Slow roll inflation

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1 What is inflation?

Inflation is a period of accelerated expansion of the universe. Historically, it was invented to solve several problems:

- Homogeneity: a causal patch at the time of recombination (i.e. at about 3×10^5 years after the big bang) subtends an angle of about 1° on the sky today. How come the universe is so homogeneous and isotropic?
- Relicts: Where are the relicts of phase transitions? Monopoles, domain walls, strings, etc? (if you believe in GUTs)
- ...

Inflation's main selling point is the generation of fluctuations (us!).

2 Equations of motions for a homogeneous scalar field in an FRW metric

We use units $8\pi G = M_p^{-2} = 1 = \hbar = c$. Take the Einstein Hilbert action plus a scalar scalar field

$$S = \int d^4x \sqrt{|g|} \left(\frac{1}{2} R + \frac{1}{2} \dot{\phi}^2 - V(\phi) \right) , \quad (1)$$

where $|g| = |\det g_{\mu\nu}|$ and $R = R^\mu_\mu(g_{\mu\nu})$ the Ricci scalar. Vary the action with respect to the metric

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad (2)$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}(\phi) \, , \quad (3)$$

where the lhs depends only on the metric and the rhs only on the scalar field.

Now let's plug in the homogeneous and isotropic FRW metric

$$ds^2 = dx^\mu dx^\nu g_{\mu\nu} \quad (4)$$

$$= dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right) , \quad (5)$$

where k determines the curvature

$$k = \begin{cases} > 0, & \text{closed universe} \\ < 0, & \text{open universe} \\ = 0, & \text{(spatially) flat} \end{cases} . \quad (6)$$

Let's focus on the flat case (which is strongly preferred by observations anyway). Letting Maple/Mathematica/Maxima perform all the tensor calculus we get

$$H^2 = \frac{1}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) - \frac{k}{a^2} \, , \quad (7)$$

$$\dot{H} = -\frac{1}{2} \dot{\phi}^2 + \frac{k}{a^2} \, , \quad (8)$$

where $H \equiv \frac{\dot{a}}{a}$ and the first equation is the Friedman equation.

The equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V(\phi) = 0 \, , \quad (9)$$

where the $\dot{\phi}$ term is like a friction term in classical mechanics. We have three equations, but only two of them are independent.

If we want to solve them numerically, use the Friedman equation

- for consistency of initial conditions
- as consistency check of the integration routine

and evolve the other 2 dynamically.

Another useful form of the FRW metric is

$$ds^2 = a(\tau)^2(d\tau^2 - d\vec{x}^2)\,, \tag{10}$$

written in conformal time

$$d\tau = \frac{1}{a}dt\,, \tag{11}$$

sometimes also called η, but don't confuse this with the slow roll parameter η. τ on the other hand is also used to denote the optical depth to the surface of last scattering...

A photon travels a coordinate distance τ during conformal time τ. Its geodesic is determined by (assuming purely radial motion)

$$ds^2 = 0 \Rightarrow dt^2 = a(t)^2 dr^2 \Rightarrow dr = \frac{1}{a}dt \tag{12}$$

$$\Rightarrow r = \int dt\,\frac{1}{a} = \tau\,. \tag{13}$$

3 Accelerated Expansion

Now that we have all basic definitions and the equations of motion, we can rigorously define what we mean by inflation/ accelerated expansion:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2\big(1 + \underbrace{\frac{\dot{H}}{H^2}}_{\equiv -\epsilon_H}\big) = H^2\,(1-\epsilon_H)\;. \tag{14}$$

The subscript H is for Hubble. ϵ is one of (many) slow roll parameters. There is an alternative definition ϵ_V in terms of the potential (see below) which can be a source of confusion. We shall drop the subscript H and will refer to all slow roll parameters defined in terms of H unless noted otherwise. Accelerated expansion refers to

$$\frac{\ddot{a}}{a} > 0 \quad \Rightarrow 0 < \epsilon_H < 1\,. \tag{15}$$

A useful quantity is the number of $efolds$ defined as

$$dN = -H dt\,. \tag{16}$$

It is counted backwards in time from the end of inflation. In other words, $N=0$ is at the end of inflation, and $N=60$ is before the end of inflation.

4 Inflation and slow roll parameters

How do we get inflation from a scalar field? This can be seen – independently of a particular model – in the following way. Take equation (9) for a very flat potential. This should mean that we can neglect the acceleration $\ddot{\phi}$. For if the field ϕ starts off with a huge acceleration $\ddot{\phi} \gg 1$, the friction term will take care of it.

$$\dot{\phi} = -\frac{1}{3H}\partial_\phi V \approx 0\,. \tag{17}$$

Then from the Friedman equation (7) we see that

$$H^2 = \frac{1}{3}V \approx \text{const} \tag{18}$$

$$\Rightarrow \epsilon_H \approx 0\,. \tag{19}$$

For the scale factor this means that $a = a_0 e^{H(t-t_0)}$ which explains why inflation is sometimes also referred to as exponential expansion.

Most inflationary models function like outlined above. Introduce the slow roll parameter

$$\eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{1}{2}\frac{\ddot{H}}{\dot{H}H}\,. \tag{20}$$

So the requirement that we can neglect the term $\ddot{\phi}$ compared to $3H\dot{\phi}$ is just the requirement that $\eta \ll 1$. (But note possible exceptions!)

In terms of the potential, ϵ_V and η_V are defined as

$$\epsilon_V \equiv \frac{1}{2}\left(\frac{\partial_\phi V}{V}\right)^2 , \quad (21)$$

$$\eta_V \equiv \frac{\partial_\phi^2 V}{V} . \quad (22)$$

They equal the Hubble slow roll parameters only if they (and the higher order ones which I did not bother defining here) are small:

$$\epsilon_V \approx \epsilon_H , \quad (23)$$

$$\eta_V \approx \epsilon_H + \eta_H , \quad (24)$$

There is higher order slow roll parameters, defined either as higher derivatives of the potential or Hubble.

Note that for successful inflation, the only criterion is $0 < \epsilon < 1$. The magnitude of η does not matter directly. Indirectly, large values of η are likely to make ϵ grow as well

$$\frac{d\epsilon}{dN} = 2\epsilon\left(\eta - \epsilon\right) , \quad (25)$$

but there are exceptions.

Observable quantities like the scalar and tensor power spectrum (see upcoming talk in two weeks) are commonly expressed in terms of Hubble or potential slow roll parameters.

5 Slow roll attractor for $m^2\phi^2$

Generically, the models of scalar field inflation possess attractor solutions. We shall now examine these in the case of $m^2\phi^2$ inflation.

We start off with the Lagrangian

$$L_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{m^2}{2}\phi^2 , \quad (26)$$

to find the equations of motion

$$\ddot{\phi} + 3H\dot{\phi} + m\phi = 0 , \quad (27)$$

$$H^2 = \frac{1}{6}\left(\dot{\phi}^2 + m^2\phi^2\right) , \quad (28)$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2 . \quad (29)$$

Plugging (28) into (27) we obtain

$$\ddot{\phi} + \sqrt{\frac{3}{2}}\left(\dot{\phi}^2 + m^2\phi^2\right)^{1/2}\dot{\phi} + m^2\phi^2 = 0 , \quad (30)$$

(31)

or

$$\frac{d\dot{\phi}}{d\phi} = -\frac{\sqrt{\frac{3}{2}}\left(\dot{\phi}^2 + m^2\phi^2\right)^{1/2}\dot{\phi} + m^2\phi^2}{\dot{\phi}} , \quad (32)$$

where we used $\ddot{\phi} = \dot{\phi}\frac{d}{d\phi}\dot{\phi}$. We shall explore its phase diagram, see Figure 1.

In the limit $\dot{\phi}^2 \gg m^2\phi^2$, (32) becomes

$$\frac{d\dot{\phi}}{d\phi} = \sqrt{\frac{3}{2}}\dot{\phi} , \quad (33)$$

$$\Rightarrow \dot{\phi} = \dot{\phi}_0 e^{\frac{3}{2}\phi} + \dot{\phi}_1 , \quad (34)$$

where we picked the right semi-plane $\phi > 0$. In other words, the field is rolling exponentially fast towards the attractor $\dot{\phi} = \dot{\phi}_1$.

Towards the end of inflation, the friction term in the eom for ϕ (9) becomes subdominant and we are left with a harmonic oscillator.

$$\ddot{\phi} + m^2\phi = 0 . \quad (35)$$

6 Power law inflation

Power law inflation is a very useful model to benchmark approximation schemes for the computation of scalar power spectra as its spectrum is exactly solvable (see talk in two weeks). The potential is given by

$$V = M^4 e^{\pm\sqrt{\frac{2}{p}}(\phi-\phi_0)} . \quad (36)$$

The scale factor in this model behaves as

$$a = t^p . \quad (37)$$

Let's construct the potential from the knowledge of the scale factor, thereby proving that this is the so-

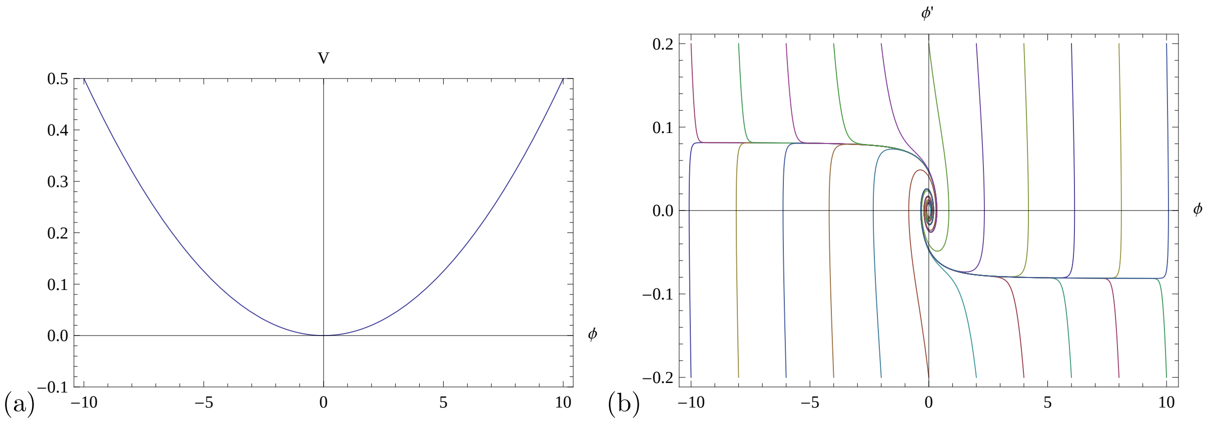

Figure 1: (a) Shape of the potential (b)Phase space for $\frac{1}{2}m^2\phi^2$ inflation. Notice the existence of the attractor solution and the oscillations around the origin.

lution.

$$
\begin{aligned}
a &= t^p\,, & (38)\\
\Rightarrow H &= H_0\frac{p}{t}\,, & (39)\\
\dot{H} &= -\frac{1}{2}\dot{\phi}^2 \Rightarrow -\frac{p}{t^2}H_0 = -\frac{1}{2}\dot{\phi}^2 & (40)\\
\Rightarrow \dot{\phi} &= \frac{\sqrt{2pH_0}}{t}\,, & (41)\\
\Rightarrow \phi &= \sqrt{2pH_0}\ln\frac{t}{t_0}\,, & (42)\\
\Rightarrow t &= t_0 e^{\sqrt{\frac{1}{2pH_0}}\phi}\,, & (43)\\
\Rightarrow V &= 3H^2 - \frac{1}{2}\dot{\phi}^2 = 3\frac{p^2}{t^2}H_0^2 - \frac{1}{2}\frac{pH_0}{t^2} & (44)\\
&= \frac{pH_0}{t^2}\left(3pH_0 - \frac{1}{2}\right) = M^4 e^{-\sqrt{\frac{2}{H_0 p}}\phi}\,. & (45)
\end{aligned}
$$

It goes to show that knowledge of Hubble as a function of time is sufficient to reconstruct the potential (modulo some integration constants).

7 Monomials

Let's turn our attention to monomial potentials

$$V = \lambda\phi^n\,. \quad (46)$$

In this case, it is best to work with the number of efolds N as time variable. Notation is $\phi' \equiv \partial_N\phi = -\frac{1}{H}\dot{\phi}$. Assume slow roll:

$$
\begin{aligned}
\dot{\phi} &= -\frac{\partial_\phi V}{3H} \Rightarrow \phi' = \frac{n\lambda\phi^{n-1}}{3H^2} & (47)\\
\Rightarrow \phi' &= \frac{n}{\phi} \Rightarrow \phi d\phi = ndN & (48)\\
\Rightarrow \phi &= \sqrt{2nN}\,. & (49)
\end{aligned}
$$

Now does this inflate? Yes:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2H^2} = \frac{1}{2}\phi'^2 = \frac{n}{4}N^{-1}\,. \quad (50)$$

8 General slow roll

For a generic potential, let us assume that the friction term in (9) will dominate over $\ddot{\phi}$ and the $\dot{\phi}$ term in the Friedman equation (7) becomes small compared to V as well. Then we have

$$
\begin{aligned}
H^2 &\approx \frac{1}{3}V\,, & (51)\\
3H\dot{\phi} + \partial_\phi V &\approx 0 \Rightarrow \dot{\phi} \approx -\frac{\partial_\phi V}{3H} \approx -\frac{\partial_\phi V}{\sqrt{3V}}\,, & (52)
\end{aligned}
$$

from which we can get the time dependence of $\phi(t)$. Then we use

$$H = \frac{\partial \ln a}{\partial t} = \dot{\phi}\partial_\phi \ln a \approx -\frac{\partial_\phi V}{3H}\partial_\phi \ln a \quad (53)$$

$$\Rightarrow H^2 \approx \frac{1}{3}V \approx \frac{1}{3}\partial_\phi V \partial_\phi \ln a \quad (54)$$

$$\Rightarrow \partial_\phi \ln a \approx \frac{V}{\partial_\phi V} = (\partial_\phi \ln V)^{-1}\,, \quad (55)$$

$$\Rightarrow a = a_0 e^{\int d\phi\,(\partial_\phi \ln V)^{-1}} \quad (56)$$

where we plug in $\phi(t)$ to obtain the time dependence of a. If we are only interested in $H(t)$, simply use

$$H(t) \approx \sqrt{\frac{1}{3}V(\phi(t))}\,. \quad (57)$$

9 Classes of inflationary models

- old inflation: Inflation proceeds via tunneling out of false vacuum
- new inflation: Coleman-Weinberg potentials $V \propto (\phi^4 \ln\frac{\phi}{\phi_0} - \frac{1}{4}\phi^4 + \frac{1}{4}\phi_0^4)$, fine-tuned initial condition: inflaton field has to sit near maximum
- chaotic inflation: "chaotic" because of arbitrary initial conditions
- k-inflation: non-canonical kinetic terms
- multifield inflation
- curvaton scenario
- $f(R)$ theories (conformally equivalent to scalar fields)
- ...