CMB-slow

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Talk at Tuesday's 'Werkstatt Seminar'

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References

[1] V. Mukhanov *Physical Foundations of Cosmology* (Cambridge University Press, 2005).

[2] V. Mukhanov *CMB-slow, or How to Estimate Cosmological Parameters by Hand*, astro-ph/0303072v1

[3] W. Hu and S. Dodelson, astro-ph/0110414 .

[4] W. Hu's homepage: http://background.uchicago.edu/∼whu/

We use the same notation as Mukhanov's textbook!

Topic of this talk: How can we understand the physics of the CMB and how is observation related to the inflationary paradigm?

1 CMB on the sky

What we actually measure are the photons that escaped the last scattering surface at the time of recombination when protons and electrons combined to build up neutral hydrogen. Thus these thermal photons obey the Plackian distribution:

$$f\left(\frac{\omega}{T}\right) \equiv \frac{2}{\exp(\omega/T)-1} \qquad , \text{ with } \omega = p_0/\sqrt{g_{00}} \tag{1}$$

The resulting black body spectrum is fully characterized by a single number, the temperature T. So, measuring the CMB means to measure the temperature in any accessible direction of the sky. This temperature turns out to be nearly isotropic throughout the whole sky with an average temperature of $T_0(\text{today}) \simeq 2.7\,\mathrm{K}$.

But there are anisotropies at the level of 10^{-5}, thus we apply linear perturbation theory:

$$T(x^\alpha, l_i) = T_0\,(\eta) \left(1 + \frac{\delta T}{T_0}(\eta, x^i, l^i)\right) \,, \tag{2}$$

where $\eta = x^0,\ x^i$ are conformal coordinates and

$$l^i \equiv -\frac{p_i}{p} \quad , \text{ with } p \equiv \sqrt{\Sigma_i p_i^2} \tag{3}$$

is the direction of motion of the photon with momentum p^α. Thus,

$$\text{today}: \quad \frac{\delta T}{T_0}(\eta_0, x_0^i, l^i) \sim 10^{-5}\,. \tag{4}$$

To display the anisotropies it is common to consider the two-point correlation function and expand it in the multipole moments C_l via Legendre polynomials $P_l(\cos\theta)$:

$$\left\langle \frac{\delta T}{T_0}\,(\boldsymbol{l}_1)\,\frac{\delta T}{T_0}\,(\boldsymbol{l}_2)\right\rangle_\theta \equiv C\,(\theta) = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)\,C_l P_l\,(\cos\theta) \tag{5}$$

The brackets $\langle\rangle_\theta$ denote the averaging over all $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$, satisfying the condition $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2 = \cos(\theta)$. The sum starts at $l = 2$ since the monopole and dipole is not of big use to us. The dipole only reveals our relative motion with respect to the CMB background. The monopole is in principle not measurable, due to the lack of a reference $C(0)$. There is no observer at a well separated point in space from whom we can obtain an additional measurement. All the same, there is a fundamental uncertainty in the measurement of C_l that gets big for small l where there are only a few samples on the sky to average over. This uncertainty is called the cosmic variance:

$$\frac{\Delta C_l}{C_l} \sim (2l+1)^{-1/2}\,. \tag{6}$$

2 Horizons

The fact that the temperature fluctuations on small scales are at the level of 10^{-5} is already a strong hint of an inflationary phase. At recombination the causally connected regions would be a lot smaller than our horizon today.

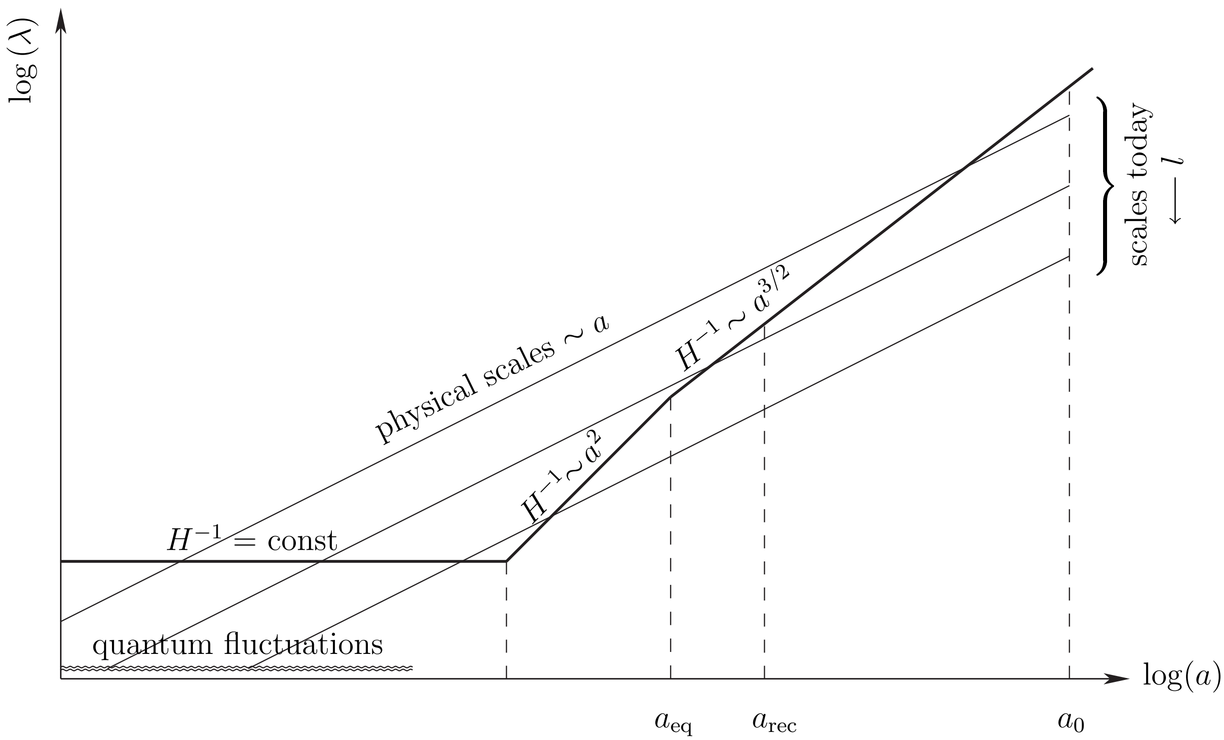

Figure 1: Behavior of the Hubble horizons H^{-1} at the different epochs of the universe: Inflationary, radiation dominated and matter dominated phase. The thin straight lines represent physical scales. The scales that have entered the horizon until today correspond to the multipoles in the CMB measurement. The middle line of the three exemplary physical scales corresponds to a position somewhere around the first peak: It enters the horizon before recombination leaving just about enough time to collapse once.

Inflationary phase: Horizon is constant $H^{-1} = \text{const}$. Quantum fluctuations that are born during inflationary phase blow up and leave the horizon. After inflation: In the radiation- and matter-dominated phase H^{-1} grows faster than $a \sim$ physical scales. Thus, scales re-enter the horizon. The scales at which we observe fluctuations today correspond to the l of the multipole expansion.

It is very important for the observed phenomena at a certain scale whether it enters before or after recombination! Why? Because recombination is the "birth" of the CMB-photons:

- After recombination: direct impact on the photon geodesics via the metric.
- Before recombination: impact on the plasma, photons inherit some properties of the plasma (they escape their potential). Harder to describe but one gets access to observables like Ω_b, Ω_m at the time of plasma oscillations.

A crucial thing is that recombination takes a "snapshot" of the plasma at the time of recombination and therefore picks up certain modes that have a certain amount of time to evolve between the entering of the scale and recombination. Inflation guarantees that for a certain mode this time is the same for all the patches on todays sky that have a size which corresponds to the horizon at the time of recombination. This is a requirement of constructive interference which in turn gives you the peaks in the C_l-spectrum.

3 Sachs-Wolfe effect

If there is a fluctuation in the metric, what is the effect on the photons and furthermore how can this be seen in the temperature fluctuations? We consider the metric

$$ds^2 = a^2 \left\{ (1 + 2\Phi)\, d\eta^2 - (1 - 2\Phi)\, \delta_{ik} dx^i dx^k \right\} \tag{7}$$

where $\Phi \ll 1$ is the gravitational potential of the scalar metric pertubations.

The momentum of a photon is defined as

$$\frac{dx^\alpha}{d\lambda} = p^\alpha \qquad , \ \lambda : \text{ affine parameter.} \tag{8}$$

Photons travel along geodesics which fulfill (be aware of lowering/raising indices in the case of derivatives):

$$\frac{dp_\alpha}{d\lambda} = \frac{1}{2} \frac{\partial g_{\gamma\delta}}{\partial x^\alpha} p^\gamma p^\delta, \tag{9}$$

What does this mean for the temperature fluctuations? Here we take the Boltzmann equation

$$\frac{Df\left(x^i(\eta), p_i(\eta), \eta\right)}{D\eta} \equiv \frac{\partial f}{\partial \eta} + \frac{dx^i}{d\eta}\frac{\partial f}{\partial x^i} + \frac{dp_i}{d\eta}\frac{\partial f}{\partial p_i} = 0 \tag{10}$$

Let's first compute the total derivatives with respect to η. From $p^\alpha p_\alpha = 0$ we obtain

$$p^0 = \frac{p}{a^2} \ , \text{ and thus } \ p_0 = (1 + 2\Phi) p \tag{11}$$

Therefore we obtain

$$\frac{dx^i}{d\eta} = \frac{p^i}{p^0} = \frac{-a^{-2}(1+2\Phi)p_i}{p^0} = l^i(1+2\Phi) \tag{12}$$

And from eq.(9) we obtain

$$\frac{dp_i}{d\eta} = \frac{1}{2}\frac{\partial g_{\gamma\delta}}{\partial x^\alpha}\frac{p^\gamma p^\delta}{p^0} = 2p\frac{\partial \Phi}{\partial x^\alpha}\,. \tag{13}$$

Thus the Boltzmann equation takes the form

$$\frac{\partial f}{\partial \eta} + l^i\,(1+2\Phi)\,\frac{\partial f}{\partial x^i} + 2p\frac{\partial \Phi}{\partial x^j}\frac{\partial f}{\partial p_j} = 0. \tag{14}$$

Taking f from eq. (1) and using the fact that the derivative of f with respect to the variable

$$y \equiv \frac{\omega}{T} = \frac{p_0}{T\sqrt{g_{00}}} \simeq \frac{p}{T_0 a}\left(1 + \Phi - \frac{\delta T}{T_0}\right) \tag{15}$$

is nonzero, the Boltzmann equation to zeroth order in pertubation reduces to the well known fact that

$$(T_0 a)' = 0, \tag{16}$$

while the first order terms lead to

$$\left(\frac{\partial}{\partial \eta} + l^i\frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T_0} + \Phi\right) = 2\frac{\partial \Phi}{\partial \eta}. \tag{17}$$

Thus, since the main contribution to Φ is constant in a matter dominated universe, we can drop the right hand side and obtain

$$\left(\frac{\delta T}{T_0} + \Phi\right) = \text{const.} \tag{18}$$

along null geodesics.

Using that the fluctuation in the temperature plus gravitational potential is constant we get a relation between the temperature fluctuations today and at recombination

$$\frac{\delta T}{T_0}(\eta_0, x_0^i, l^i) = \frac{\delta T}{T_0}(\eta_r, x_r^i(\eta_r), l^i) + \Phi(\eta_r, x_r^i(\eta_r)) - \Phi(\eta_0, x_0^i). \tag{19}$$

The last term is independent of the direction in the sky and therefore only contributes to the monopole, so we drop this term. The temperature fluctuations today depend on the temperature fluctuations and gravitational potential at recombination.

4 Temperature Density Relation

The fluctuations in the temperature at recombination can be related to the fluctuations in the energy density of the photons at large scattering, $\delta\gamma \equiv \delta\epsilon\gamma/\epsilon$. This is achieved by matching the energy-momentum tensor (EMT) for the radiation at the time of recombination. The EMT for the free photons have the form

$$T^{\alpha}_{\beta} = \frac{1}{\sqrt{-g}} \int f \frac{p^{\alpha} p_{\beta}}{p^0} d^3 p \tag{20}$$

For the 00 component this gives us

$$T^0_0 = \frac{1}{\sqrt{-g}} \int f p_0 d^3 p \simeq \frac{(1+2\Phi)}{a^4} \int d^3 p f(\omega/T) p_0 \tag{21}$$

Changing variables to $y = \frac{\omega}{T}$, and using $p \simeq \frac{y T_0 a}{\left(1+\Phi-\frac{\delta T}{T_0}\right)}$ and $p_0 = (1+2\Phi)p$, eq. (15) gives

$$\begin{aligned} T^0_0 \simeq & (1+2\Phi) \int (1+2\Phi) \left(1 - 4\Phi + 4\frac{\delta T}{T_0}\right) T_0^4 y^3 f(y) dy d^2 l \\ \simeq & T_0^4 \int \left(1 + 4\frac{\delta T}{T_0}\right) f(y) y^3 dy d^2 l. \end{aligned} \tag{22}$$

Matching this to the EMT before recombination, expressed as

$$T^0_0 = \epsilon\gamma(1+\delta\gamma) \tag{23}$$

give the first relation between the fluctuations in the photon density and temperature.

$$\delta_\gamma = 4 \int \frac{\delta T}{T_0} \frac{d^2 l}{4\pi}. \tag{24}$$

Using also the matching condition for $T^i_{0,i}$ give

$$\delta'_\gamma = -4 \int l^i \nabla_i \left(\frac{\delta T}{T_0}\right) \frac{d^2 l}{4\pi} \tag{25}$$

Taking the Fourier transformations and dropping the γ index give

$$\delta_{\boldsymbol{k}} = 4 \int \left(\frac{\delta T}{T_0}\right)_{\boldsymbol{k}} \frac{d^2 l}{4\pi} \tag{26}$$

$$\delta'_{\boldsymbol{k}} = -4i \int k_i l^i \left(\frac{\delta T}{T_0}\right)_{\boldsymbol{k}} \frac{d^2 l}{4\pi} \tag{27}$$

Both of these equations are satisfied by

$$\left(\frac{\delta T}{T_0}\right)_k (\eta_r) = \frac{1}{4}\left(\delta_k + \frac{3i}{k^2}\left(k_i l^i\right)\delta'_k\right) \tag{28}$$

5 Power Spectrum

Combining the solution with eq. (19), Fourier transforming the gravitational potential as well gives

$$\frac{\delta T}{T_0}(\eta_0, \boldsymbol{x}_0, \boldsymbol{l}) = \int \left(\left(\Phi + \frac{\delta}{4} \right)_{\boldsymbol{k}} + \frac{3i\delta'_{\boldsymbol{k}} \boldsymbol{k} \cdot \boldsymbol{l}}{4k^2} \right)_{\eta_r} e^{i\boldsymbol{k}\cdot\boldsymbol{x}(\eta_r)} \frac{d^3k}{(2\pi)^{3/2}} \tag{29}$$

The first term corresponds to redshift due to the gravitational potential when the photons travel out of potential wells. The second term describes the increase of temperature with density, while the third term is an effect of the doppler shift caused by the radial velocity in the plasma.

The photons coming to us from direction $\boldsymbol{l}$ have traveled along the geodesic since recombination

$$\boldsymbol{x}(\eta_r) \simeq \boldsymbol{x}_0 + \boldsymbol{l}(\eta_r - \eta_0) \tag{30}$$

This enables us to substitute the $\boldsymbol{x}$ in the exponential and also exchange the $i\boldsymbol{k} \cdot \boldsymbol{l}$ for a derivative with respect to the present time.

$$\frac{\delta T}{T_0}(\eta_0, \boldsymbol{x}_0, \boldsymbol{l}) = \int \left(\left(\Phi + \frac{\delta}{4} \right)_{\boldsymbol{k}} - \frac{3\delta'_{\boldsymbol{k}}}{4k^2} \frac{\partial}{\partial \eta_0} \right)_{\eta_r} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_0 + \boldsymbol{l}(\eta_r - \eta_0))} \frac{d^3k}{(2\pi)^{3/2}} \tag{31}$$

Plugging this into eq. (5) it is possible to obtain

$$C_l = \frac{2}{\pi} \int \left| \left(\Phi_k(\eta_r) + \frac{\delta_k(\eta_r)}{4} \right) j_l(k\eta_0) - \frac{3\delta'_k(\eta_r)}{4k} \frac{dj_l(k\eta_0)}{d(k\eta_0)} \right|^2 e^{-2(\sigma k \eta_r)^2} k^2 dk. \tag{32}$$

for the coefficients of the multipole moments, applying a damping factor due to the finite width of recombination. $j_l(k\eta_0)$ are spherical Bessel functions of order l. At large scales, $l < 200$, i.e. larger than horizon at recombination, perturbations have been frozen since they left the horizon during inflation, therefore the fluctuations in density should be proportional to the gravitational potential and the derivative approximately zero.

$$\delta_k(\eta_r) \simeq -\frac{8}{3}\Phi_k, \quad \delta'_k(\eta_r) \simeq 0. \tag{33}$$

These perturbations are directly related to the spectrum generated from inflation, $|(\Phi_k^0)^2 k^3| = Bk^{(1-n_s)}$. B gives the amplitude for the primordial perturbations and $n_s \simeq 1$ a scale invariant spectrum, which we can observe in the power spectrum.

For modes that enter the horizon just before recombination the solution is more complicated and can not be obtained analytically. We will merely state them and discuss their content.

$$\Phi_k + \frac{\delta_k}{4} \simeq \left[T_p \left(1 - \frac{1}{3c_s^2} \right) + T_o \sqrt{c_s} \cos\left(k \int_0^{\eta_r} c_s d\eta \right) e^{-(k/k_D)^2} \right] \Phi_k^0 \tag{34}$$

and

$$\delta_k' \simeq -4T_o k c_s^{3/2} \sin\left(k\int_0^{\eta_r} c_s d\eta\right) e^{-(k/k_D)^2}\Phi_k^0. \tag{35}$$

The two transfer functions cannot be solved analytically but depend on $k\eta_e q$ and baryon density.

$$\begin{aligned} c_s^2 &= \frac{1}{3(1+\xi)} \\ \xi &\simeq 17(\omega_b h_{75}^2 \\ \int_0^{\eta_r} c_s d\eta &\simeq 0.014(1+0.13\xi)^{-1}(\omega_m h_{75}^{3.1})^{0.16} \end{aligned} \tag{36}$$

There is both an oscillating and a non-oscillating part of the fluctuations. The oscillations create the peaks in the power spectrum. The Doppler effect oscillates exactly out of phase with the density perturbations. The exponential factor is damping due to dissipation in the plasma.

Trying to understand the different features, the oscillation can be seen as an oscillation of the plasma in a gravitational potential well created by the dark matter. The gravity of plasma enhances contraction peaks and lower rarification peaks. An increase in the total matter density will decrease the entire spectrum, while an increase in baryon density will increase odd peaks and decrease even peaks. Curvature will move the peaks towards smaller/larger l and the amplitude of the primordial perturbations will give height of the first plateau, while the spectral index will give the slope. This strength of the CMB fluctuations is that this rich dependence enables the determination of many cosmological parameters from the power spectrum alone.