Supergravity in the Jordan frame

Werkstattseminar WS 2010/11

Michael Grefe 11 January 2011

Literature:

- D. G. Cerdeño and C. Muñoz, "An Introduction to Supergravity", 6th Hellenic School and Workshop on Elementary Particle Physics (1998), see references herein for more exhaustive discussions of supergravity

- S. Ferrara, R. Kallosh, A. Linde, A. Marrani, and A. Van Proeyen, "Jordan frame supergravity and inflation in the NMSSM", Phys. Rev. D 82, 045003 (2010)

Outline:

1. Supergravity in the Einstein frame
   - Why gravity naturally appears in local supersymmetry
   - the supergravity Lagrangian in the Einstein frame

2. Supergravity in the Jordan frame
   - The supergravity Lagrangian in an arbitrary Jordan frame
   - Canonical kinetic terms for scalars

In the last seminar we discussed Higgs inflation in the Standard Model. In order to study Higgs inflation in supersymmetric theories we need to introduce supergravity.

1/ Supergravity in the Einstein frame

Why gravity naturally appears in local supersymmetry

Local supersymmetry, i.e. supergravity, naturally incorporates gravity. This can be seen explicitly in the simple case of a theory containing only a chiral supermultiplet (ψ^α, ϕ), i.e. a scalar field ϕ and its supersymmetric partner, the spin-1/2 fermion ψ. The Lagrangian

$$\mathcal{L} = -(\partial_\mu \phi)(\partial^\mu \phi^*) - \frac{1}{2}\bar{\psi}\gamma^\mu \partial_\mu \psi$$

is invariant under the global supersymmetry transformations

$$\delta\phi = \varepsilon\psi \qquad \text{and} \qquad \delta\psi = -i\sigma^\mu \bar{\varepsilon}\,\partial_\mu\phi \,.$$

However, going to local supersymmetry, i.e. $\varepsilon \to \varepsilon(x)$, the Lagrangian is not invariant anymore, but

$$\delta\mathcal{L} = (\partial_\mu \varepsilon^\alpha) K^\mu_\alpha + \text{h.c.},$$

where K^μ_α is a function of ψ and $\partial_\mu\phi$. In order to keep the action invariant, a gauge field has to be introduced: the gravitino, a spin-3/2 vector-spinor field with coupling

$$\mathcal{L}_{3/2} = k\, K_\mu^\alpha \Psi^\mu_\alpha \quad \text{and transformation} \quad \delta\Psi^\mu_\alpha = \frac{1}{k}\partial^\mu \varepsilon_\alpha \,.$$

However, also the Lagrangian $\mathcal{L} + \mathcal{L}_{3/2}$ is not invariant:

$$\delta(\mathcal{L} + \mathcal{L}_{3/2}) = k\,\Psi^\alpha_\mu \gamma_\nu \varepsilon_\alpha T^{\mu\nu},$$

where $T^{\mu\nu}$ is the energy-momentum tensor.

This term can only be canceled by adding gravity to the theory:

$$\mathcal{L}_g = -g_{\mu\nu} T^{\mu\nu} ,$$

with the tensor field $g_{\mu\nu}$, the graviton, transforming as

$$\delta g_{\mu\nu} = k \bar{\Psi}^{\alpha}_{\mu} \gamma_\nu \varepsilon_\alpha .$$

Therefore, a locally supersymmetric theory always contains, in addition to the chiral supermultiplets (ψ^α, ϕ) and the vector supermultiplets (V_μ, λ^α) of the globally supersymmetric Standard Model, the gravity supermultiplet $(g_{\mu\nu}, \Psi^\alpha_\mu)$ consisting of the graviton and the gravitino. The latter is the gauge field of local supersymmetry.

The supergravity Lagrangian in the Einstein frame

The complete supergravity Lagrangian can be constructed using e.g. the superspace formalism. In component fields the Einstein frame supergravity Lagrangian reads

$$\mathcal{L}_E = \sqrt{-g_E}\Big(-\frac{1}{2\kappa^2} R(g_E) - G_{i\bar{j}}(\phi,\phi^*)(D_\mu \phi^i)(D_\nu \phi^{*j}) g_E^{\mu\nu} - V_E(\phi,\phi^*)$$
$$+ \text{vectors} + \text{fermionic terms}\Big),$$

where $g_E = -\det g_E^{\mu\nu}$ is the negative determinant of the Einstein frame space-time metric and $\kappa^2 = 8\pi G_N \equiv \frac{1}{M_{Pl}^2}$ is the inverse of the reduced Planck mass squared.

The covariant derivative of the scalars is given by

$$D_\mu \phi^i = \partial_\mu \phi^i - A^a_\mu k^i_a ,$$

where k^i_a is a holomorphic Killing vector field corresponding to an isometry of the metric $G_{i\bar{j}}$.

In general the kinetic term of the scalar fields is not canonical. However, in the case of supergravity the non-linear sigma model of scalars is not arbitrary. Invariance under supergravity transformations requires that the field space of scalars is a Kähler manifold with the metric given by

$$G_{i\bar{j}}(\phi,\phi^*) = \frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\phi^{*j}} K(\phi,\phi^*),$$

where $K(\phi,\phi^*)$ is the real Kähler potential.

The scalar potential is given by

$$V_E = V_E^F + V_E^D = e^{\kappa^2 K}\left(\nabla_i W\, G^{i\bar{j}}\, \nabla_{\bar{j}} W^* - 3\kappa^2 |W|^2\right) + \frac{1}{2}\left(\mathrm{Re}\, f\right)^{-1}{}_{ab}\, D^a D^b,$$

where $\nabla_i W = \frac{\partial W}{\partial\phi^i} + \kappa^2 \frac{\partial K}{\partial\phi^i} W$ is the Kähler covariant derivative and $D^a = -i\frac{\partial K}{\partial\phi^i} k^{ai}$ is the real Killing prepotential.

The complete Lagrangian is determined by three functions that can be chosen arbitrarily:

- The real Kähler potential $K(\phi,\phi^*)$
- The holomorphic superpotential $W(\phi)$
- The holomorphic gauge kinetic function $f_{ab}(\phi)$

The supergravity Lagrangian is Kähler invariant, i.e. it is invariant under transformations

$$K(\phi,\phi^*) \longrightarrow K(\phi,\phi^*) + F(\phi) + F^*(\phi^*),$$
$$W(\phi) \longrightarrow e^{-\kappa^2 F(\phi)} W(\phi),$$

where $F(\phi)$ is an arbitrary holomorphic function. The Kähler covariant derivative of the superpotential transforms as

$$\nabla_i W(\phi) \longrightarrow e^{-\kappa^2 F(\phi)} \nabla_i W(\phi).$$

In the flat / renormalizable limit, i.e. $\kappa^2 \to 0$ or equivalently $M_{Pl} \to \infty$, we have:

- $K(\phi, \phi^*) = \delta_{i\bar{j}} \phi^i \phi^{*\bar{j}}$ leading to $G_{i\bar{j}} = \delta_{i\bar{j}}$
- $W(\phi) = \lambda_i \phi^i + \frac{1}{2} m_{ij} \phi^i \phi^j + \frac{1}{3} Y_{ijk} \phi^i \phi^j \phi^k$
- $\nabla_i W = \dfrac{\partial W}{\partial \phi^i}$
- $f_{ab} = \delta_{ab}$
- $k^{ai} = i T^{a i}{}_j \phi^j$ leading to $\mathcal{D}^a = \phi^*_i T^{a i}{}_j \phi^j$.

The Einstein frame is the frame where the gravity part of the theory is canonically normalized, i.e. the curvature, expressed by the Ricci scalar R, appears in the action only through the Einstein-Hilbert term $\sqrt{-g_E} \frac{1}{2\kappa^2} R(g_E)$. In this frame the scalars are minimally coupled to curvature.

2 Supergravity in the Jordan frame

A Jordan frame theory is related to the Einstein frame theory by an arbitrary Weyl rescaling of the metric and the fermions:

$$g_J^{\mu\nu} = \Omega^2 g_E^{\mu\nu}, \quad \Omega^2 = -\frac{1}{3}\Phi(\phi,\phi^*) > 0.$$

The real frame function $\Phi(\phi,\phi^*)$, together with the Kähler potential, the superpotential and the gauge kinetic function, uniquely defines a particular Jordan frame supergravity. As before, we neglect the fermionic part.

The supergravity Lagrangian in an arbitrary Jordan frame

The Lagrangian in an arbitrary Jordan frame with a nonminimal scalar-curvature coupling of the form $\Phi(\phi,\phi^*)R$ can be found using the above Weyl rescaling. A more elegant way, using a symmetry-inspired approach, is to start with an $N=1$, $d=4$ superconformal supergravity. The complete supergravity Lagrangian in a generic Jordan frame is then constructed through a suitable gauge fixing of the superconformal supergravity theory.

Here we only give the final result:

$$\mathcal{L}_J = \sqrt{-g_J}\Big(-\frac{1}{6\kappa^2}\Phi R(g_J) + \frac{1}{3}\Phi G_{i\bar{j}}(D_\mu\phi^i)(D_\nu\phi^{*\bar{j}})g_J^{\mu\nu} - \frac{1}{4\Phi}(\partial_\mu\Phi)(\partial_\nu\Phi)g_J^{\mu\nu} - V_J + \text{vectors} + \text{fermionic terms}\Big),$$

where the scalar potential in the Jordan frame is given by

$$V_J = \frac{\Phi^2}{9}V_E.$$

The frame function $\Phi(\phi, \phi^*)$ is in general invariant under gauge transformations of the scalar fields. Therefore

$$\partial_\mu \Phi = D_\mu \Phi$$

so that we can rewrite the term

$$-\frac{1}{4\Phi}(\partial_\mu \Phi)(\partial^\mu \Phi) = -\frac{1}{4\Phi}(D_\mu \Phi)(D^\mu \Phi).$$

In addition

$$D_\mu \Phi = \frac{\partial \Phi}{\partial \phi_i} D_\mu \phi^i + \frac{\partial \Phi}{\partial \phi^{*\bar{\jmath}}} D_\mu \phi^{*\bar{\jmath}}.$$

Defining $\Phi_i \equiv \frac{\partial \Phi}{\partial \phi_i}$ and $\Phi_{\bar{\jmath}} \equiv \frac{\partial \Phi}{\partial \phi^{*\bar{\jmath}}}$ the term can be written as

$$-\frac{1}{4\Phi}(D_\mu \Phi)(D^\mu \Phi) = -\frac{1}{4\Phi}\left(\Phi_i D_\mu \phi^i - \Phi_{\bar{\jmath}} D_\mu \phi^{*\bar{\jmath}}\right)^2 - \frac{\Phi_i \Phi_{\bar{\jmath}}}{\Phi}(D_\mu \phi^i)(D^\mu \phi^{*\bar{\jmath}})$$

leading to the Lagrangian

$$\mathcal{L}_J = \sqrt{-g_J}\left(-\frac{1}{6\kappa^2}\Phi R(g_J) + \left(\frac{1}{3}\Phi G_{i\bar{\jmath}} - \frac{\Phi_i \Phi_{\bar{\jmath}}}{\Phi}\right)(D_\mu \phi^i)(D_\nu \phi^{*\bar{\jmath}}) g_J^{\mu\nu} + \Phi \mathcal{A}_\mu \mathcal{A}^\mu - V_J + \text{vectors} + \text{fermionic terms}\right),$$

where $\mathcal{A}_\mu = -\frac{i}{2\Phi}\left(\Phi_i D_\mu \phi^i - \Phi_{\bar{\jmath}} D_\mu \phi^{*\bar{\jmath}}\right)$

is the bosonic part of the auxiliary field of supergravity.

Due to the contribution of the term $-\frac{1}{4\Phi}(\partial_\mu \phi)(\partial^\mu \phi)$ to the kinetic term of the scalars their nonlinear sigma model is of a modified Kähler type. The metric is not Hermitian due to terms proportional to $d\phi\, d\phi$ and $d\phi^* d\phi^*$. Therefore the Kähler geometry is not apparent.

The Einstein frame Lagrangian is recovered from the general Jordan frame Lagrangian by setting the frame function to $\Phi = -3$.

Canonical kinetic terms for scalars

In the Einstein frame the scalars have a Kähler metric:

$$G_{i\bar{j}} (D_\mu \phi^i)(D^\mu \phi^{*\bar{j}}).$$

Therefore, canonical kinetic terms are achieved for a Kähler potential of the form

$$K(\phi, \phi^*) = \delta_{i\bar{j}} \phi^i \phi^{*\bar{j}} + F(\phi) + F^*(\phi^*),$$

where $F(\phi)$ is a holomorphic function.

In the Jordan frame it is more complicated to achieve this goal. We therefore concentrate on a particular class of Jordan frames, where the frame function is related to the Kähler potential in the following way:

$$\Phi = -3\, e^{-\frac{\kappa^2}{3} K} \quad \Leftrightarrow \quad K = -3\kappa^{-2} \ln\left(-\tfrac{1}{3}\Phi\right).$$

This relation appears for instance in the superspace derivation of the supergravity Lagrangian.

In this case the Kähler metric is given by

$$G_{i\bar{j}} = -\frac{3}{\kappa^2}\frac{\Phi_{i\bar{j}}}{\Phi} + \frac{3}{\kappa^2}\frac{\Phi_i \Phi_{\bar{j}}}{\Phi^2}, \qquad \Phi_{i\bar{j}} \equiv \frac{\partial}{\partial \phi^i}\frac{\partial}{\partial \phi^{*\bar{j}}} \Phi$$

so that the Lagrangian simplifies to

$$\mathcal{L}_J^{\Phi(K)} = \sqrt{-g_J}\left(-\frac{1}{6\kappa^2}\Phi R(g_J) - \Phi_{i\bar{j}}(D_\mu \phi^i)(D^\mu \phi^{*\bar{j}}) + \Phi \mathcal{A}_\mu \mathcal{A}^\mu - \frac{\Phi^2}{9} V_E + \ldots\right)$$

The kinetic term of the scalars is determined by the second and third term in the Lagrangian. In order to have canonical kinetic terms it is therefore sufficient to fulfill the following two conditions:

a) The frame function is of the form

$$\Phi(\phi, \phi^*) = -3 + \kappa^2 \delta_{ij} \phi^i \phi^{*j} + \kappa^2 \mathcal{J}(\phi) + \kappa^2 \mathcal{J}^*(\phi^*)$$

with a holomorphic function $\mathcal{J}(\phi)$. This corresponds to the Kähler potential

$$K(\phi, \phi^*) = -3\kappa^{-2} \ln\left(1 - \frac{\kappa^2}{3} \delta_{ij} \phi^i \phi^{*j} - \frac{\kappa^2}{3} \mathcal{J}(\phi) - \frac{\kappa^2}{3} \mathcal{J}^*(\phi^*)\right)$$

b) Only scalar configurations for which the contribution from the bosonic part of the auxiliary vector field vanishes are considered:

$$\mathcal{A}_\mu = 0 .$$

This latter condition is in general not fulfilled for complex scalar fields.

A Jordan frame supergravity theory with canonical scalars can be easily transformed to the Einstein frame:

$$\mathcal{L}_E = \sqrt{-g_E}\left(\frac{1}{2\kappa^2} R(g_E) - G_{ij} (D_\mu \phi^i)(D^\mu \phi^{*j}) - V_E\right)$$

with $K(\phi, \phi^*)$ as given above, so that

$$G_{ij} = -3\frac{\delta_{ij}}{\Phi} + 3\frac{(\phi_i^* + \mathcal{J}_i)(\phi_j + \mathcal{J}_j^*)}{\Phi^2}, \quad V_E = \frac{9}{\Phi^2} V_J .$$

This can be used to study Higgs inflation models in supersymmetry starting from a Jordan frame theory with canonical scalars.