}} = \mathcal{N}\left[\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\right]^{-1/2}$$

$$\times \left[\frac{\det'\left(-\frac{d^2}{d\tau^2}+V''(x)\right)}{\det\left(-\frac{d^2}{d\tau^2}+\omega^2\right)}\right]^{-1/2} e^{-\overbrace{S_E[x_{-\eta\to\eta}]}^{S_0}}\,(1+\text{corrections})$$

$$= \ \ldots$$

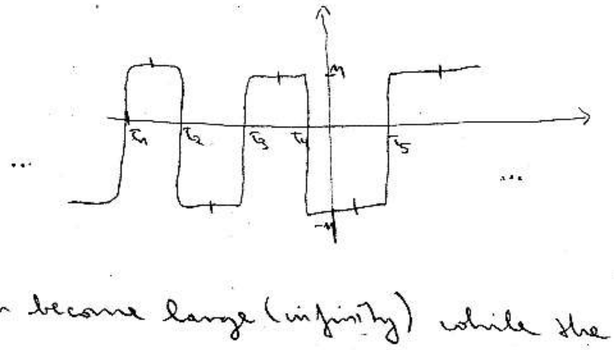
$$= \left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right)\left(\sqrt{\frac{6}{\pi}}\,\sqrt{S_0}\, e^{-S_0}\right)\omega\, d\tau_c$$

where the ~~differential mode~~ $dc_0 = \sqrt{S_0}\, d\tau_c$ remains because c_0 precisely corresponds to the degree of freedom that shifts τ_c which we did not want to integrate over (indicated as $\det'$, the eigenvalue of $x_0(\tau)$ is $\varepsilon_0 = 0$). The first factor is the one from the harmonic oscillator. Now, integrating the result over τ_c, i.e. $\int_{-\tau_0/2}^{\tau_0/2} d\tau_c$, which is trivial as the integrand does not depend on τ_c, one finds that the semi-classical approximation is only valid for

$$\sqrt{S_0}\, e^{-S_0}\,\omega\tau_0 \ll 1,$$

because it is supposed to be an amplitude of probability. Apparently, τ_0 cannot be chosen arbitrarily large without violating this bound.

The solution is to string together instantons and antiinstantons ($\rightsquigarrow$ instanton gas)



because then τ_0 can become large (infinity) while the many instantons / antiinstantons always satisfy the semi-classical bound.

For n (anti)instantons with centers $\tau_1, \tau_2, \ldots, \tau_n$ with $-\frac{\tau_0}{2} < \tau_1 < \tau_2 < \ldots < \tau_n < \frac{\tau_0}{2}$ and $|\tau_i - \tau_j| \gg \omega^{-1}$ (sufficiently spaced) the n-instanton transition has the amplitude

$$\left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right)\underbrace{\left(\sqrt{\frac{6}{\pi}}\sqrt{S_0}\, e^{-S_0}\right)^n}_{=:\, d^n} \int_{-\tau_0/2}^{\tau_0/2}\omega\, d\tau_n \int_{-\tau_0/2}^{\tau_n}\omega\, d\tau_{n-1} \cdots \int_{-\tau_0/2}^{\tau_2}\omega\, d\tau_1$$

$$= \left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right) d^n \, \frac{(\omega\tau_0)^n}{n!}$$

Finally,

$$\langle -\eta | e^{-H\tau_0} | \eta \rangle = \sum_{n=1,3,\dots} \left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right) \frac{(\omega\tau_0)^n}{n!}$$

$$= \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2} \sinh(\omega\tau_0 d)$$

$$\langle \eta | e^{-H\tau_0} | \eta \rangle = \sum_{n=0,2,\dots} \left(\sqrt{\frac{\omega}{\pi}}\, e^{-\frac{\omega\tau_0}{2}}\right) \frac{(\omega\tau_0)^n}{n!}$$

$$= \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2} \cosh(\omega\tau_0 d),$$

where we summed over all n-instanton transitions that are appropriate for $\pm\eta \to \eta$, i.e. the ones with an even / odd number of (anti) instantons.

In the limit $\tau_0 \to \infty$,

$$\lim_{\tau_0\to\infty} \langle \pm\eta | e^{-H\tau_0} | \eta \rangle = \sqrt{\frac{\omega}{\pi}}\, e^{-\omega\tau_0/2} \frac{e^{+\omega\tau_0 d}}{2}$$

$$= \sqrt{\frac{\omega}{4\pi}}\, e^{-\tau_0\left(\frac{\omega}{2} - \frac{\omega}{2}\sqrt{\frac{2}{\pi}\frac{\omega^3}{\lambda}}\, e^{-\frac{\omega^3}{12\lambda}}\right)}$$

$$d = \sqrt{\frac{6}{\pi}}\sqrt{S_0}\, e^{-S_0} = \sqrt{\frac{1}{2\pi}\frac{\omega^3}{\lambda}}\, e^{-\frac{\omega^3}{12\lambda}}$$

As a result, $E_0 = \frac{\omega}{2} - \frac{\omega}{2}\sqrt{\frac{2\omega^3}{\pi\lambda}}\, e^{-\frac{\omega^3}{12\lambda}}$

$$\psi_0(\pm\eta) = \sqrt[4]{\frac{\omega}{4\pi}}\, e^{i\varphi_\pm} \quad \text{with } \varphi_+ = +\varphi_-$$

Note that while any choice of a classical vacuum breaks the $\mathbb{Z}_2$-symmetry, the quantum vacuum does not break this symmetry.

The semi-classical computation requires

$$1 \ll \frac{\omega^3}{\lambda} \sim \frac{V(0)}{\omega}$$

which amounts to a high barrier between the two classical minima.

So for fixed curvature $\omega^2 = 8\lambda\eta^2$, one needs a small coupling $\lambda \to 0$.

In this regime, the correction to E_0 is small but cannot be approximated perturbatively in $\frac{\omega^3}{\lambda}$:

$$e^{-\frac{\omega^3}{12\lambda}} = 1 - \frac{\omega^3}{12\lambda} + \frac{1}{24}\left(\frac{\omega^3}{\lambda}\right)^2 + \dots$$

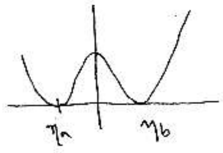
Hence, need all orders and the correction is non-perturbative!

Ib) Relation between instantons and tunneling (extracted from Bitar & Chang)

For a moment we refrain from the investigation of the ground state and focus on the computation of transition amplitudes between classical vacua but this time we stay in Minkowski space. The transition amplitude

$$\langle \eta_f | e^{-iHt_0} | \eta_i \rangle$$

describes tunneling through the barrier between η_i and η_f.



Let $\mathcal{L} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$ with $V(x)$ the double-well potential. A tunneling path from η_a to η_b can be written as the composition of a path and a reparametrization:

$$x(\lambda(t)) \qquad \text{with} \quad x(\lambda(t_a)) = x(\lambda_a) = \eta_a$$

$$\text{and} \quad x(\lambda(t_b)) = x(\lambda_b) = \eta_b$$

For such a path: $\mathcal{L} = \frac{1}{2} m(\lambda)\, \dot{\lambda}^2 - V(x(\lambda(t)))$

$$= \frac{1}{2}\left(\frac{dx}{d\lambda}\right)^2 \dot{\lambda}^2 - V(x(\lambda(t)))$$

According to quantum mechanics the tunneling amplitude is:

$$P = e^{-R} \qquad \text{with} \qquad R = \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{2\, m(\lambda)\, V(x(\lambda))} \qquad (E=0)$$

$$= \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{2\left(\frac{dx}{d\lambda}\right)^2 V(x(\lambda))} = R\left[\frac{dx}{d\lambda}, x\right]$$

The question now is what is the most probable escape path (MPEP)? We therefore look at stationary points of the functional $R[\frac{dx}{d\lambda}, x]$, i.e. $\delta R \stackrel{!}{=} 0$. By means of the Euler-Lagrange equations

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial\left(\frac{dx}{d\lambda}\right)} \sqrt{2\left(\frac{dx}{d\lambda}\right)^2 V(x(\lambda))}\right) = \frac{\partial}{\partial x}\sqrt{2\left(\frac{dx}{d\lambda}\right)^2 V(x(\lambda))}$$

$$\text{RHS} = \frac{1}{2\sqrt{2\left(\frac{dx}{d\lambda}\right)^2 V(x(\lambda))}}\, 2\left(\frac{dx}{d\lambda}\right)^2 \frac{dV}{dx}\Big|_{x(\lambda)} = \sqrt{\frac{\left(\frac{dx}{d\lambda}\right)^2}{2V(x(\lambda))}}\, \frac{dV}{dx}$$

$$\text{LHS} = \frac{\partial}{\partial\lambda}\left[\frac{1}{2\sqrt{2\left(\frac{dx}{d\lambda}\right)^2 V(x(\lambda))}}\, 4\, \frac{dx}{d\lambda}\, V(x(\lambda))\right] = \frac{\partial}{\partial\lambda}\left[\sqrt{\frac{2V(x(\lambda))}{\left(\frac{dx}{d\lambda}\right)^2}}\, \frac{dx}{d\lambda}\right].$$

Introducing a new parametrization $\tau(t) = \tau(\lambda(t))$ with $\tau(t_{a/b}) = \tau(\lambda_{a/b}) = \lambda_{a/b}$ via:

$$\frac{d\tau(\lambda)}{d\lambda} = \sqrt{\frac{\left(\frac{dx}{d\lambda}\right)^2}{2V(\lambda)}}$$

one now finds:

$$\text{RHS} = \frac{d\tau}{d\lambda} \cdot \frac{dV}{dx}$$

$$\text{LHS} = \frac{\partial}{\partial\lambda}\left[\frac{d\lambda}{d\tau}\frac{dx}{d\lambda}\right] = \frac{\partial}{\partial\lambda}\left[\frac{dx}{d\tau}\right] = \frac{d\tau}{d\lambda}\frac{d^2x}{d\tau^2}$$

Hence, $\delta R[\frac{dx}{d\lambda}, x] = 0 \iff \exists$ parametrization $\tau(t) = \tau(\lambda(t))$ such that $x(\tau)$ obeys the Euclidean equation of motion:

$$\frac{d^2x}{d\tau^2} = -\frac{d(-V)}{dx}$$

As a result, the most probable escape path can be reparametrized such that it formally becomes an instanton.

II a) Yang-Mills instantons

The starting point is classical Yang-Mills theory with a (simple) gauge group G.

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

where the field strengths are given as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \underbrace{f^{abc}}_{\text{structure constants}} A^b_\mu A^c_\nu$$

In order to parallel the instanton discussion in quantum mechanics we pick the temporal gauge

$$A^a_0 = 0$$

but <u>must not impose</u> the additional Coulomb gauge $\partial_i A^a_i = 0$. So gauge transformations that depend on $\vec{x} \in \mathbb{R}^3$ but not on t are still possible. In this gauge the Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int d^3x \left(E^a \cdot E^a + B^a \cdot B^a \right). \qquad E^a_i = G^a_{0i}, \quad B^a_i = \frac{1}{2} \varepsilon_{ijk} G^a_{jk}$$

Classical vacua (at time t) are configurations $A^a_i(t, \vec{x})$ that are absolute minima of $\mathcal{H}$. Obviously, they must be pure gauge,

$$A_i(\vec{x}) = \frac{i}{g} U(\vec{x})\, \partial_i U^{-1}(\vec{x}) \qquad A_i(\vec{x}) = A^a_i(\vec{x})\, T^a \in \mathfrak{g}$$

where the gauge function is a continuous map

$$U: \mathbb{R}^3 \longrightarrow G.$$

NB: Any Lie group is a topological space.

Among those vacua we restrict ourselves to the ones with

$$\lim_{|\vec{x}| \to \infty} U(\vec{x}) = \text{const} = \underbrace{1}_{\text{WLOG}}$$

because it turns out that only ~~between such vacua~~ instanton transitions exist, see Callan, Dashen, Gross. (The Euclidean action for transitions would otherwise be infinite and, thus, we would not learn much about the quantum vacuum)

Using the general result from topology

$$\mathbb{R}^n \cup \{\infty\} \cong S^n \leftarrow \text{n-dim sphere}$$

homeomorphic, i.e. topologically equivalent

our gauge functions can be considered as continuous maps

$$U: S^3 \longrightarrow G$$

where "the infinitely distant point ∞" is mapped to $1 \in G$. Two such functions $U_1(\vec{x})$ and $U_2(\vec{x})$ are called equivalent if one can be continuously deformed into the other while always mapping $S^3 \ni \infty \mapsto 1 \in G$, i.e. if there exists a homotopy

$$f: S^3 \times [0,1] \longrightarrow G \quad \text{continuous s.t.}$$

$$f(\vec{x}, 0) = U_1(\vec{x})$$
$$f(\vec{x}, 1) = U_2(\vec{x})$$
$$f(\infty, t) = 1$$

The set of such equivalence classes is defined as the third homotopy group of G

$$\pi_3(G)$$

(which can be endowed with a group structure...). For all simple Lie groups

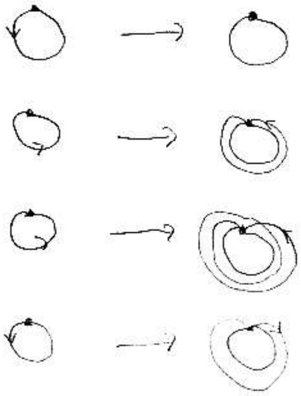
$$\pi_3(G) \cong \mathbb{Z}$$

and, thus, equivalence classes are characterized in terms of the winding number

$$\nu \in \mathbb{Z}.$$

e.g. $\pi_1(S^1) = \mathbb{Z}$:

$S^1 \to S^1$



$U(\varphi) = \text{const}$ $\quad \nu = 0$

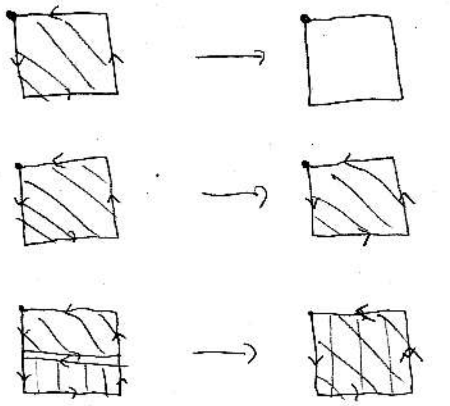
$U(\varphi) = e^{i\varphi}$ $\quad \nu = 1$

$U(\varphi) = e^{2i\varphi}$ $\quad \nu = 2$

$U(\varphi) = e^{-i\varphi}$ $\quad \nu = -1$

$\pi_2(S^2) = \mathbb{Z}$: $\quad$ S^2 will be illustrated as a square with identified boundary

$S^2 \to S^2$



$\nu = 0$

$\nu = 1$

$\nu = 2$

The "simplest" simple compact Lie group is $SU(2)$. Any element $U \in SU(2)$ can be uniquely written as

$$U = a + i\vec{b}\cdot\vec{\sigma}, \qquad a \in \mathbb{R},\ \vec{b} \in \mathbb{R}^3,\ \sigma_i \text{ Pauli matrices}$$

with

$$a^2 + \vec{b}^2 = 1.$$

Thus, topologically we have

$$SU(2) \cong S^3,$$

and it is not surprising that $\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$.

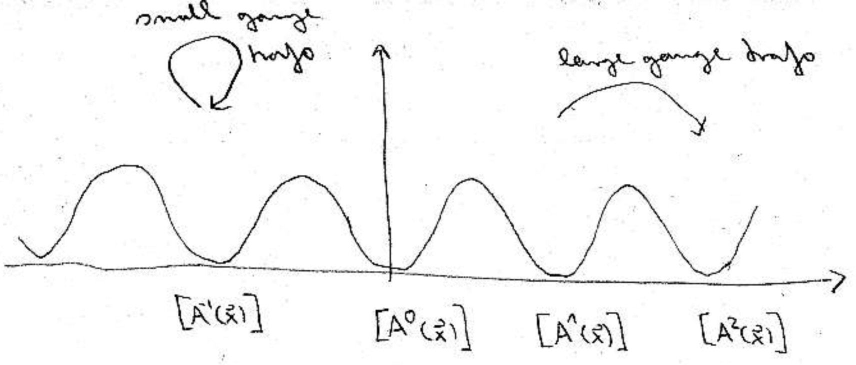
Equivalent functions $U_1(\vec{x})$ and $U_2(\vec{x})$ can (by definition) be continuously deformed into each other and the same is true for our classical vacua

$$A_{1/2}(\vec{x}) = \frac{i}{g} U(\vec{x})\, \partial_i U^{-1}(\vec{x}).$$

On the other hand, inequivalent functions and their associated vacua cannot be continuously deformed into each other. Denoting the equivalence class of winding number $\nu \in \mathbb{Z}$ as

$$[A^\nu(\vec{x})]$$

one can schematically draw a potential as



Note: All classical vacua are pure gauge and, thus, are related to each other by gauge transformations. In fact, any vacuum $A(\vec{x}) = \frac{i}{g} U(\vec{x})\, \partial_i U^{-1}(\vec{x})$ can be gauge transformed into the class with $\nu = 0$:

$$A(\vec{x}) = \frac{i}{g} U(\vec{x})\ \partial_i \left(U^{-1}(\vec{x})\right)$$

$$\longrightarrow U^{-1}(\vec{x})\, A(\vec{x})\, U(\vec{x}) + \frac{i}{g} U^{-1}(\vec{x})\, \partial_i\left(U(\vec{x})\right)$$

$$= \frac{i}{g} U^{-1} U \left(\partial_i U^{-1}\right) U + \frac{i}{g} U^{-1}\, \partial_i U$$

$$= \frac{i}{g}\, \partial_i \left(U^{-1} U\right) = 0$$

Gauge transformations that change the winding number are called <u>large gauge transformations</u>, and otherwise <u>small gauge transformations</u>.

There is a way to compute the winding number ν for a given classical vacuum: Using the <u>Chern-Simons current</u>

$$K_\mu = 2\varepsilon_{\mu\nu\alpha\beta}\left(A_\nu^a\, \partial_\alpha A_\beta^b + \frac{g}{3} f^{abc} A_\nu^a A_\alpha^b A_\beta^c\right)$$

The claim is that ν is now given as the charge of this current,

$$K = \frac{g^2}{32\pi^2} \int_{\mathbb{R}^3} K_0(x)\, d^3x$$

$$= \nu \qquad \text{for } A_i(\vec{x}) = \frac{i}{g} U^{(\nu)}(\vec{x})\, \partial_i U^{(\nu)-1}(\vec{x}),\quad A_0(\vec{x}) = 0$$

Pf: $K_0 = 2\varepsilon_{0ijk}\left(A_i^a\, \partial_j A_k^a + \frac{g}{3} f^{abc} A_i^a A_j^b A_k^c\right)$

$$= 2\varepsilon_{0ijk}\left(\tfrac{1}{2} A_i^a \left(\partial_j A_k^a - \partial_k A_j^a + g f^{abc} A_j^b A_k^c\right) - \frac{g}{6} f^{abc} A_i^a A_j^b A_k^c\right)$$

$$= \varepsilon_{0ijk} A^a_i \underbrace{F^a_{jk}}_{=0 \text{ for pure gauge configurations}} - \frac{1}{3}\varepsilon_{0ijk}\, g\, f^{abc} A^a_i A^b_j A^c_k$$

$$\Rightarrow K = -\frac{g^3}{3\cdot 32\pi^2}\varepsilon_{0ijk} f^{abc}\int A^a_i A^b_j A^c_k \, d^3x$$

$$= -\frac{g^3}{3\cdot 32\pi^2}\varepsilon_{ijk}\, \xi \int d^3x\, \underbrace{\mathrm{Tr}[A_i A_j A_k]}_{=A^a_i A^b_j A^c_k \underbrace{\mathrm{Tr}[T^a T^b T^c]}_{=\frac{1}{2}\mathrm{Tr}[(\underbrace{T^aT^b + T^bT^a}_{\text{vanishes due to symmetry under } a\leftrightarrow b} + \underbrace{[T^a, T^b]}_{=f^{abd}T^d})T^c]}}$$

$\mathrm{Tr}[T^d, T^c] \propto \delta^{dc}$

$$= -\frac{g^3}{3\cdot 32\pi^2}\frac{i^3}{g^3}\,\xi \int \underbrace{\varepsilon_{ijk}\, d^3x\, \mathrm{Tr}[U(\partial_i U^{-1}) U(\partial_j U^{-1}) U(\partial_k U^{-1})]}_{\text{invariant measure, Cartan-Maurer form}}$$

$\propto n$ see Weinberg II, 23.4

Having found countably many classical vacua all separated by an energy barrier we would like to apply the instanton methods as in I). In particular, we are interested in transition amplitudes in Euclidean time

$$\langle \nu_f | e^{-H\tau_0} | \nu_i \rangle = N \int \mathcal{D}A_i(\vec{x},t)\, e^{-S_E}$$

where the path integral (mainly) runs over all continuous configurations with

$$A_i(\vec{x}, -\tau_0/2) = A_i^{(\nu_i)}(\vec{x}) \qquad A_0 = 0$$
$$A_i(\vec{x}, \tau_0/2) = A_i^{(\nu_f)}(\vec{x}).$$

Among those, the least action configurations with finite action are the <u>Yang-Mills or BPST n-instantons</u> where (for $\nu_f \neq \nu_i$)

$$n = \nu_f - \nu_i .$$

However, such stationary points of the Euclidean action are usually not computed in temporal gauge but rather without any gauge fixing.

The transition from Yang-Mills theory without gauge fixing in Minkowski spacetime to Euclidean spacetime requires some care. One possible way is

$$\hat{x}_i = x^i, \qquad x_0 = -i\hat{x}_4$$
$$A^m = -\hat{A}_m, \quad A_0 = i\hat{A}_4$$

for which $\hat{G}^a_{4n}$ becomes purely imaginary and, hence, the Euclidean action is real.

$$S_E = \int d^4\hat{x}\, \tfrac{1}{4}\, \hat{G}^a_{\mu\nu}\, \hat{G}^a_{\mu\nu}$$

where again

$$\hat{G}^a_{\mu\nu} = \frac{\partial}{\partial \hat{x}_\mu} \hat{A}^a_\nu - \frac{\partial}{\partial \hat{x}_\nu} \hat{A}^a_\mu + g\, f^{abc}\, \hat{A}^b_\mu\, \hat{A}^c_\nu \qquad (\mu,\nu = 1,\dots,4).$$

The question is: When is the Euclidean action S_E finite?

Obviously, $G^a_{\mu\nu}$ must decrease more rapidly than $\frac{1}{|x|^2}$, and so the action is finite for continuous fields A^a_μ vanishing faster than $\frac{1}{|x|}$. However, there is another possibility: A^a_μ may decrease with $\frac{1}{|x|}$ as long as it approaches pure gauge

$$A_\mu = T^a A^a_\mu \xrightarrow{|x|\to\infty} \frac{i}{g} S\, \partial_\mu S^{-1} \qquad \left(= \mathcal{O}\left(\tfrac{1}{|x|}\right)\right)$$

where <u>S depends on the directions/angles</u> and is an element of the gauge group G. Under a gauge transformation $U(x)$ (smooth on $\mathbb{R}^4$),

$$A_\mu \to U^{-1}(x)\, A_\mu\, U(x) + \frac{i}{g} U^{-1}(x)\, \partial_\mu U(x),$$

S on the remote sphere is replaced by $U^{-1}(|x|\to\infty)\, S$. In general, there is no smooth $U: \mathbb{R}^4 \to G$ such that $U^{-1}(|x|\to\infty) = S^{-1}$. However, at least [illegible], one can always arrange that for a given direction $\lim_{|x|\to\infty} U(|x|\hat{x}) = S$ so that S in this direction can without loss of generality be chosen to be $1 \in G$.

Hence, finite action solutions can be characterized in terms of continuous maps

$$\underset{\substack{\uparrow \\ \text{remote 3-sphere}}}{S:\ S^3} \longrightarrow G \qquad \text{with } S(\hat{x}) = 1 \text{ for a fixed unit vector } \hat{x}.$$

that are again classified according to the third homotopy group of G,

$$\pi_3(G) \cong \mathbb{Z}$$

i.e. in terms of a winding number $n \in \mathbb{Z}$.

e.g. $G = SU(2)$

$$S^{(1)} = \lim_{|x|\to\infty} \frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{|x|} \qquad n = 1$$

$$S^{(n)} = \lim_{|x|\to\infty} \left(\frac{x_4 + i\vec{x}\cdot\vec{\sigma}}{|x|}\right)^n \qquad n = n$$

As before, there is a gauge-independent way to compute the topological charge n:

$$n = \frac{g^2}{32\pi^2}\int_{\mathbb{R}^4} d^4x\, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \qquad \text{where } \tilde{G}^a_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\gamma\delta}G^a_{\gamma\delta}, \quad \varepsilon_{1234} = 1$$

$$= \frac{g^2}{32\pi^2}\int_{\mathbb{R}^4} d^4x\, \partial_\mu K_\mu \qquad K_\mu = 2\varepsilon_{\mu\nu\gamma\delta}\left(A^a_\nu\partial_\gamma A^a_\delta + \tfrac{1}{3}g f^{abc}A^a_\nu A^b_\gamma A^c_\delta\right)$$

or $\displaystyle\int_{\substack{S^3 \text{ with}\\ |x|\to\infty}} d\sigma_\mu\, K_\mu$ using $A_\mu \to \frac{i}{g}S\partial_\mu S^{-1}$.

Note $\tilde{G}^a_{\mu\nu}$ is the dual field strength of $G^a_{\mu\nu}$.

A sufficient condition for stationary finite action configurations (instantons) can be given in terms of (anti)-selfdual field strength as follows:

$$0 \le S_E = \int d^4x\, \tfrac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$$

$$= \int d^4x\left[\tfrac{1}{4}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} + \tfrac{1}{8}\left(G^a_{\mu\nu} - \tilde{G}^a_{\mu\nu}\right)^2\right] \qquad \text{because } (G^a_{\mu\nu})^2 = (\tilde{G}^a_{\mu\nu})^2$$

$$= \frac{8\pi^2}{g^2}n + \tfrac{1}{8}\int d^4x\left(G^a_{\mu\nu} - \tilde{G}^a_{\mu\nu}\right)^2 \qquad (*)$$

Here n is the winding number associated to the finite action configuration A^a_μ.

For $n \geq 0$ the action is minimised if

$$\boxed{G^a_{\mu\nu} = \tilde{G}^a_{\mu\nu}},$$

i.e. if the field strength $G^a_{\mu\nu}$ is selfdual.

For $n<0$, the second term in (*) cannot be set to zero because then $0 \leq n < 0$. $\lightning$

However, in this case we use that the action S_E is invariant under space reflections

$$x_{1,2,3} \to -x_{1,2,3} .$$

On the other hand field strengths transform as:

$$G_{ij} \to G_{ij}, \qquad G_{i4} \to -G_{i4},$$

$$\tilde{G}_{ij} = \tfrac{1}{2}\varepsilon_{ij\rho\sigma} G_{\rho\sigma} \to -\tilde{G}_{ij}, \qquad \tilde{G}_{i4} = \tfrac{1}{2}\varepsilon_{i4\rho\sigma} G_{\rho\sigma} \to \tilde{G}_{i4},$$

$$\Rightarrow G_{\mu\nu}\tilde{G}_{\mu\nu} \to -G_{\mu\nu}\tilde{G}_{\mu\nu}, \qquad \Rightarrow \text{winding number flips!}$$

$$G_{ij} - \tilde{G}_{ij} \to G_{ij} + \tilde{G}_{ij},$$

$$G_{4i} - \tilde{G}_{4i} \to -(G_{4i} + \tilde{G}_{4i})$$

Hence, for $n<0$,

$$0 \leq -\frac{8\pi^2}{g^2} n + \frac{1}{8}\int d^4x \left(G^a_{\mu\nu} + \tilde{G}^a_{\mu\nu}\right)^2$$

and minima are realized for antiselfdual field strengths $G^a_{\mu\nu}$, i.e.

$$\boxed{G^a_{\mu\nu} = -\tilde{G}^a_{\mu\nu}} .$$

As a result, instanton actions are given in terms of their winding number n:

$$\boxed{S_E = \frac{8\pi^2}{g^2} n}$$

It can be checked that (anti)selfdual field strength automatically imply the equation of motion (which is 2nd order in derivatives) by means of the Bianchi identity:

$$D_\mu G^a_{\mu\nu} = D_\mu \tilde{G}^a_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} D_\mu G^a_{\rho\sigma}$$

$$= \frac{1}{6}\varepsilon_{\mu\nu\rho\sigma}\underbrace{\left(D_\mu G^a_{\rho\sigma} + D_\rho G^a_{\sigma\mu} + D_\sigma G^a_{\mu\rho}\right)}_{=0 \text{ (Bianchi identity)}} = 0$$

Rem: 1) Recall that in the quantum mechanic example before we also did not need to solve the Euclidean equation of motion because we used conservation of energy $0 = E = \frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)$ (also a first order differential eq.)

2) Usually, in mathematics, instantons are defined as (anti) selfdual connections in G-principal bundles over a 4-dim Riemannian manifold.

3) It can be shown that instantons with winding number n, when expressed in temporal gauge, are the instanton configurations* interpolating an initial classical vacuum $A_i(\vec{x}, \tau=-\infty)^{(\nu)}$ with winding number ν into a final vacuum $A_i(\vec{x}, \tau=\infty)^{(\nu+n)}$ of winding number $\nu+n$, see e.g. "ABC of instantons". In retrospective, this justifies the restriction to only those classical vacua that are constant on the infinitely remote 3-sphere (cf. p. 10).

<u>Example</u> ($n=1$ instanton for $G = SU(2)$)

$$A^a_\mu = \frac{2}{g}\eta_{a\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2}$$

$$G^a_{\mu\nu} = -\frac{4}{g}\eta_{a\mu\nu}\frac{\rho^2}{[(x-x_0)^2+\rho^2]^2} \qquad \text{(selfdual)}$$

Here $\eta_{a\mu\nu} \in \mathbb{R}$ are the so-called 't Hooft symbols (defined by mapping the Lie algebra generators of $so(3,1)$ to $\underbrace{su(2) \oplus su(2)}_{\text{the ones of}}$). Moreover, x_0 and ρ are arbitrary collective coordinates. Note that in the limit $|x| \to \infty$ A^a_μ vanishes $\frac{1}{|x|}$ in fact as $\frac{1}{|x|}$.

II b) The emergence of the Θ-angle

As in Weinberg II, 236 we want to demonstrate that the so-called Θ-term

$$\Theta\, \varepsilon^{\mu\nu\varsigma\sigma}\, G^a_{\mu\nu}\, G^a_{\varsigma\sigma}$$

can (at least qualitatively) be thought of as induced by instantons.

The expectation value of a local observable $\mathcal{O}$ located in a large Euclidean spacetime volume Ω is

$$\langle \mathcal{O} \rangle_\Omega = \frac{\sum_\nu f(\nu) \int_\nu D\phi\, e^{-S_E^\Omega(\phi)}\, \mathcal{O}(\phi)}{\sum_\nu f(\nu) \int_\nu D\phi\, e^{-S_E^\Omega(\phi)}}$$

where ϕ are all fields of the theory to which a winding number ν can be assigned, $S_E^\Omega(\phi)$ is the integral of the Lagrangian over the spacetime volume Ω, and $f(\nu)$ is a weighting function to be determined. So this is a general ansatz.

Let Ω now be divided into very large volumes Ω_1 and Ω_2, and $\mathcal{O}$ shall be located in the volume Ω_1. Fields with winding number ν can then be separated into fields in Ω_1 with ν_1 and fields in Ω_2 with ν_2, s.t. $\nu_1 + \nu_2 = \nu$. Hence,

$$\langle \mathcal{O} \rangle_\Omega \simeq \frac{\sum_{\nu_1,\nu_2} f(\nu_1+\nu_2) \int_{\nu_1} D\phi\, e^{-S_E^{\Omega_1}}\, \mathcal{O} \int_{\nu_2} D\phi\, e^{-S_E^{\Omega_2}}}{\sum_{\nu_1,\nu_2} f(\nu_1+\nu_2) \int_{\nu_1} D\phi\, e^{-S_E^{\Omega_1}} \int_{\nu_2} D\phi\, e^{-S_E^{\Omega_2}}}$$

It is by means of the cluster decomposition principle

$$\langle 0 | T\, \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) | 0 \rangle \xrightarrow{|x_1 - x_2| \to \infty} \langle \mathcal{O}_1(x_1) \rangle \langle \mathcal{O}_2(x_2) \rangle$$

that in $\langle \mathcal{O} \rangle_\Omega$ the distant Ω_2 dependence drops out which is only possible if

$$f(\nu_1 + \nu_2) = f(\nu_1)\, f(\nu_2)$$

and one therefore finds

$$f(\nu) = \exp i\Theta\nu$$

for $\Theta \in \mathbb{R}$ (it could also be complex, couldn't it?).

Representing the winding number as

$$\nu = \alpha \int d^4x \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} ,$$

as before, one finds

$$\begin{aligned} f(\nu)\, e^{-S_E^{\Omega}} &= e^{i\Theta\nu} \, e^{-S_E^{\Omega}} \\ &= e^{i\Theta\xi \int d^4x \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}} \, e^{-S_E^{\Omega}} \\ &= e^{-S_E^{\Omega} + i\Theta\xi \int d^4x \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}} \end{aligned}$$

and, hence, the weighting function is equivalent to adding what is called the Θ-term in the literature. Such a term breaks the discrete spacetime symmetries P and T (the effect of P has already been used before on p. 17, T is similar.)

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