Starobinsky's model – f(R) theories

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V. Mukhanov, Cosmology p. 257, 258

A. A. Starobinsky, Phys. Lett. 91B (1980) 99

Introduction:

Basic idea: Generate inflation in the theory of gravity itself by including higher order curvature invariants

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\,(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^3 + \dots)$$

$$\equiv \frac{1}{2}\int d^4x \sqrt{-g}\, f(R) \qquad (M_p = 1)$$

- Einstein gravity in 4dim is the only metric theory where the eq. of motion are 2nd order. Any modification of the action introduces higher derivative terms. So in addition to gravitational waves the gravitational field has an extra degree of freedom which generally includes a Spin-0 field.

- In String Theory all kind of higher order curvature invariants can arise. Restrictions on the appearance of such terms increase with the number of imposed supersymmetries.

- Starobinsky's original model (1980) is <u>not</u> an f(R) - theory but is based on Vacuum quantum effects that lead to non-vanishing $\langle T_{\mu\nu}^{Vac.} \rangle$ in the Einstein equations. (anomaly induced inflation)

2. Equations of motion

Vary the following action S with respect to $g^{\mu\nu}$

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\, f(R) + \int d^4x\, \mathcal{L}_M(g_{\mu\nu}, \Psi_M)$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \Rightarrow f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' = T^{(M)}_{\mu\nu} \quad (1)$$

with $f' = \frac{\partial f}{\partial R}$, $\Box = \nabla_\mu \nabla^\mu$, $T^{(M)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}}$

For $f(R) = R$ we obtain the Einstein equations

$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(M)}_{\mu\nu}$. We take the trace of (1):

$$3\Box f' + f' R - 2f = T \qquad \text{with } T = T^{\mu(M)}_{\mu}$$

If the $\Box f'$ term does not vanish there is a propagating scalar degree of freedom, $f'(R)$. Consider de Sitter point, i.e. a vacuum solution ($T=0$) at which R is constant. At this point $\Box f' = 0$ so

$$f' R - 2f = 0$$

For $f(R) = R + \alpha R^2$ this is approximately satisfied as long as the αR^2 term dominates. When the linear term dominates the oscillation of R leads to gravitational particle production (Reheating)

Now consider FLRW spacetime with $k=0$:

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$$

Then we obtain with $R = 6(2H^2 + \dot{H})$:

$$\boxed{\begin{aligned} 3f'H^2 &= (f'R - f)/2 - 3H\dot{f}' + \rho_M &(2)\\ -2f'\dot{H} &= \ddot{f}' - H\dot{f}' + \rho_M + P_M &(3)\\ \text{and} \quad \dot{\rho}_M &+ 3H(\rho_M + P_M) = 0 &(4)\end{aligned}}$$

3. Inflationary dynamics in the Jordan-frame

Consider $f(R) = R + \alpha R^n \quad (\alpha > 0,\ n > 0)$

Then (2) becomes for $\rho_M = 0$:

$$3(1 + n\alpha R^{n-1}) H^2 = \frac{1}{2}(n-1)\alpha R^n - 3n(n-1)\alpha H R^{n-2} \dot{R}$$

Inflation can be realized for $f' = 1 + n\alpha R^{n-1} \gg 1$ ($\Rightarrow f' \simeq n\alpha R^{n-1}$)

$$\Rightarrow H^2 \simeq \frac{n-1}{6n}\left(R - 6nH\frac{\dot{R}}{R}\right)$$

H evolves slowly during inflation: $\left|\frac{\dot{H}}{H^2}\right| \ll 1$, $\left|\frac{\ddot{H}}{H\dot{H}}\right| \ll 1$

$$\Rightarrow \epsilon \simeq -\frac{\dot{H}}{H^2} \simeq \frac{2-n}{(n-1)(2n-1)} \quad (5)$$

tegrating for $\varepsilon > 0$ gives

$$H \simeq \frac{1}{\varepsilon t} \qquad a \propto t^{1/\varepsilon}$$

$\varepsilon(n)$ dependence in (5):

$n = 2 \Rightarrow \varepsilon = 0 \Rightarrow \dot{H} = 0 \Rightarrow$ inflation

$n > (1+\sqrt{3})/2 \simeq 1{,}4 \Rightarrow \varepsilon < 1 \Rightarrow$ inflation

$n > 2 \Rightarrow \varepsilon < 0 \Rightarrow \dot{H} > 0 \Rightarrow$ super-inflation

$n < (1+\sqrt{3})/2 \Rightarrow \varepsilon > 1 \Rightarrow$ no inflation

$\Rightarrow$ Choose $(1+\sqrt{3})/2 < n \leq 2$

In the following choose $f(R) = R + \frac{R^2}{6M^2}$. Inserting into (2) & (3) gives ($g_M = 0$):

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{1}{2}M^2 H = -3H\dot{H} \qquad (6)$$

$$\ddot{R} + 3H\dot{R} + M^2 R = 0 \qquad (7)$$

Compare the last eq. to a scalar field $\varphi = R$ with $V(\varphi) = \frac{1}{2}M^2\varphi^2$. The first two terms in (6) can be neglected during inflation so that

$$\dot{H} \simeq -\frac{M^2}{6}$$

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$$\Rightarrow \quad H \simeq H_i - \frac{M^2}{6}(t - t_i) \qquad (8)$$

$$a \simeq a_i \exp\left[H_i(t-t_i) - \frac{M^2}{12}(t-t_i)^2\right] \quad (9)$$

$$R \simeq 12H^2 - M^2 \ (10) \quad (\text{remember } R = 6(2H^2 + \dot{H}))$$

where i denotes the onset of inflation.
Inflation continues until $\varepsilon = -\frac{\dot{H}}{H^2} \simeq \frac{M^2}{6H^2} < 1$.
$\varepsilon_f = 1 \Leftrightarrow H_f \simeq \frac{M}{\sqrt{6}}$ so M sets the energy scale where inflation ends.
The normalization of CMB temperature anisotropies constrains this scale to $M \simeq 10^{13}$ GeV. Inflation ends at $t_f \simeq t_i + \frac{6H_i}{M^2}$ (see (8)). Then the number of e-folds from t_i to t_f is

$$\boxed{N = \int_{t_i}^{t_f} H\,dt \simeq \frac{3H_i^2}{M^2} \simeq \frac{1}{2\varepsilon_1(t_i)}}$$

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t. Conformal Transformation

erive an action in the Einstein frame (linear action
n R) by the conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

hen $R = \Omega^2 (\tilde{R} + 6\tilde{\Box}\ln\Omega - 6\tilde{g}^{\mu\nu}\partial_\mu \ln\Omega\, \partial_\nu \ln\Omega)$

nd $\sqrt{-g} = \Omega^{-4}\sqrt{-\tilde{g}}$ so that

$$S_R = \int d^4x \sqrt{-g}\, f$$

$$= \int d^4x \sqrt{-g}\, \left(\tfrac{1}{2} f' R - U\right) \quad \text{with} \quad U = \frac{f'R - f}{2}$$

$$= \int d^4x \sqrt{-\tilde{g}} \left[\tfrac{1}{2} f' \Omega^{-2} (\tilde{R} + 6\tilde{\Box}\ln\Omega - 6\tilde{g}^{\mu\nu}\partial_\mu \ln\Omega\, \partial_\nu \ln\Omega) - \Omega^{-4} U \right]$$

e want: – Einstein frame $\Rightarrow \Omega^2 = f' > 0$

– Canonical kinetic term $\Rightarrow \phi \equiv \sqrt{6}\ln\Omega = \sqrt{\tfrac{3}{2}}\ln f'$

$$\Rightarrow S_R = \int d^4x \sqrt{-\tilde{g}} \left[\tfrac{1}{2}\tilde{R} + \tfrac{1}{2}\sqrt{6}\,\tilde{\Box}\phi - \tfrac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \right]$$

ith $\boxed{V(\phi) = \dfrac{U}{f'^2} = \dfrac{f'R - f}{2f'^2}}$. The integral

$\int d^4x \sqrt{-\tilde{g}}\, \tilde{\Box}\phi$ vanishes (Gauss theorem)

$$\Rightarrow \boxed{\begin{aligned} S = &\int d^4x \sqrt{-\tilde{g}} \left[\tfrac{1}{2}\tilde{R} - \tfrac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi) \right] \\ &+ \int d^4x\, \mathcal{L}_M (f^{-1}(\phi)\tilde{g}^{\mu\nu}, \psi_M) \end{aligned}}$$

$f^{-1}(\phi)\tilde{g}^{\mu\nu}$ induces the coupling of ϕ to matter in the Einstein frame.

Metric in the Einstein frame:

$$d\tilde{s}^2 = \Omega^2 ds^2 = f'(-dt^2 + a^2(t)\, d\vec{x}^2)$$

$$= -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})\, d\vec{x}^2$$

$$\Rightarrow d\tilde{t} = \sqrt{f'}\, dt, \quad \tilde{a} = \sqrt{f'}\, a$$

$$\tilde{H} = \frac{1}{\tilde{a}}\frac{d\tilde{a}}{d\tilde{t}} = \frac{1}{\sqrt{f'}}\left(H + \frac{\dot{f}'}{2f'}\right)$$

(A) K.D

Dynamics in the Einstein Frame :

$$V(\phi) = \frac{f' R - f}{2\varkappa^2 F^2} \qquad \varkappa^2 = 8\pi G = M_{pl}^{-2}$$

with $\phi = \sqrt{\frac{3}{2}}\, \varkappa \, \ln\left[f'(R)\right]$

$$f(R) = R + \frac{R^2}{6M^2} \qquad = \sqrt{\frac{3}{2}}\, \varkappa\, \ln\left[1 + \frac{R}{3M^2}\right]$$

sometimes we use $F(R) = f'(R)$

Thus

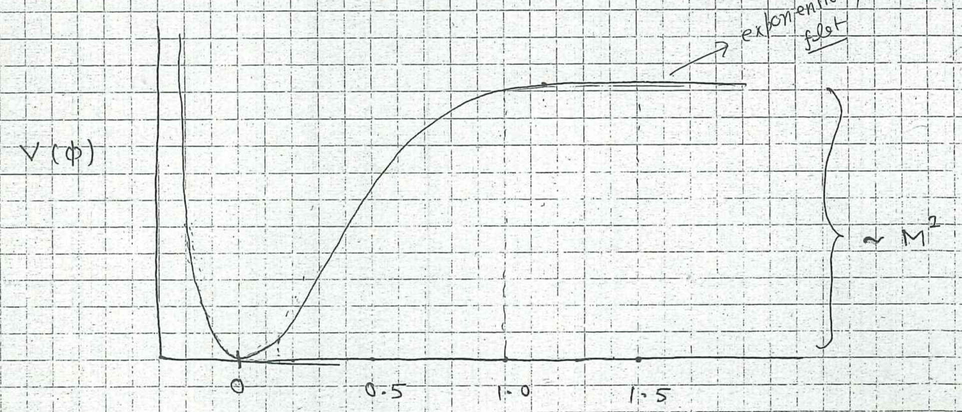
$$V(\phi) = \frac{3M^2}{4\varkappa^2}\left[1 - e^{-\sqrt{\frac{2}{3}}\varkappa\varphi}\right]^2$$

exponentially flat potential for $\varkappa\varphi \gg 1$

i.e. $\phi \gg M_{pl}$ ✓

$$\simeq \frac{3M^2}{4\varkappa^2} \qquad \text{for } \phi \gg M_{pl}$$

$$\simeq \frac{1}{2} M^2 \varphi^2 \qquad \varphi \ll M_{pl}$$



(B) K.D

Some Relations :

$$\begin{cases} R = 6(2H^2 + \dot{H}) \\ \quad \simeq 12H^2 - M^2 \\ \quad \simeq 12H^2 \\ F = 1 + R/3M^2 \simeq \dfrac{R}{3M^2} \\ \quad = 4H^2/M^2 \end{cases} \qquad \epsilon_1 \simeq \frac{M^2}{6H^2} \ll 1$$

$d\tilde{t} = \sqrt{F}\, dt$

Einstein frame ↲ ↳ Jordan frame

$$\Rightarrow \tilde{t} = \int_{t_i}^{t_f} \sqrt{F}\, dt \simeq \frac{2}{M}\left[H_i(t - t_i) - \frac{M^2}{12}(t - t_i)^2\right]$$

using $H = H_i - \frac{M^2}{6}(t - t_i)$

$$\tilde{a}(\tilde{t}) \simeq \left(1 - \frac{M^2}{12H_i^2} M\tilde{t}\right) \tilde{a}_i\, e^{M\tilde{t}/2}$$

where $\tilde{a}_i = \left(\frac{2H_i a_i}{M}\right)$

exponential expansion ✓

$$\tilde{H}(\tilde{t}) \simeq \frac{M}{2}\left[1 - \frac{M^2}{6H_i^2}\left(1 - \frac{M^2}{12H_i^2} M\tilde{t}\right)^{-2}\right]$$

$$\simeq \frac{M}{2}$$

standard $E^{n}s$ as we expect

$$\begin{cases} 3\tilde{H}^2 = \varkappa^2\left[\frac{1}{2}\left(\frac{d\phi}{d\tilde{t}}\right)^2 + V(\varphi)\right] \\ \dfrac{d^2\varphi}{d\tilde{t}^2} + 3H\dfrac{d\varphi}{d\tilde{t}} + \dfrac{dV}{d\varphi} = 0 \end{cases}$$

SR blah... blah... etc.

you know ...

C

K.D

Define SR parameters

Pascal Notation

$$\tilde{\epsilon}_1 \equiv - \frac{\frac{d\tilde{H}}{d\tilde{t}}}{\tilde{H}^2} \simeq \frac{1}{2\varkappa^2}\left(\frac{\frac{dV}{d\phi}}{V}\right)^2 = \epsilon_H$$

see last talk

$$\tilde{\epsilon}_2 \equiv \frac{\left(\frac{d^2\phi}{d\tilde{t}^2}\right)^2}{\tilde{H}\left(\frac{d\phi}{d\tilde{t}}\right)^2} \simeq \tilde{\epsilon}_1 - \underbrace{\frac{V_{,\phi\phi}}{3\tilde{H}^2}}_{\eta_V} = -\eta_H$$

$$\begin{cases} \tilde{\epsilon}_1 \simeq \frac{4}{3}\left(e^{\sqrt{\frac{2}{3}}\varkappa\varphi} - 1\right)^{-2} & \ll 1 \text{ for } \phi \gg M_p \\ \tilde{\epsilon}_2 \simeq \tilde{\epsilon}_1 + \frac{M^2}{3\tilde{H}^2} e^{-\sqrt{\frac{2}{3}}\varkappa\varphi}\left(1 - 2e^{-\sqrt{\frac{2}{3}}\varkappa\varphi}\right) & \end{cases}$$

$\ll 1$ for $\varphi \gg M_p$

Inflation ends: $\left\{\tilde{\epsilon}_1, |\tilde{\epsilon}_2|\right\} = \mathcal{O}(1)$

$$\tilde{\epsilon}_1 = 1 \Rightarrow \boxed{\varphi_f \simeq 0.19\ M_{pl}}$$

of e-folds:

$$\tilde{N} = \int_{\tilde{t}_i}^{\tilde{t}_f} \tilde{H}\, d\tilde{t} \underset{\text{using SR}}{\simeq} \varkappa^2 \int_{\phi_f}^{\phi_i} \frac{V}{V'}\, d\phi$$

$$\tilde{H}\, d\tilde{t} = \frac{1}{\sqrt{F}}\left(H + \frac{\dot{F}}{2F}\right)\sqrt{F}\, dt$$

use $F = \frac{4H^2}{M^2}$

$$= H\, dt\left(1 + \frac{\dot{F}}{2HF}\right) \quad \overset{SR}{\underset{SR}{\ }} \quad \frac{\dot{H}}{H^2} \ll 1$$

$$\simeq H\, dt\left(1 + \frac{\dot{H}}{H^2}\right) \simeq H\, dt$$

i.e. $N = \tilde{N}$ in SR

D

K.D

Under SR condition:-

$$\tilde{N} \simeq \frac{3}{4} e^{\sqrt{\frac{2}{3}}\varkappa\phi_i}$$

$$\Rightarrow \text{ for } \tilde{N} = 70 \rightarrow \boxed{\phi_i \simeq 1.1\ M_{pl}}$$

look at the potential — inflation (observable) happens at the last part of the flat potential

$$\boxed{\varphi_f \simeq 0.19\ M_{pl}}$$

Plug back the expression for $\tilde{N}$ to $\tilde{\epsilon}_1$ & $\tilde{\epsilon}_2$

$$\boxed{\tilde{\epsilon}_1 \simeq \frac{3}{4}\frac{1}{\tilde{N}^2}}$$

$$\tilde{\epsilon}_2 \simeq \frac{3}{4}\frac{1}{\tilde{N}^2} + \frac{1}{\tilde{N}}\left(1 - \ldots\right)$$

$$\Rightarrow \boxed{\tilde{\epsilon}_2 \simeq \frac{1}{\tilde{N}}}$$

Density Perturbations & Comparision to Data

Details of density perturbation – next talk

in the Jordon frame "hidden" DOF is $F = f'(R)$

– therefore in the perturbation theory

$$F = \bar{F} + \delta F$$

in the Einstein frame scalar field appears explicitly

$$P_{\mathcal{R}} \simeq \frac{1}{3\pi F}\left(\frac{H}{M_{pl}}\right)^2 \frac{1}{\epsilon_1^2}$$

$$= \frac{1}{12\pi}\left(\frac{M}{M_{pl}}\right)^2 \frac{1}{\epsilon_1^2}$$

$$N_k \simeq \frac{1}{2\epsilon_1(t_k)}$$

$$\simeq \frac{N_k^2}{3\pi}\left(\frac{M}{M_{pl}}\right)^2$$

WMAP 5 year $P_{\mathcal{R}} = (2.445 \pm 0.096) \times 10^{-9}$

at $k = 0.002\ \mathrm{Mpc}^{-1}$

$N_k \sim 55$ $\rightarrow$ $\boxed{M \simeq 3 \times 10^{-6}\ M_{pl}}$ $\sim 10^{13}$ GeV

$$P_T \simeq \frac{16}{\pi}\left(\frac{H}{M_{pl}}\right)^2 \frac{1}{F} \simeq \frac{4}{\pi}\left(\frac{M}{M_{pl}}\right)^2$$

$\mathcal{R} = \tilde{\mathcal{R}}$ curvature perturbation is conformal invariant quantity

Scalar Spectral Index :

$$\tilde{n}_s = 1 - 4\tilde{\epsilon}_1 - 2\tilde{\epsilon}_2$$

$$\simeq 1 - \frac{2}{\tilde{N}_k}$$

$$= 1 - 3.6 \times 10^{-2}\left(\frac{N_k}{55}\right)^{-1}$$

for $N_k = 55$ $n_s = 0.964$

WMAP 5 $n_s = 0.960 \pm 0.013$ at 68% CL ✓

Consistent

Scalar to Tensor Ratio :

$$\tilde{r} = \frac{64\pi}{M_{pl}^2}\left(\frac{d\phi}{d\tilde{t}}\right)^2 \frac{1}{\tilde{H}^2}$$

: SR condition

$$\simeq 16\tilde{\epsilon}_1 \simeq \frac{12}{\tilde{N}_k^2}$$

$$\simeq 4.0 \times 10^{-3}\left(\frac{N_k}{55}\right)^{-2}$$

Current bound $r < 0.22$ ✓ Consistent ✓

Comparisions:

...milar
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... Einstein
...rame

A) Starobinsky $\quad n_s = 1 - \frac{2}{N} \simeq 0.96$

$$r = \frac{12}{N^2} \simeq 0.004$$

B) chotic

$V = \frac{1}{2} m^2 \varphi^2 \quad n_s = 1 - \frac{2}{N} \simeq 0.96$

$$r = \frac{8}{N} \simeq 0.13 \quad \text{LARGE}$$

C) Higgs Inflation $\quad n_s \simeq 0.97$

$$r \simeq 0.0034$$

Planck sensitivity $\quad r \sim 10^{-2}$

Current limit $\quad r < 0.22$

— o —

$$P_{grav} \sim \frac{2}{M_{pl}^2} \left(\frac{H}{2\pi}\right)^2$$

$$\tilde{H}_{stero} \simeq \frac{M^2}{4} \quad ; \quad H^2_{chotic} \simeq \frac{1}{6 M_{pl}^2} m^2 \phi^2$$

$$\frac{P_{grav}^{chotic}}{P_{grav}^{stero}} \equiv \frac{H^2_{chotic}}{\tilde{H}_{stero}} \simeq \frac{2}{3} \left(\frac{m}{M}\right)^2 \left(\frac{\phi}{M_{pl}}\right)^2$$

obeservationally $\quad m \sim M \sim 10^{13}$ GeV

$\left(\frac{\phi}{M_{pl}}\right)^2$ factor $\Rightarrow$ large r for chaotic

whereas Starobinsky (Higgs inflation) potential becomes flat for large field value !!