slower than the matter term:

$$\rho_0 \delta \sim \frac{1}{a^3}\delta$$

if ρ_Λ or $\rho_{curv.}$ are present.

In the growing $k << k_J$, we can solve (97) if we are in matter domination:

$$\rho_0(t) \sim \frac{1}{a^3} \sim \frac{1}{t^2}, \quad H = \frac{2}{3t}$$

$$(97) \Rightarrow \ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0$$

This is solved by:

$$\delta \sim t^{2/3} \sim a$$

thus δ has grown since recombination & CMB production at $z_* = 1100$ only by a factor 10^3:

$$\delta_0 = \delta_{CMB} \cdot \frac{a_0}{a_{CMB}} = \delta_{CMB} \cdot (1+z_*) \simeq 10^3 \cdot \delta_{CMB} \qquad (97')$$

k_J corresponds to critical mass M_J of overdensity δ with size $a(t)/k_J$:

$$M_J = \frac{4\pi}{3}\rho \cdot d_J^3 = \frac{4\pi}{3}\rho_m \left(\frac{2\pi \cdot a(t)}{k_J}\right)^3$$

$$(99) \quad = \frac{4\pi^{5/2} \cdot c_s^{3/2}}{3G^{3/2} \cdot \rho_m^{1/2}} \sim 3.4 \cdot 10^{20} \cdot c_s^3 \cdot M_\odot$$

now for the non-relativistic hydrogen gas after photon decoupling:

(100) $c_s^2 \sim \frac{T}{m_N}$, T : gas temperature

As mentioned at the beginning, this discussion is only valid for scales : $\lambda_{phys.} \ll \frac{1}{H}$

$$\Leftrightarrow k_{phys.} \gg H$$

for $k_{phys.} \gtrsim H$ one has to GR perturbation theory, with the focus

central result, that perturbation modes with :

$$k_{phys.} < H \Leftrightarrow \lambda_{phys.} > H$$

i.e. which are 'super-horizon', freeze, and do not evolve (grow, decline or oscillate) any more !

For the earliest evidence of density perturbations, let us take a look at the <u>cosmic microwave background</u> (CMB) radiation.

<u>recall</u>: photons should decouple after hydrogen recombination $e^- + p \to {}^1_1H + \gamma$ at 13.6 eV — however, baryon asymmetry: 10^9 γ's for 1 baryon, delays recombination to ~ 0.3 eV, $1 + z_* = 1100$ (Saha equation).

If there is a source Φ of primordial density fluctuation, it will drive fluctuations θ in the tightly coupled baryon-photon fluid, which are just the soundwaves of (97) in the oscillating regime $k \gg k_J$ — a driven oscillator:

$$\ddot{\theta} + H \cdot \dot{\theta} + c_s^2 \frac{k^2}{a^2} \theta = \Phi$$

speed of sound

in conformal time: $d\eta = \frac{dt}{a}$, $\frac{\partial}{\partial t} = \frac{1}{a} \frac{\partial}{\partial \eta}$

$$\Rightarrow \theta'' + c_s^2 k^2 \cdot \theta = \Phi, \quad (\,)' \equiv \frac{\partial}{\partial \eta}(\,)$$

solutions are soundwaves:

$$\theta_k(s) \sim \cos\left(k \cdot s + \varphi_k\right)$$

initial phase

where e.g. a driving source Φ which puts:

$$\theta_k(0) = \text{const.}, \quad \theta_k'(0) = 0$$

gives coherent initial phases:

$$\varphi_k = 0$$

s is the comoving distance from $t=0$ to time t, which these sound waves travel at

speed c_s:

$$s = a_0 \cdot \int_{\eta(0)}^{\eta(t)} d\eta \cdot c_s \quad , \quad a_0 = 1$$

$$= \int_{\infty}^{z} \frac{dz}{H(z)} \cdot c_s$$

the speed of sound c_s in the tightly coupled baryon-photon fluid before recombination:

$$c_s = \frac{1}{\sqrt{3}}$$

now, these sound waves imprint their momentary profile at recombination!

$$z = z_* = 1100$$

onto the freed photons. The sound waves have then reached the 'comoving sound horizon' of recombination:

$$s = s_* = c_s \int_\infty^{z_*} \frac{dz}{H(z)}$$

matter dominated most of the time, & we assume a <u>flat universe</u>, so:

$$\frac{1}{H(z)} = \frac{1}{H_0} \cdot (1+z)^{-3/2}$$

$$\Rightarrow s_* = 2 \cdot \frac{c_s}{H_0} (1+z_*)^{-1/2} \simeq c_s \cdot t_*$$

$$\frac{1}{H_0} \simeq 10^{10}\, ly \simeq 3000\ MPc$$

$$\Rightarrow s_* \simeq 120\ MPc$$

in a <u>flat universe</u>

now, after the photons got the imprint of the soundwaves at $s = s_*$ at recombination:

$$(\text{CMB}) \quad \theta_k(s_*) \sim \cos(k s_* + \varphi_k)$$

the photons decoupled, and froze the soundwave field (CMB) until today – this is the CMB field we see today (roughly)!

now, <u>if</u> the initial phases were set coherently at:

$$\varphi_k = 0$$

by the primordial fluctuations Φ, then waves with:

$$k_n = \frac{n\pi}{s_*}$$

where captured in peaks at recombination.

The corresponding half-wavelength:

$$\frac{\lambda_n}{2} = \frac{s_*}{n}$$

forms an angle at the sky:

$$\vartheta_n = \frac{\lambda_n}{2D_*} = \frac{s_*}{n D_*}$$

where D_* is the distance from us to decoupling:

$$D_* = \underbrace{a_0}_{=1}(1+\underbrace{z_0}_{=0}) \cdot \int_{\eta_*}^{\eta_0} d\eta \simeq \eta_0 \,, \quad \eta_* \ll \eta_0$$

$$\Rightarrow \vartheta_n = \frac{1}{n} c_s \cdot \frac{\eta_*}{\eta_0} = \frac{1}{n} \frac{c_s}{\sqrt{1+z_*}}$$

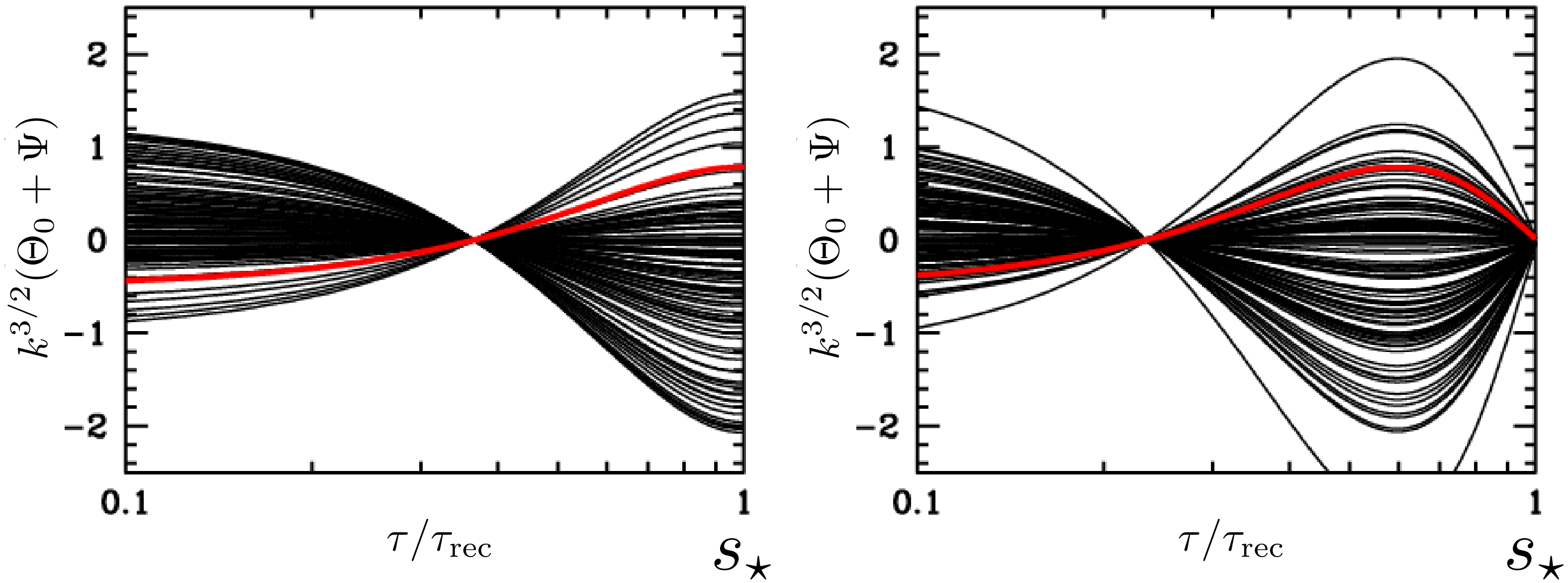

Figure 25: Evolution of an infinite number of modes all with the same wavelength. Recombination is at $\tau = \tau_{\rm rec}$. (Left) Wavelength corresponding to the first peak in the CMB angular power spectrum. (Right) Wavelength corresponding to the first trough. Although the amplitudes of all these different modes differ from one another, since they start with the same phase, the ones on the left all reach maximum amplitude at recombination, the ones on the right all go to zero at recombination. This leads to the acoustic peaks of the CMB power spectrum.

source - arXiv:0907.5424

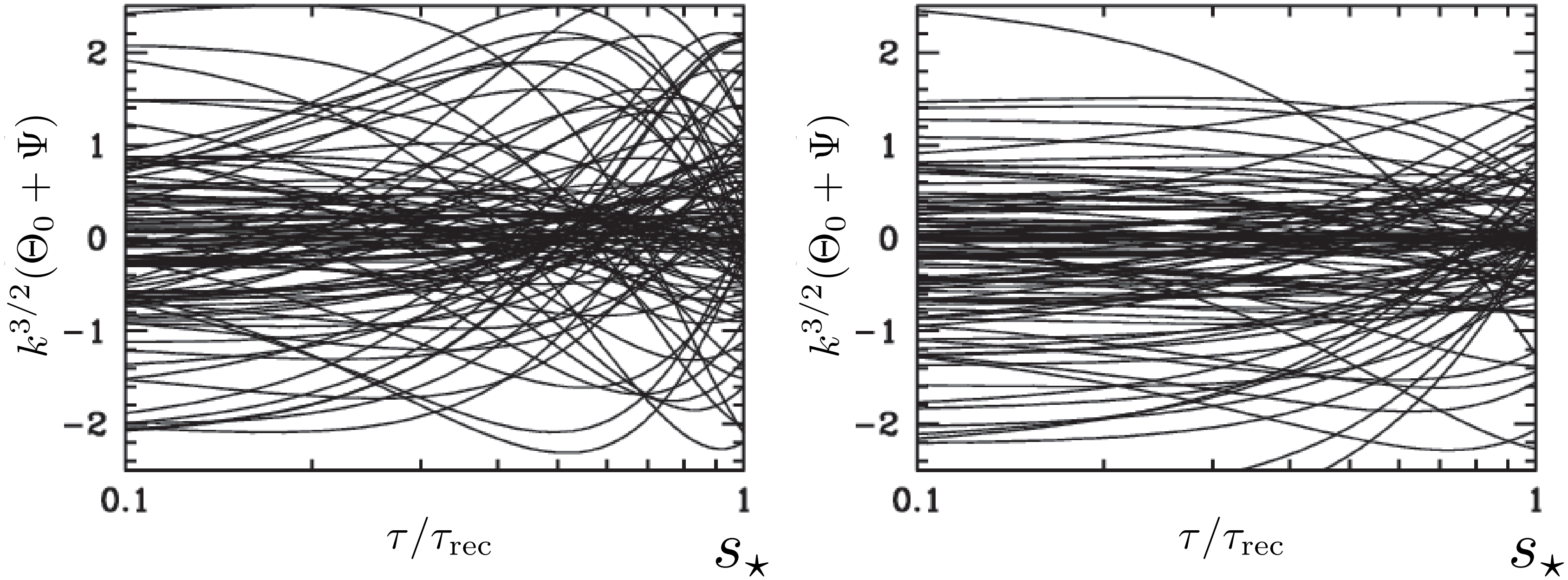

Figure 26: Modes corresponding to the same two wavelengths as in Fig. 25, but this time with random initial phases. The anisotropies at the angular scales corresponding to these wavelengths would have identical rms's if the phases were random, *i.e.* the angular peak structure of the CMB would be washed away.

source - arXiv:0907.5424

if you now project the CMB temperature field onto a spherical sky 'surface', you can decompose it into spherical harmonics:

$$\theta_{l,m} \sim \int d\vec{n}\, Y_{l,m}(\vec{n}) \cdot \theta_k(\vec{n})$$

$Y_{l,m}(\vec{n})$: spherical harmonics

one usually plots then the 2-point correlation function:

$$l(l+1)\, C_l \sim l(l+1) \cdot T^2 \sum_m \langle |\theta_{l,m}|^2 \rangle$$

this displays power as a function of angular separation ϑ expressed

by multipole number l:

$$l = \frac{\pi}{\vartheta}$$

now we expect power maxima at $\vartheta = \vartheta_n$, the angular size today of soundwaves with wavelengths λ_n which captured at peaks at recombination:

maxima of $l(l+1)\, C_l$

at : $l_n = \dfrac{\pi}{\vartheta_n} = n \cdot \pi \dfrac{\sqrt{1+z_*}}{c_s}$

$\Rightarrow$ 1st power maximum

at:

$$\ell_1 = \pi \frac{\sqrt{1+z_*}}{c_s}$$

$$\Rightarrow \boxed{\ell_1 = \pi\sqrt{3}\sqrt{1+z_*} \simeq 200}$$

$$\hat{=} \quad \boxed{\vartheta_1 \simeq 1^\circ}$$

compare to the WMAP figure of $\ell(\ell+1)\,C_\ell$ – this fits very well with the position of the 1st observed power maximum!

s_* would be different for a positively or negatively curved universe, compared to the value derived here for a flat universe ... and so would ℓ_1 differ – thus the observed $\ell_1 \simeq 200$ is very good evidence for a spatially flat universe!

note, that after recombination the sound speed of the formed hydrogen gas drops precipitously, compared to before, to:

$$c_s^2 \sim \frac{T}{m_N} \ll 1$$

which happens at $T_* \sim 1\,\mathrm{eV}$ of recombination.

This means, that the gas distribution at large scales

maintains the same sound wave imprints as the CMB, and indeed the 2-point function of large-scale galaxy cluster distribution data shows 'baryon acoustic oscillations' with the 1st peak in power at length scales $\sim 100\,\mathrm{MPc} \sim S_*$.

A final note:

The CMB shows $\frac{\Delta T}{T} \sim 10^{-5}$

This is thus also the level of the density contrast in the hydrogen gas after recombination:

$$\delta_* \sim 10^{-5}$$

We saw from (97') that large scale δ grew after recombination only by:

$$\frac{\delta_0}{\delta_*} = \left(\frac{t_0}{t_*}\right)^{2/3} = 1 + z_* = 10^3$$

$$\Rightarrow \delta_0 \sim 10^{-2}$$

but we know that on scales < 8 MPc, matter has gone into non-linear collapse, forming clusters of galaxies!

$\Longrightarrow$ need extra dark matter, uncoupled to photons & baryons, that grew its perturbations already <u>before</u> recombination!

<u>mandated by CMB</u>

$\rightarrow$ need again $\Omega_{DM} \sim 0.3$