27.6.

to obtain freeze-out value of n_X, go to limit:

$$\frac{dn_X}{dt} \to 0 \text{ for } t \to \infty$$

We have:

$$\frac{1}{a^3}\frac{d(n_X a^3)}{dt} = \frac{dn_X}{dt} + 3\frac{\dot{a}}{a}\cdot n_X = \frac{dn_X}{dt} + 3H\cdot n_X$$

thus we demand:

$$\frac{dn_X}{dt} = 0 \text{ at late times}$$

furthermore, at late times:

$$\frac{m}{T} \to \infty$$

$$\Rightarrow \quad n_X^{(0)} \sim e^{-\frac{m}{T}} \to 0$$

exponentially fast at late times

$\Rightarrow$ (74) becomes:

$$3H\cdot n_X \simeq \langle \sigma_X v\rangle \cdot n_X^2 \quad (75)$$

if we realize:

$$\langle \sigma_X v\rangle n_X = \langle \Gamma_X \rangle$$

thermally averaged reaction rate

this is nothing else than:

$$H \simeq \langle \Gamma_X \rangle$$

our old freeze-out criterion!

We get:

$$(75) \Leftrightarrow n_X \sim \frac{H}{\langle \sigma_X v \rangle}$$

thus we get for ρ_X:

$$\rho_X = m_X n_X \sim \frac{m_X H}{\langle \sigma_X v \rangle} \quad (76)$$

and at freeze-out:

$$H \sim \frac{T_{f.o.}^2}{M_P}, \; T_{f.o.} \sim m_X$$

$$(76) \Rightarrow \rho_X \sim \frac{m_X T_{f.o.}^2}{M_P \langle \sigma_X v \rangle}$$

$$\sim \frac{T_{f.o.}^3}{M_P \langle \sigma_X v \rangle} \quad (77)$$

now, ρ_X is ρ_X at freeze-out, dilutes until now $\rightarrow$ non-relat. matter:

$$\rho_{X,0} = \rho_X \left(\frac{a_{f.o.}}{a_0} \right)^3 \sim$$

$$\sim \frac{T_0^3}{M_P \langle \sigma_X v \rangle} \left(\frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \right)^3 \quad (78)$$

now, because:

$$S = \text{const.} = s \cdot a^3 \Rightarrow T \sim \frac{1}{a}$$

$$\Rightarrow \frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \sim \mathcal{O}(1)$$

$$\Rightarrow \rho_{x,0} \sim \frac{T_0^3}{M_p \langle \sigma_x v \rangle} \qquad (79)$$

we can now calculate to-day's density parameter $\Omega_{x,0}$ of the X particles:

$$(80) \quad \Omega_{x,0} = \frac{\rho_{x,0}}{\rho_{cr,0}} \sim \frac{T_0^3}{M_p^3} \cdot \frac{1}{H_0^2 \langle \sigma_x v \rangle}$$

now we have:

$$T_0 \sim 10^{-4}\,\text{eV}, \quad M_p \sim 2 \cdot 10^{18}\,\text{GeV}$$

$$\frac{H_0}{c} \simeq 3 \cdot 10^{-9} \frac{s}{m} \cdot \frac{72 \cdot 10^3\,m}{s \cdot \text{MPc}} \simeq \frac{2 \cdot 10^{-4}}{\text{MPc}} \simeq \frac{2 \cdot 10^{-4}}{3 \cdot 10^6 \cdot 10 \cdot 10^{12}\,\text{km}} \simeq 6 \cdot 10^{-29} \frac{1}{\text{cm}}$$

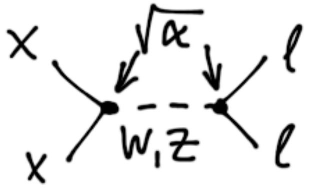
plug into (80):

$$\Omega_{x,0} \sim 10^{-94} \cdot \frac{10^{58}}{36} \cdot \frac{\text{cm}^2}{\langle \sigma_x v \rangle}$$

$$\Rightarrow \Omega_{\chi,0} \sim \frac{3\cdot 10^{-38}\,\mathrm{cm}^2}{\langle \sigma_\chi v\rangle} \quad (81)$$

for $\langle \sigma_\chi v\rangle$ we get:

σ_χ :



$\alpha \sim 0.01$ (EW interaction!)

$$\Rightarrow \mathcal{M} \sim \frac{\alpha}{M_W^2} \sim \alpha G_F$$

$$\Rightarrow \langle \sigma_\chi v\rangle \sim \alpha^2 G_F^2 m_\chi^2$$

$$= 10^{-4}\cdot\left(\frac{1}{(300\,\mathrm{GeV})^2}\right)^2\cdot(100\,\mathrm{GeV})^2\cdot\left(\frac{m_\chi}{100\,\mathrm{GeV}}\right)^2$$

$$\simeq 10^{-10}\,\mathrm{GeV}^{-2}\cdot\left(\frac{m_\chi}{100\,\mathrm{GeV}}\right)^2 \quad (82)$$

little conversion help:

$$\mathrm{GeV}^{-1} \sim 0.2\,\mathrm{fm} = 2\cdot 10^{-14}\,\mathrm{cm}$$

$$\Rightarrow \langle \sigma_\chi v\rangle \sim \left(\frac{m_\chi}{100\,\mathrm{GeV}}\right)^2\cdot 10^{-37}\,\mathrm{cm}^2$$

plug this into (81):

$$\Rightarrow \Omega_{\chi,0} \sim 0.3\cdot\left(\frac{100\,\mathrm{GeV}}{m_\chi}\right)^2 \quad (83)$$

$\rightsquigarrow$ A WIMP with mass of ~ 100 GeV, natural in EW-scale SUSY, is a perfect DM candidate!

III Structure formation, density perturbations & CMB

We will discuss the Newtonian theory of small perturbations, as the relevant length scales will be always be $\ll H^{-1}$ ...

Relevant equations of motion: 'fluid of non-relativistic dust'

$$\Rightarrow \begin{cases} \dot{\rho} + \vec{\nabla}(\rho \vec{v}) = \dot{\rho} + \rho \cdot \vec{\nabla} \vec{v} = 0 \quad , \rho = \rho(t) \\ \qquad\qquad \text{continuity eq.} \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\dfrac{1}{\rho} \vec{\nabla} p + \vec{g} \\ \qquad\qquad \text{Euler fluid e.o.m.} \end{cases} \qquad (84)$$

supplemented by the gravitational field equation:

$$\begin{cases} \vec{\nabla} \times \vec{g} = 0 \\ \vec{\nabla} \cdot \vec{g} = \Delta \phi = -4\pi G \rho \end{cases} \qquad (85)$$

using a few little tricks, like:

$$\vec{\nabla} \cdot \vec{r} = 3 \quad \text{and} \quad (\vec{r} \cdot \vec{\nabla}) \cdot \vec{r} = \vec{r}$$

one can show starting with an ansatz for $\vec{v}$ (Hubble's law):

$$\vec{v}_0(\vec{r}, t) = \vec{r} \cdot H(t) \, , \; H(t) = \frac{\dot{a}}{a} \qquad (86)$$

... that the continuity eq. is solved by:

$$\rho_0(t) = \rho_0 \cdot \left(\frac{a_0}{a(t)}\right)^3 \quad (87)$$

as befits non-relativistic matter.

For such ρ then (85) has the solution:

$$\vec{g}_0(t) = -\vec{r} \cdot \frac{4\pi G}{3} \cdot \rho_0(t) \quad (88)$$

and if we plug this into the Euler equation, we get the Friedmann eq.:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho_0(t) \quad (89)$$

which is solved by:

$$a(t) \sim t^{2/3} \quad (90)$$

again befitting non-relativistic matter.

Now perturb this background solution (86) – (88):

$$\left.\begin{aligned} \rho(\vec{r},t) &= \rho_0(t) + \rho_1(\vec{r},t) \\ v(\vec{r},t) &= \vec{v}_0(\vec{r},t) + \vec{v}_1(\vec{r},t) \\ \vec{g}(\vec{r},t) &= \vec{g}_0(t) + g_1(\vec{r},t) \end{aligned}\right\} \quad (91)$$

and write the perturbation in a spatial Fourier decomposition:

$$\rho_1(\vec{r},t) = \rho_1(t)\cdot e^{i\vec{k}_{phys.}\cdot\vec{r}} = \rho_1(t)\cdot e^{i\frac{\vec{k}}{a}\cdot\vec{r}} \qquad (92)$$

$\vec{k}$: comoving wave number

etc. ...

and plug this into the e.o.m. (84) & (85):

$$\Rightarrow \begin{cases} \dot{\rho}_1 + 3H\rho_1 + i\rho_0\cdot\dfrac{\vec{k}\cdot\vec{v}_1}{a(t)} = 0 \\ \dot{\vec{v}}_1 + H\cdot\vec{v}_1 = -i\,\dfrac{c_s^2}{\rho_0(t)}\cdot\dfrac{\vec{k}}{a(t)}\cdot\rho_1 + \vec{g}_1 \\ \vec{g}_1 = i\cdot 4\pi G\rho_1\cdot a(t)\cdot\dfrac{\vec{k}}{\vec{k}^2} \end{cases} \qquad (93)$$

here:

$$c_s = \sqrt{\frac{dp}{d\rho}} \quad \text{'speed of sound'} \qquad (94)$$

We see that:

$$\vec{v}_{1\perp} : \vec{v}_{1\perp}\cdot\vec{k} = 0,\ \vec{v}_{1\perp} \perp \vec{k}$$

falls as: $\sim \dfrac{1}{a(t)}$

For the other perturbations define:

$$\delta(t) \equiv \frac{\rho_1(t)}{\rho_0(t)} \quad \text{'density contrast'}$$

$$(95) \qquad \epsilon(t) = -i\cdot a(t)\cdot\frac{\vec{k}\cdot\vec{v}_1}{\vec{k}^2} \sim \vec{v}_{1\parallel}$$

in terms of these:

$$\left.\begin{aligned}\dot{\delta} &= \frac{\vec{k}^2}{a^2(t)}\cdot\epsilon \\ \dot{\epsilon} &= \left(-c_s^2 + \frac{4\pi G\rho_0(t)}{\vec{k}^2}\right)\cdot\delta\end{aligned}\right\} \quad (96)$$

can be combined:

$$\ddot{\delta} + 2H\cdot\dot{\delta} + \left[c_s^2\cdot\frac{\vec{k}^2}{a^2} - 4\pi G\cdot\rho_0(t)\right]\cdot\delta = 0 \tag{97}$$

The bracket [...] in (97) defines a critical "Jeans wave number" k_J:

$$\frac{k_J}{a(t)} = \sqrt{\frac{4\pi G\rho_0(t)}{c_s^2}} \tag{98}$$

its meaning:

i) $\delta \sim \cos\left(c_s\frac{k}{a}t\right)$ 'oscillatory'

for $k \gg k_J$

ii) $\delta \sim t^p$, $p>0$ 'growing and non-oscillatory'

for $k \ll k_J$

---

exist only in matter- or radiation-dominated regime, because for ρ_Λ or $\rho_{curv.}$ we have the damping term:

$2H\cdot\dot{\delta} \sim 2H^2\delta \sim \rho_{tot.}\,\delta$ falling