11.7.

how does the metric fluctuate with ϕ? $\rightsquigarrow$ need fluctuations of gravitational potential to seed $\frac{\delta\rho}{\rho}$ 'density perturbations' ...

guidance: Schwarzschild metric of gravitating mass

$$ds^2 = (1-2\zeta)dt^2 - \frac{dr^2}{1-2\zeta} - d\vec{x}_2^2$$

at weak fields:

$$ds^2 = (1-2\zeta)dt^2 - (1+2\zeta)dr^2 - \dots$$

(full argument long $\leftrightarrow$ gauge inv. of GR)

now, in slow-roll:

inflaton jump $\delta\phi$

$\Rightarrow$ need $\delta N = H \cdot \delta t = \frac{H}{\dot{\phi}} \cdot \delta\phi$ more/less e-folds to reach reheating

$$\Rightarrow ds^2 = dt^2 - e^{2H \cdot t} \cdot d\vec{x}_3^{\,2}$$

$$\rightarrow ds^2 = dt^2 - e^{2H(t+\delta t)} \cdot d\vec{x}_3^{\,2}$$

$$= dt^2 - e^{2Ht} \cdot (1 + 2\cdot\delta N) \cdot d\vec{x}_3^{\,2}$$

compare:

$$\zeta = \delta N = \frac{H}{\dot\phi} \cdot \delta\phi$$

'curvature perturbation' induced by inflaton fluctuation $\delta\phi$

$$\curvearrowright \quad \zeta^2 = \frac{H^2}{\dot\phi^2}\,\delta\phi^2$$

$$\Rightarrow \Delta_\zeta^2 = \frac{H^2}{\dot\phi^2}\,\Delta_\phi^2 = \frac{H^4}{4\pi^2\dot\phi^2}$$

in slow-roll: $\dot\phi = -\frac{V'}{3H}$

$$\Rightarrow \Delta_\zeta^2 = \frac{1}{12\pi^2} \cdot \frac{V^3}{V'^2} = \frac{1}{24\pi^2} \cdot \frac{V}{\epsilon}$$

can show:

$$\frac{\delta\rho}{\rho} = \frac{2}{5}\sqrt{\Delta_\zeta^2} \longrightarrow \frac{\Delta T}{T} \text{ of CMB}$$

$$\boxed{\begin{array}{l}\text{CMB:} \\ \frac{\Delta T}{T} \sim 10^{-5} \\ \text{measures } \frac{V}{\epsilon}\end{array}}$$

in exact dS:

$$\Delta_\phi^2 = \text{const.}$$

in slow-roll can parametrize:

$$\Delta_\zeta^2(k) = \Delta_\zeta^2(k_0) \cdot \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$\nwarrow$ spectral tilt

expand:

$$\ln \Delta_\zeta^2(k) = \ln \Delta_\zeta^2(k_0) + \frac{d\ln \Delta_\zeta^2(k_0)}{d\ln k} \cdot \ln\frac{k}{k_0} + \ldots$$

$$\Rightarrow n_S - 1 = \left.\frac{d\ln\Delta_\zeta^2}{d\ln k}\right|_{k=k_0=aH}$$

relation between comoving wave number k and physical wave number $k_{phys.}$:

$$k = \frac{k_{phys.}}{a} = k_{phys.}\cdot e^{-N}$$

$$\Rightarrow d\ln k = -dN$$

$$\Rightarrow n_S - 1 = \left.\frac{d\ln\Delta_\zeta^2}{dN}\right|_{N\simeq 60}$$

$$\frac{d\ln\Delta_\zeta^2}{dN} = \frac{1}{\Delta_\zeta^2}\cdot\frac{d\Delta_\zeta^2}{dN}$$

$$= 12\pi^2\cdot\frac{V'^2}{V^3}\cdot\frac{d\phi}{dN}\cdot\frac{d}{d\phi}\left(\frac{1}{12\pi^2}\frac{V^3}{V'^2}\right)$$

$$\frac{d\phi}{dN} = \dot\phi\cdot\frac{dt}{dN} = \frac{\dot\phi}{H} = -\frac{V'}{3H^2}$$

$$= -\frac{V'}{V}$$

$$= -\frac{V'^3}{V^4}\cdot\left(3\frac{V^2}{V'} - 2\frac{V^3}{V'^3}V''\right)$$

$$= -6\cdot\left(\frac{1}{2}\frac{V'^2}{V^2}\right) + 2\cdot\frac{V''}{V} = -6\epsilon + 2\eta$$

$$\Rightarrow n_S = 1 - 6\epsilon + 2\eta$$

WMAP: $n_s = 0.963 \pm 0.013$, 1σ

PLANCK: $\Delta n_s = 0.005$ at 2σ

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inflation also seeds primordial gravitational waves:

$\rightsquigarrow$ tensor perturbations of the dS metric

$$ds^2 = (1-2\zeta)dt^2 - \left[(1+2\zeta)\delta_{ij} + h_{ij}\right]dx^i dx^j$$

get a free wave e.o.m. like inflaton scalar field in the limit $V = \text{const}$.

no further 'translation factor' unlike Δ_ζ^2 – h_k is already a metric perturbations ...

define tensor-to-scalar ratio r:

$$r = \frac{\Delta_h^2}{\Delta_\zeta^2} = \frac{\dot\phi^2}{H^2} = 2\epsilon$$

... doing a better job on normalization:

$$\boxed{r = 16\,\epsilon}$$

example: $V \sim \phi^p$

$$n_s = 1 - \frac{2+p}{2N_e} \simeq 0.97$$

$$r = \frac{4p}{N_e} = 0.13 \text{ for } p = 2.$$

measuring r determines ϵ, and via $\delta\rho/\rho$ from $\Delta T/T$ the scale of inflation V!

2^{nd} significance of r:

compute $N_e = \int H\,dt = \int \frac{d\varphi}{\sqrt{2\epsilon}}$

$$\Rightarrow\ N_e \simeq \frac{\Delta\phi}{M_P}\cdot\frac{1}{\sqrt{2\epsilon}}$$

$$\Leftrightarrow\ r = 16\epsilon \simeq \frac{8}{N_e^2}\cdot\left(\frac{\Delta\phi}{M_P}\right)^2$$

$$\Rightarrow\ \boxed{r \simeq 0.003\cdot\left(\frac{50}{N_e}\right)^2\cdot\left(\frac{\Delta\phi}{M_P}\right)^2}$$

'Lyth bound'

$\rightsquigarrow$ $r \sim 0.01$ corresponds to boundary between large-field and small-field inflation.

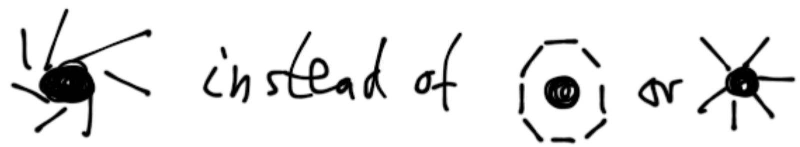
how to measure r?

i) Δ_h^2 converts into $\frac{\Delta T}{T}$ at large angular scales $> 10^0$

$\to$ WMAP bound, $r \lesssim 0.2$

ii) B-mode polarization of CMB:

$\to$ look for curl-like pattern of polarization vectors around cold spot in CMB:



$\rightsquigarrow$ PLANCK: $r \lesssim 0.03 \ldots 0.05$ (extended run, 2.5 yr)

similar range for ground-based: QUIET, Keck array, Spider ...

by 2013–2014 : 3 yrs !

$\rightsquigarrow$ observational reach on r

$\hat{=}$ small-field / large-field boundary ...