The fate of the false vacuum

29.11.2011

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- Coleman Phys. Rev. D15 (1977) 2929–2936 (1)
- Callan, Coleman Phys. Rev. D16 (1977) 1762–1768 (2)
- Coleman, De Luccia Phy. Rev. D21 (1980) 3305 (3)

Introduction

Scalar field ϕ with action

$$S = \int d^4x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \right]$$

potential of the form:

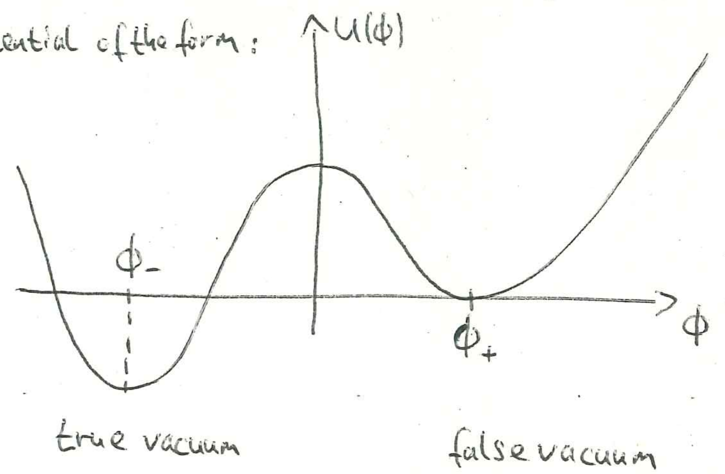


fig. 1

false vacuum is unstable by quantum effects $\Longrightarrow$ decays through barrier penetration

compute probability of decay per unit time per unit volume

$$\Gamma/V = A\, e^{-B/\hbar} \left[1 + O(\hbar) \right]$$

first compute B in (1) then A in (2)

Set-up

$\phi = \phi_+$ false vacuum

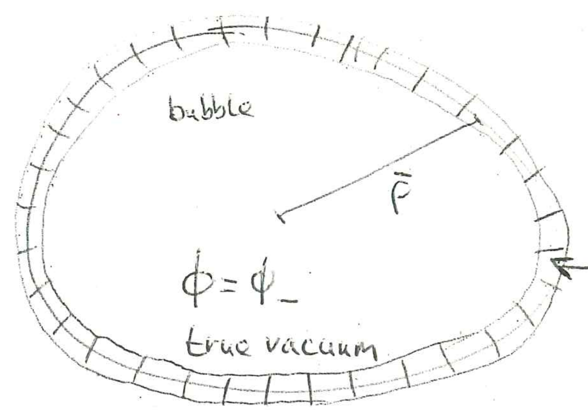
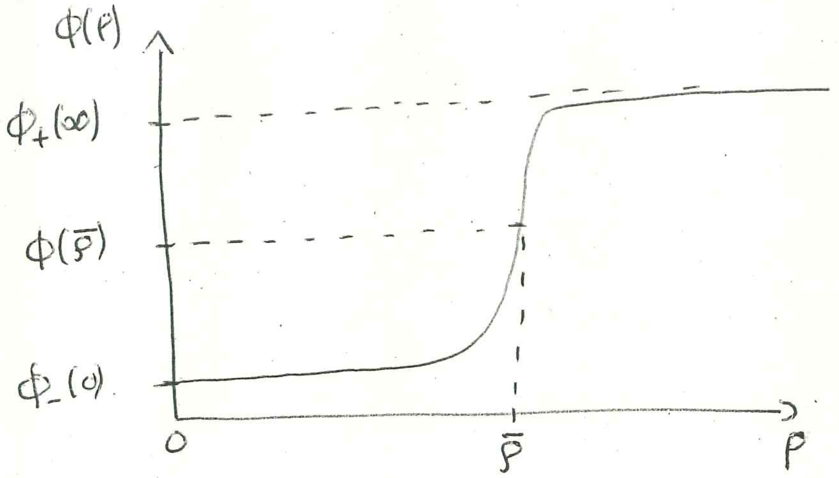


fig. 2



$\phi(x)$ is $O(4)$ symmetric
$\phi = \phi(\rho)$ with $\rho^2 = -t^2 + \vec{x}^2$

fig. 3

probability that such a bubble emerges given by Euclidean action

$$S_E = \int d^4x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + U(\phi) \right] \qquad \eta^{\mu\nu} = \mathrm{diag}(+1,+1,+1,+1)$$

physical intuition now $-U(\phi)$

$$= 2\pi^2 \int_0^\infty \rho^3 \, d\rho \left[\tfrac{1}{2}(\phi')^2 + U(\rho) \right] \qquad \phi' = \frac{\partial \phi}{\partial \rho}$$

equation of motion

$$\phi'' + \frac{3}{\rho}\phi' = \frac{\partial U}{\partial \phi} \qquad \text{eq. (1)}$$

<u>remark:</u>

compare to e.o.m for uniform scalar field in FRW

$$\phi'' + 3H\phi' + \frac{\partial V}{\partial \phi} = 0$$

where Hubble parameter

$$H^2 = \frac{1}{3}\left(\tfrac{1}{2}(\phi')^2 + V(\phi) \right) - \frac{k}{a^2}$$

with $k = -1$ for open universe

and $a \sim t$ scale factor of FRW metric

$\Rightarrow$ for small times $H \sim \frac{1}{t}$

$$(2) \quad \phi'' + \frac{3}{t}\phi' + \frac{\partial V}{\partial \phi} = 0$$

eq. (1)

neglect term $\frac{3}{\rho}\phi'$ in e.o.m. because

- away from the wall $\phi' \approx 0$
- in the wall $\bar{\rho}$ is large

compute B:

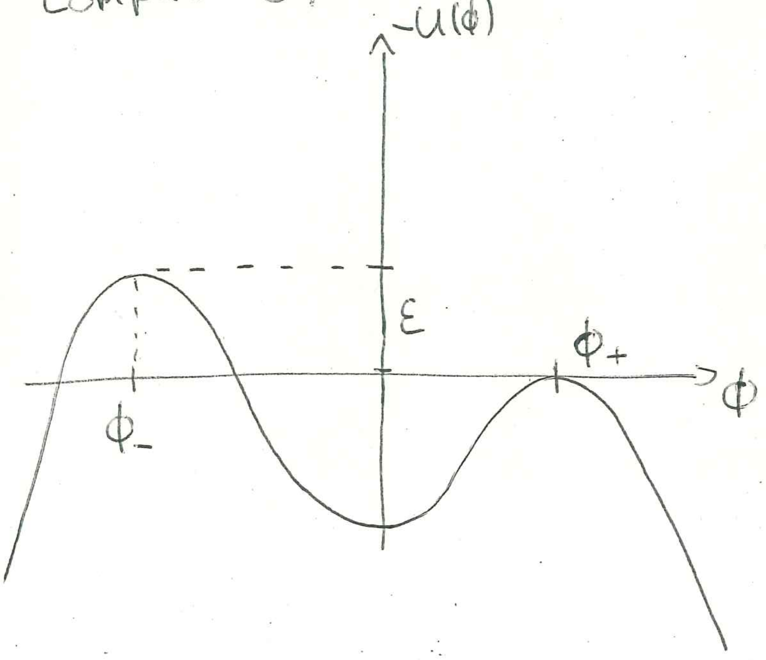


fig. 4

$\varepsilon = U(\phi_+) - U(\phi_-)$ small $\Rightarrow$ $U(\phi) = U_0(\phi) + O(\varepsilon)$

with $U_0(\phi_-) = U_0(\phi_+)$

and $\left.\frac{\partial U_0}{\partial \phi}\right|_{\phi_\pm} = 0$

$\Rightarrow$ approximate e.o.m.

$$\phi'' = \frac{\partial U_0}{\partial \phi} \qquad | \cdot \phi' \text{ and integrate}$$

$$\frac{1}{2}\frac{\partial}{\partial \rho}(\phi')^2 = \frac{\partial U_0}{\partial \phi}\frac{\partial \phi}{\partial \rho} = \frac{\partial U_0}{\partial \rho}$$

$$\Rightarrow \frac{\partial}{\partial \rho}\underbrace{\left[\frac{1}{2}(\phi')^2 - U_0\right]}_{= \text{const}} = 0$$

fix constant at $\phi = \phi(\infty) = \phi_+$

$$\frac{1}{2}(\phi')^2 - U_0 = -U_0(\phi_+)$$

$$\Rightarrow \phi' = \sqrt{2\left(U_0(\phi) - U_0(\phi_+)\right)} \geq 0 \qquad \text{eq. (2)}$$

ϕ goes monotonically from ϕ_- to ϕ_+ as ρ increases from 0 to ∞

determine $\phi(\rho)$

$$d\phi = \sqrt{2\left(U_0(\phi) - U_0(\phi_+)\right)}\, d\rho$$

$$\int_{\phi(\bar{\rho})}^{\phi(\rho)} d\tilde{\phi} \left[2\left(U_0(\tilde{\phi}) - U_0(\phi_+)\right)\right]^{-1/2} = \int_{\bar{\rho}}^{\rho} d\tilde{\rho} = \rho - \bar{\rho}$$

$\bar{\rho}$ integration constant defined by $\phi(\bar{\rho}) = \dfrac{\phi_+ + \phi_-}{2}$

example

$$U_0 = \frac{1}{8}\lambda\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2$$

$$\Rightarrow \phi = \frac{\mu}{\sqrt{\lambda}} \tanh\left[\frac{1}{2}\mu(\rho - \bar{\rho})\right] \qquad \text{as in fig. 3}$$

now we can evaluate S_E and determine $\bar{\rho}$ by demanding that S_E is stationary under variations of $\bar{\rho}$

$$B = S_E(\phi) - S_E(\phi_+)$$

for S_E integrate $\int_0^\infty d\rho$

split into three regions:

1) outside the wall $\quad \phi = \phi_+$
2) inside the bubble $\quad \phi = \phi_-$
3) at the wall

1) $\underline{\phi = \phi_+}$:

$$B_{outside} = 0$$

2) $\underline{\phi = \phi_-}$:

$\phi' \approx 0$ inside

$$S_E^{inside} = 2\pi^2 \int_0^{\bar{\rho}} \rho^3 d\rho \underbrace{U(\phi_-)}_{= U(\phi_+) - \varepsilon} = 2\pi^2 \frac{1}{4} \bar{\rho}^4 (U(\phi_+) - \varepsilon)$$

and

$$S_E^{inside}(\phi_+) = 2\pi^2 \int_0^{\bar{\rho}} \rho^3 d\rho \, U(\phi_+)$$

$$\Rightarrow B_{inside} = -\frac{\pi^2}{2} \bar{\rho}^4 \varepsilon$$

3) $\underline{\text{at the wall}}$:

$$B_{wall} = S_E(\phi) - S_E(\phi_+) = 2\pi^2 \int_{\bar{\rho}-\delta\bar{\rho}}^{\bar{\rho}+\delta\bar{\rho}} \rho^3 d\rho \left(\underbrace{\tfrac{1}{2}(\phi')^2}_{\substack{= U_0(\phi) - U_0(\phi_+) \\ \text{from eq. (2)}}} + U_0(\phi) - U_0(\phi_+) \right)$$

$$= 2\pi^2 \int_{\bar{\rho}-\delta\bar{\rho}}^{\bar{\rho}+\delta\bar{\rho}} \rho^3 d\rho \; 2 \left(U_0(\phi) - U_0(\phi_+) \right)$$

$$\approx 2\pi^2 \bar{\rho}^3 \int_{\phi_-}^{\phi_+} d\phi \underbrace{\frac{\partial \rho}{\partial \phi}}_{= \left(2(U_0(\phi) - U_0(\phi_+))\right)^{-1/2}} 2 \left(U_0(\phi) - U_0(\phi_+) \right)$$

$$= 2\pi^2 \bar{\rho}^3 \underbrace{\int_{\phi_-}^{\phi_+} d\phi \left(2 (U_0(\phi) - U_0(\phi_+)) \right)^{1/2}}_{\equiv S_1}$$

$$= 2\pi^2 \bar{\rho}^3 S_1$$

$$\Rightarrow \quad B = B_{outside} + B_{wall} + B_{inside}$$

$$= -\frac{\pi^2}{2}\bar{\rho}^4 \varepsilon + 2\pi^2 \bar{\rho}^3 S_1$$

$$\cancel{\pi^2 \bar{\rho}^3 \left(-\frac{1}{2}\bar{\rho}\varepsilon + 2 S_1\right)}$$

$$\frac{\partial B}{\partial \bar{\rho}} = -2\pi^2 \bar{\rho}^3 \varepsilon + 6\pi^2 \bar{\rho}^2 S_1$$

$$= 2\pi^2 \bar{\rho}^2 \left(-\bar{\rho}\varepsilon + 3 S_1\right) \overset{!}{=} 0$$

$$\boxed{\bar{\rho} = \frac{3 S_1}{\varepsilon}} \qquad (\bar{\rho} \text{ large if } \varepsilon \text{ small})$$

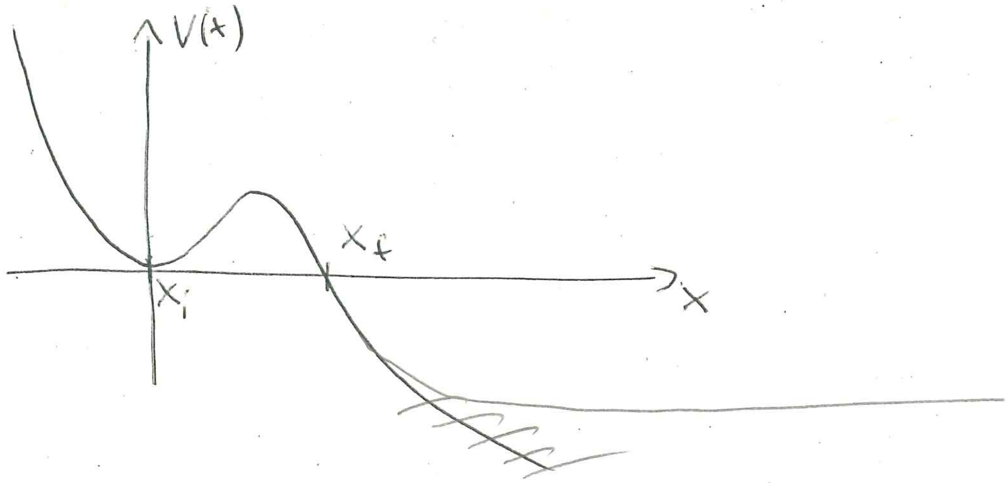
insert into B:

$$B = -\frac{\pi^2}{2}\frac{81 S_1^4}{\varepsilon^4}\varepsilon + 2\pi^2 \frac{27 S_1^3}{\varepsilon^3} S_1$$

$$= \frac{\pi^2 S_1^4}{2\varepsilon^3}(-81 + 108) = \frac{27\pi^2 S_1^4}{2\varepsilon^3}$$

Some intuition for A and B

consider quantum mechanics of particle in potential



(one spatial dimension)

Euclidean version of Feynman's sum over histories

$$\langle x_f | e^{-HT/\hbar} | x_i \rangle = \mathcal{N} \int [dx]\, e^{-S/\hbar}$$

normalization

$$S = \int_{-T/2}^{T/2} dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V \right]$$

$$x\left(-\frac{T}{2}\right) = x_i \quad \text{and} \quad x\left(\frac{T}{2}\right) = x_f$$

Denote by $\bar{x}(t)$ any function obeying the boundary conditions

$$\Rightarrow x(t) = \bar{x}(t) + \sum_n c_n x_n(t)$$

$x_n(t)$ complete set of real orthonormal functions

$$\int_{-T/2}^{T/2} dt\, x_n(t)\, x_m(t) = \delta_{mn}$$

$$x_n(\pm T/2) = 0$$

$$\Rightarrow [dx] = \prod_n (2\pi\hbar)^{-1/2} dc_n$$

then we can evaluate $\int [dx] e^{-S/\hbar}$ in the small $\hbar$ limit ;

we know

$$\frac{\delta S}{\delta \bar{x}} = -\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0$$

eq.(3)
first variation vanishes
$\Rightarrow$ e.o.m.

$$\frac{\delta^2 S}{\delta \bar{x}^2} \Rightarrow -\frac{d^2 x_n}{dt^2} + V''(\bar{x})\, x_n = \lambda_n x_n$$

choose x_n to be eigen fcts of second variation

$$S = \int dt \left[\frac{1}{2}\left(\frac{d}{dt}\left(\bar{x}(t) + \sum_n c_n x_n(t)\right)\right)^2 + V\left(\bar{x}(t) + \sum_n c_n x_n(t)\right)\right]$$

$$= \int dt \left[\frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 + \frac{d\bar{x}}{dt}\sum_n c_n \frac{dx_n}{dt} + \frac{1}{2}\sum_{n,m} c_n c_m \frac{dx_n}{dt}\frac{dx_m}{dt}\right.$$

$$\left. + V(\bar{x}) + \sum_n c_n x_n(t) V'(\bar{x}) + \frac{1}{2}\sum_{n,m} c_n c_m x_n(t) x_m(t) V''(\bar{x}) \right]$$

$$= S(\bar{x}) + \int dt \left[-\frac{d^2\bar{x}}{dt^2}\sum_n c_n x_n - \frac{1}{2}\sum_{n,m} c_n c_m x_n \frac{d^2 x_m}{dt^2}\right.$$

$$\left. + \sum_n c_n x_n V'(\bar{x}) + \frac{1}{2}\sum_{n,m} c_n c_m x_n x_m V''(\bar{x})\right]$$

$$= S(\bar{x}) + \int dt \Bigg[\underbrace{\sum_n c_n x_n \left(-\frac{d^2\bar{x}}{dt^2} + V'(\bar{x})\right)}_{=0 \text{ e.o.m.}}$$

$$+ \frac{1}{2}\sum_{n,m} c_n c_m x_n \underbrace{\left(-\frac{d^2 x_m}{dt^2} + x_m V''(\bar{x})\right)}_{\lambda_m x_m}\Bigg]$$

$$= S(\bar{x}) + \frac{1}{2}\int dt \sum_{n,m} c_n c_m \lambda_m x_n(t) x_m(t) = S(\bar{x}) + \frac{1}{2}\sum_n \lambda_n c_n^2$$

$$\boxed{\int dt \cdot x_n(t) x_m(t) = \delta_{mn}}$$

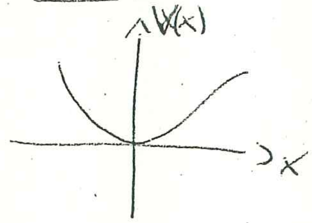
$$\Rightarrow \int [dx]\, e^{-S/\hbar} = e^{-S(\bar{x})/\hbar} \int (2\pi\hbar)^{-N/2} \prod_n dc_n\, e^{-\frac{1}{2\hbar}\sum_{n,m} c_n c_m \lambda_m x_n x_m} + \ldots$$

$$= e^{-S(\bar{x})/\hbar} \prod_n \lambda_n^{-1/2} + \ldots$$

$$= e^{-S(\bar{x})/\hbar}\left[\det\left(-\partial_t^2 + V''(\bar{x})\right)\right]^{-1/2} + \ldots$$

one eigenvalue λ_n must be negative $\Rightarrow$ complex $\Rightarrow$ decay

from eq. (3) $E = \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x})$ constant of motion example



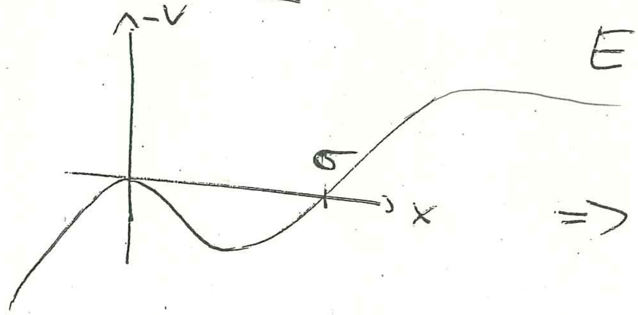
$x_i = x_f = 0$

$\bar{x}(t) = 0$

$V''(0) = \omega^2$

$$\Rightarrow N\left[\det(-\partial_t^2 + \omega^2)\right]^{-1/2} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}$$

"the bounce"



$$E = 0 \quad \Leftrightarrow \quad \frac{1}{2}\left(\frac{d\bar{x}}{dt}\right)^2 = V(\bar{x})$$

$$\Rightarrow B \equiv S(\bar{x}) = \int_{-\infty}^{\infty} dt \left(\frac{d\bar{x}}{dt}\right)^2 = \int_0^\sigma dx \left[2V(x)\right]^{1/2}$$

several bounces

- $S = nB$
- each bounce from det $\Rightarrow \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \; K^n$
  (bounces separated, potential like $V''(0) = \omega^2$)
  fix later
- $\det[\ldots]$ eigen fct x_1 with $\lambda_1 = 0$

$$x_1 = B^{-1/2} \frac{d\bar{x}}{dt} \qquad x_1(\pm T/2) = 0$$

$$\text{and} \quad -\frac{d^2 x_1}{dt^2} + V''(\bar{x})\, x_1 = 0$$

$\Rightarrow$ do not integrate over c_1

$$\ldots \Rightarrow (2\pi\hbar)^{-1/2} dc_1 = (B/2\pi\hbar)^{1/2} dt$$

$$\int dc_1 \longrightarrow \int dt_1 \ldots \int dt_n = \frac{T^n}{n!} \quad \text{for } n \text{ bounces}$$

Sum:

$$\langle x_f | x_i \rangle = \sum_{n=0}^{\infty} \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \frac{\left(K e^{-B/\hbar} T\right)^n}{n!}$$

$$= \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\omega T/2 + K e^{-B/\hbar} T\right)$$

$\Rightarrow$ ground state energy

$$E_0 = \left(\frac{\hbar\omega}{2} - \hbar K e^{-B/\hbar}\right)\left(1 + O(\hbar)\right) \qquad \text{eq. (4)}$$

determine K using one bounce solution:

$$\underbrace{\left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}}_{= N\left[\det(-\partial_t^2 + \omega^2)\right]^{-1/2}} \mathrm{Im}(K)\, e^{-B/\hbar}\, T = \underbrace{\mathrm{Im}\left(N \int [dx]\, e^{-S/\hbar}\right)_{\text{one bounce}}}_{= \frac{1}{2} N e^{-B/\hbar} (B/2\pi\hbar)^{1/2} T \cdot \left|\det'[-\partial_t^2 + V''(\bar{x})]\right|^{-1/2}}$$

half a Gaussian; problem in path integral

$$\Rightarrow \mathrm{Im}\, K = \frac{1}{2}(B/2\pi\hbar)^{1/2} \left|\frac{\det'(-\partial_t^2 + V''(\bar{x}))}{\det(-\partial_t^2 + \omega^2)}\right|^{-1/2}$$

$$\Rightarrow \Gamma = -2\, \mathrm{Im}\, E_0/\hbar \qquad \text{using (4)}$$

$$= (B/2\pi\hbar)^{1/2} e^{-B/\hbar} \left|\frac{\det'[-\partial_t^2 + V''(\bar{x})]}{\det[-\partial_t^2 + \omega^2)}\right|^{-1/2}$$

$$\Rightarrow \boxed{\Gamma/V = \frac{B^2}{4\pi^2\hbar^2} e^{-B/\hbar} \left|\frac{\det'[-\partial^2 + U''(\phi)]}{\det[-\partial^2 + U''(\phi_+)]}\right|^{-1/2}}$$