Density Fluctuations

Cecelie Hector

2010-11-9

Contents

1 Why density fluctuations?

Primordial inhomogeneities served as the seeds for structure formation. Inflation can explain the origin and predict their spectrum. Primordial perturbations originated from quantum fluctuation. Their size is blown up from Planckian length to superhorizon scales during inflation. Of the inflaton field $\phi = \phi_{cl} + \delta\phi$, only the perturbation $\delta\phi$ will be quantised. The field itself stays classical. The resulting scalar and tensor perturbations enable us to test inflation, thus it is important to calculate the spectra correctly and more precisely. These perturbations split up into three different groups: scalar, vector and tensor perturbations.

Scalar perturbations are induced by energy density inhomogenities. These perturbations are most important because they exhibit gravitational instability and may lead to the formation of structure in the universe.

Vector perturbations decay quickly and are not very interesting for us here.

Tensor perturbations describe gravitational waves, which are the degrees of freedom of the gravitational field itself. In the linear approximation the gravitational waves do not induce any perturbations in the perfect fluid.

Scalar, vector and tensor perturbations are decoupled and thus can be studied seperately.

Important for the calculations is ratio of the physical wavelength of the perturbations λ_{ph} a/k and the Hubble horizon $1/H$. The Hubble scale does not change much during inflation, while λ_{ph} grows strongly.

Standard results to first order in the slow roll approximation are

$$P_S^{\frac{1}{2}}(k) = \left(\frac{H^2}{2\pi\dot{\phi}}\right)\Bigg|_{aH=k} \tag{1}$$

for the curvature perturbation spectrum and

$$P_T^{\frac{1}{2}}(k) = \left(\frac{H}{\pi}\right)\Bigg|_{aH=k} \tag{2}$$

for the gravitational wave spectrum.

We will use as slow roll parameters

$$\epsilon = \frac{-\dot{H}}{H^2}, \quad \eta = \frac{\ddot{\phi}}{H\dot{H}}. \tag{3}$$

τ is the conformal time, the background metric is given by

$$\mathrm{d}s^2 = a(\tau)^2(\mathrm{d}\tau^2 - \mathrm{d}\mathbf{x}^2). \tag{4}$$

The effective action during inflation is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \tag{5}$$

2 Scalar Perturbations

First, we will look at the scalar linear perturbations. The most general expression for this metric is

$$\mathrm{d}s^2 = a(\tau)^2\left\{(1+2A)\mathrm{d}\tau^2 - 2\partial_i B\mathrm{d}x^i\mathrm{d}\tau - \left[(1+2R)\delta_{ij} + 2\partial_i\partial_j E\right]\mathrm{d}x^i\mathrm{d}x^j\right\} \tag{6}$$

were A, B, R, E are real scalars representing the four scalar degrees of freedom. After fixing the gauge, we insert the perturbed quantities into the action leading to

$$S_{\mathrm{pert}}^{\mathrm{scalar}} = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\left[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2\right], \tag{7}$$

prime denoting the derivation with respect to conformal time, using also

$$u = -z\mathcal{R} = -z\left(R - \frac{H}{\dot{\phi}}\delta\phi\right), \quad z \equiv a\frac{\dot{\phi}}{H}. \tag{8}$$

$\mathcal{R}$ is the intrinsic curvature perturbation of comoving hypersurfaces. Its spectrum is defined by

$$\mathcal{R} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\mathcal{R}_{\mathbf{k}}(\tau)e^{i\mathbf{k}.\mathbf{x}}, \quad \langle\mathcal{R}_{\mathbf{k}}\mathcal{R}^*_{\mathbf{l}}\rangle = \frac{2\pi^2}{k^3}P_S\delta^3(\mathbf{k}-\mathbf{l}) \tag{9}$$

Now we quantise the perturbations

$$\hat{u}(\tau,\vec{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left\{u_k(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}.\mathbf{x}} + u^*_k(\tau)\hat{a}^\dagger_{\mathbf{k}}e^{-i\mathbf{k}.\mathbf{x}}\right\} \tag{10}$$

imposing the standard relations for the creation and annihilation operators

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{l}}\right] = \delta^3(\mathbf{k}-\mathbf{l}) \;\;,\;\; \hat{a}_{\mathbf{k}}|0\rangle = 0, \ldots. \tag{11}$$

The Fourier componets of momentum k are decoupled from other momenta and the equation of motion for u_k is simply

$$u''_k + \left(k^2 - \frac{z''}{z}\right)u_k = 0 \tag{12}$$

As $aH/k \to 0$ (wavelength of fluctuations much smaller than horizon), we can approximate the modes by the free field solution in flat space

$$u_k = \frac{1}{\sqrt{2k}}e^{-ik\tau}. \tag{13}$$

For $aH/k \gg 1$ (wavelength much larger than horizon), $u_k \propto z$. Now we can use the slow roll parameters to express the z"/z term

$$\frac{z''}{z} = 2a^2H^2\left(1 + \frac{3}{2}\eta + \epsilon + \frac{1}{2}\eta^2 + \frac{1}{2}\epsilon\eta + \frac{1}{2H}\dot{\epsilon} + \frac{1}{2H}\dot{\eta}\right). \tag{14}$$

Assuming ϵ and η are constant, (12) can be solved by

$$\frac{z''}{z} = \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right), \quad \nu = \frac{1+\delta+\epsilon}{1-\epsilon} + \frac{1}{2} \tag{15}$$

with

$$\tau = \frac{-1}{aH}\left(\frac{1}{1-\epsilon}\right). \tag{16}$$

Approximating u_k for a wavelength much larger than the horizon

$$\begin{aligned} u_k =& \frac{1}{2}\sqrt{\pi}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}(-\tau)^{\frac{1}{2}}H_\nu^{(1)}(-k\tau) \\ &\to e^{i(\nu-\frac{1}{2})\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu} \quad \text{as} \quad aH/k \to \infty \end{aligned} \tag{17}$$

we finally find from (9) for the scalar perturbation spectrum

$$\begin{aligned} P_S^{1/2}(k) =& \sqrt{\frac{k^3}{2\pi^2}}\left|\frac{u_k}{z}\right| \\ =& 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}(1-\epsilon)^{\nu-\frac{1}{2}}\left.\frac{H^2}{2\pi|\dot{\phi}|}\right|_{aH=k}. \end{aligned} \tag{18}$$

3 Tensor perturbations

Now we turn to tensor perturbations corresponding to gravity waves. Tensor linear perturbations can be most generally expressed as

$$\mathrm{d}s^2 = a(\tau)^2\left[\mathrm{d}\tau^2 - (\delta_{ij} + 2h_{ij})\,\mathrm{d}x^i\mathrm{d}x^j\right], \tag{19}$$

notice that here we do not have gauge degrees of freedom as in the scalar case, since tensor perturbations are gauge invariant. The tensor h_{ij} is symmetric $h_{ij} = h_{ji}$, traceless $\delta^{ij}h_{ij} = 0$ and transverse $\partial^i h_{ij} = 0$. Inserting the perturbed metric in the Einstein-Hilbert action yields the perturbed action

$$S_{\text{pert}}^{\text{tensor}} = \frac{1}{8}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}a^2\left[(h'_{ij})^2 - (\partial_l h_{ij})^2\right]. \tag{20}$$

Defining $v_{ij} \equiv \frac{a}{2}h_{ij}$ leads to

$$S_{\text{pert}}^{\text{tensor}} = \frac{1}{2}\int \mathrm{d}\tau\mathrm{d}^3\mathbf{x}\left[(v'_\lambda)^2 - \partial_l v^{ij}\partial^l v_{ij} - \frac{a''}{a}v_\lambda^2\right] \tag{21}$$

The two point correlation function is

$$\sum_\lambda \left\langle h_{k,\lambda}h^*_{k',\lambda}\right\rangle = \frac{2\pi^2}{k^3}P_T(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{22}$$

with $\lambda \in (+, \times)$ being the two polarisations. Quantizing v_{ij} similar to the previous chapter gives

$$\hat{v}_{ij}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left\{ (v_k)_{ij}(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} + (v_k)^*_{ij}(\tau)\hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{x}} \right\}. \tag{23}$$

with

$$(v_k)_{ij} = v_k^+ e_{ij}^+(\mathbf{k}) + v_k^\times e_{ij}^\times(\mathbf{k}) \tag{24}$$

and the polarisation tensors satisfying

$$\begin{aligned} &e_{ij} = e_{ji}, \quad k^i e_{ij} = 0, \quad e_{ii} = 0, \\ &e_{ij}(-\mathbf{k}, \lambda) = e^*_{ij}(\mathbf{k}, \lambda) \\ &e^*_{ij}(\mathbf{k}, \lambda) e^{ij}(\mathbf{k}, \lambda') = \delta^\lambda_{\lambda'}. \end{aligned} \tag{25}$$

Again we derive the equation of motion from the action

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0 \tag{26}$$

and we make the approximations

$$\begin{aligned} &v_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \quad \text{as} \quad aH/k \to 0 \\ &v_k \propto a \quad \text{for} \quad aH/k \gg 1. \end{aligned} \tag{27}$$

Again we express the mass term in terms of slow roll parameters (again assuming ϵ to be constant) to solve the equation of motion

$$\begin{aligned} \frac{a''}{a} =& 2a^2H^2 \left(1 - \frac{1}{2}\epsilon\right) \\ =& \frac{1}{\tau^2}\left(\mu^2 - \frac{1}{4}\right), \quad \mu = \frac{1}{1-\epsilon} + \frac{1}{2}. \end{aligned} \tag{28}$$

Similar to the scalar spectrum we find

$$P_{\mathrm{T}}^{\frac{1}{2}}(k) = 2^{1+\mu-\frac{3}{2}} \frac{\Gamma(\mu)}{\Gamma(\frac{3}{2})} (1-\epsilon)^{\mu-\frac{1}{2}} \left.\frac{H}{2\pi}\right|_{aH=k}. \tag{29}$$

4 Power law inflation

As already mentioned in Pascal's talk two weeks ago the model of power law inflation is particularly nice for these calculations since both spectra are exactly solvable. In power law inflation we have $a \propto t^p$. For the slow roll parameters this gives

$$\epsilon = -\eta = \frac{1}{p} = \text{constant} \tag{30}$$

and further

$$\nu = \mu = \frac{3}{2} + \frac{1}{p-1}. \tag{31}$$

Inserting this in (18) gives us the exact curvature spectrum

$$P_S^{1/2}(k) = \left[2^{\frac{1}{p-1}} \frac{\Gamma\left(\frac{3}{2} + \frac{1}{p-1}\right)}{\Gamma\left(\frac{3}{2}\right)} (1 - \frac{1}{p})^{\frac{p}{p-1}} \right] \sqrt{\frac{p}{2}} \frac{H_1}{2\pi} \left(\frac{k_1}{k}\right)^{\frac{1}{p-1}} \tag{32}$$

where $H_1 = H|_{aH=k_1}$ is the Hubble parameter at the epoch when the scale k/a leaves the horizon. For the gravitation waves we find the relation

$$P_T^{\frac{1}{2}}(k) = \frac{2}{\sqrt{p}} P_S^{1/2}(k). \tag{33}$$

5 Slow-roll inflation in general to second order

Now I will extend the results for slow-roll (1), (2) to second order. To obtain the results to first order, we had neglected ϵ and η, now we will only assume them to be small and neglect the quadratic terms. From

$$\frac{1}{H}\dot{\epsilon} = \epsilon(2\epsilon + \eta), \quad \frac{1}{H}\dot{\eta} = \eta(\epsilon - \eta - \dot{\eta}) \tag{34}$$

we see that ϵ and η are approximately constant under this assumption. We can thus use the results from Section 3.

$$\nu \simeq \frac{3}{2} + 2\epsilon + \eta \quad \text{and} \quad \mu \simeq \frac{3}{2} - \epsilon \tag{35}$$

To second order the spectra are

$$
\begin{aligned}
P_S^{1/2}(k) &\simeq [1 + (2 - \ln 2 - b)(2\epsilon + \eta) - \epsilon] \frac{H^2}{2\pi|\dot{\phi}|}\bigg|_{aH=k} \\
P_T^{\frac{1}{2}}(k) &\simeq [1 - (\ln 2 + b - 1)\epsilon] \frac{H}{2\pi}\bigg|_{aH=k} .
\end{aligned}
\tag{36}
$$

where b is the Euler-Mascheroni constant with $2 - \ln 2 - b \simeq 0.7296$ and $\ln 2 + b - 1 \simeq 0.2704$.

6 Everything and more

Tensor to Scalar ratio An interesting parameter is the tensor to scalar ratio r

$$
r = \frac{P_{\mathrm{T}}(k_0)}{P_S(k_0)}. \tag{37}
$$

The scale dependence of the spectra is given by the spectral indices n_S and n_T. Since probably $n_S \neq n_T$, r will be scale dependent. n_S is in most models expected to be slightly smaller than 1. For $\dot{H} = 0$, P is independent of k and is called Harrison-Zeldovich-spectrum.

With (1), (2) and $\dot{\phi}^2 = -2\dot{H}$ we have

$$
r = 16\epsilon|_{k_0=aH}, \tag{38}
$$

so the tensor perturbations are expected to be much smaller than the scalar perturbations. r could also be defined at a given multipole l for a more observer-friendly approach (see next talk). WMAP 7 data give $r < 0.22$.

Consistency relation

If gravitational waves were detected, a strong evidence for inflation could be given by the so called consistency relation. The spectral index for the tensor modes can be written as

$$
n_T = -2\epsilon. \tag{39}
$$

Expressed through the spectra, we find

$$
n_T = -\frac{P_T}{8P_S} = -\frac{r}{8}. \tag{40}
$$

This means, if we manage to measure P_T at least for two values good enough for a proper estimation of the slope, we can check the consistency relation.

If it holds, it would strongly suggest inflation.

Lyth bound
Lyth derived a lower bound on the variation in the inflaton field during inflation in terms of the ratio r, known as the Lyth bound. In terms of ϵ the derivative of ϕ with respect to the number of e-foldings to the end of inflation is given by

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = \frac{m_{pl}}{2\sqrt{\pi}}\sqrt{\epsilon}. \tag{41}$$

Inserting (38) with ϵ roughly constant gives

$$\Delta\phi = \frac{m_{pl}}{8\sqrt{\pi}}\sqrt{r}|\Delta N|. \tag{42}$$

In his original paper, Lyth considered $\Delta\phi$ as scales exiting the horizon were corresponding to $1 < l \simeq 100$ (see next talk). This happens at $|\Delta N| \approx 4$. So the lower limit on the total field variation, called the Lyth bound, is

$$\Delta\phi \gtrsim m_{pl}\sqrt{\frac{r}{4\pi}}. \tag{43}$$

Lyth and others argue that inflation cannot be described by a low energy effective field theory if $\Delta\phi \gtrsim m_{pl}$ and so high values of $r \sim 1$ are possible only in models for which no rigorous theoretical framework exists. For models of inflation based on well-motivated particle physics it has been argued, that the Lyth bound requires $r \ll 1$.

References

[1] E. W. Kolb and M. S. Turner, "The Early universe," Front. Phys. **69**, 1 (1990).

[2] V. Mukhanov, "Physical foundations of cosmology," *Cambridge, UK: Univ. Pr. (2005) 421 p*

[3] E. D. Stewart and D. H. Lyth, "A more accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation," Phys. Lett. B **302** (1993) 171 [arXiv:gr-qc/9302019].