18.6.

II The Thermal Universe

thermal plasma of mostly relativistic particles at early times ...

$\leadsto$ need to recall a bit of thermodynamics and cross sections of scattering

density n, ρ and p of dilute gas with g internal d.o.f.:

$$\left.\begin{aligned} n &= \frac{g}{(2\pi)^3}\cdot\int d^3p\cdot f(\vec{p}) \\ \rho &= \frac{g}{(2\pi)^3}\cdot\int d^3p\cdot E(\vec{p})\, f(\vec{p}) \\ p &= \frac{g}{(2\pi)^3}\cdot\int d^3p\cdot\frac{|\vec{p}|^2}{3E(\vec{p})}\, f(\vec{p}) \end{aligned}\right\} \quad (23)$$

now: $|\vec{p}| = \sqrt{E^2 - m^2}$

$$\Rightarrow d|\vec{p}| = \frac{E\,dE}{\sqrt{E^2 - m^2}}$$

and $d^3p = 4\pi\,|\vec{p}|^2\, d|\vec{p}|$

$$\Rightarrow \quad n = \frac{g}{2\pi^2} \cdot \int_m^\infty dE \cdot E \cdot \sqrt{E^2 - m^2} \cdot f(E)$$

$$\rho = \frac{g}{2 \cdot \pi^2} \cdot \int_m^\infty dE \cdot E^2 \sqrt{E^2 - m^2} \cdot f(E) \tag{24}$$

$$p = \frac{g}{6\pi^2} \int_m^\infty dE \cdot \left(E^2 - m^2\right)^{3/2} \cdot f(E)$$

$f(E)$: partition function of particles

$$f(E) = \left(e^{\frac{E-\mu}{kT}} \pm 1\right)^{-1} \tag{25}$$

$+$: Fermi statistics

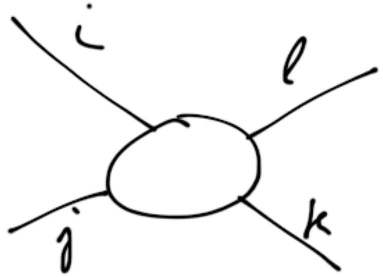
$-$: Bose statistics

μ: chemical potential

$\hat{=}$ change of partition function by $\sim e^{-\frac{\mu}{T}}$ for a newly added particle

consider a reaction of particle species i, j, k, l:

(26)



$i + j \to k + l$

$\rightsquigarrow \quad \mu_i + \mu_j = \mu_k + \mu_l$ chemical equilibrium relativistic limit $T \gg m$:

$$n = \begin{cases} \zeta(3)/\pi^2 \cdot g \cdot T^3 & \text{Bose} \\ \frac{3}{4} \cdot \zeta(3)/\pi^2 \cdot g T^3 & \text{Fermi} \end{cases}$$

$$\rho = \begin{cases} \frac{\pi^2}{30} \cdot g T^4 & \text{Bose} \\ \frac{7}{8} \cdot \frac{\pi^2}{30} \cdot g T^4 & \text{Fermi} \end{cases} \qquad (27)$$

$$p = \frac{1}{3} \rho$$

non-relativistic limit $T \ll m$:

$$n = g \left(\frac{mT}{2\pi} \right)^{3/2} \cdot e^{-\frac{m-\mu}{T}} \qquad (28)$$

$$\rho = m \cdot n \quad , \quad p = nT \ll \rho$$

instructive application:

$n_+ - n_- = n_B$ "baryon number density"
→ net particle over antiparticle excess

$\rightsquigarrow$ reaction $\bullet_+ + \bar{\bullet}_- \to 2\gamma$

if fast $\Rightarrow \mu_+ = -\mu_- \equiv \mu$

$$\Rightarrow n_+ - n_- = \frac{g}{2\pi^2} \int_m^\infty dE \cdot E \sqrt{E^2 - m^2} \qquad (29)$$

$$\times \left[\frac{1}{1 + e^{\frac{E-\mu}{T}}} - \frac{1}{1 + e^{\frac{E+\mu}{T}}} \right]$$

$$= \begin{cases} \frac{gT^3}{6\pi^2} \cdot \left[\pi^2 \cdot \frac{\mu}{T} + \mathcal{O}\left(\frac{\mu^2}{T^2} \right) \right], & T \gg m \\ 2g \left(\frac{mT}{2\pi} \right)^{3/2} \sinh(\mu T) \cdot e^{-\frac{m}{T}}, & T \ll m \end{cases}$$

measurable quantity:

$$\frac{n_B}{n_\gamma} \equiv \eta_B \quad \text{"baryon asymmetry"}$$

in the non-relativistic limit:

$$\eta_B = \frac{n_B}{n_\gamma} \sim \frac{e^{-m/T}}{T^3} \longrightarrow 0$$

for $T \ll m$ exponentially fast ...

$$\Rightarrow \eta_B(T_0) \sim \frac{e^{-\frac{m}{T_0}}}{T_0^3} \ll \eta_B^{obs.}$$

for $\eta_B^{obs.} \sim 10^{-9}$, $T_0 \simeq 3K$, $m \sim m_p \simeq 1 GeV$

$\rightsquigarrow$ need B-violation to generate matter-antimatter asymmetry!

total energy density of all species:

$\rightarrow$ species i with temperature T_i

$$\Rightarrow \rho_R = \sum_i \frac{g_i}{2\pi^2} \int_{m_i}^{\infty} dE \cdot E^2 \cdot \frac{\sqrt{E^2 - m_i^2}}{1 + e^{\frac{E-\mu_i}{T_i}}}$$

$$E \to u_i = \frac{E}{T_i}, \quad m_i \to x_i = \frac{m_i}{T_i}$$

$$\mu_i \to y_i = \frac{\mu_i}{T_i}$$

$$= T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} du_i\, u_i^2 \frac{\sqrt{u_i^2 - x_i^2}}{1 + e^{u_i - y_i}} \tag{30}$$

$\rightsquigarrow$ P_R analogously.

non-relat. species exponentially damped in ρ (eq. (28)) ...

$\Rightarrow$ good approximation:

ρ_R only from relativstic species

$$\Rightarrow \quad \rho_R = \frac{\pi^2}{30}\, g_* \cdot T^4 \qquad (31)$$

with:

$$g_* = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{j=\text{fermions}} g_j \cdot \left(\frac{T_j}{T}\right)^4 \qquad (32)$$

species count:

i) $T \ll m_e \Rightarrow$ only γ, ν's

$$\Rightarrow g_* = \underbrace{1}_{\gamma} \cdot \underbrace{2}_{\text{spin}} + \frac{7}{8} \cdot \underbrace{3}_{\substack{\text{righthanded } \nu\text{'s} \\ \text{3 families}}} \cdot \underbrace{2}_{\text{spin}} \cdot \left(\frac{4}{11}\right)^{4/3} \simeq 3.36$$

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$$

from neutrino decoupling (see below)

ii) $100\,\text{MeV} \gtrsim T \gtrsim m_e \simeq \text{MeV} \Rightarrow \gamma, \nu\text{'s}, e^\pm$

$$\Rightarrow g_* = \underbrace{1}_{\gamma}\cdot\underbrace{2}_{\text{spin}} + \frac{7}{8}\Big(\underbrace{2}_{e^- \& e^+}\cdot\underbrace{2}_{\text{spin}} + \underbrace{3}_{\text{rh } \nu\text{'s, 3 families}}\cdot\underbrace{2}_{\text{spin}}\Big)$$

$$= 10.75$$

$\leadsto g_*$ increases with T ...

iii) $T \gtrsim 300\,\text{GeV} \Rightarrow$ the whole SM

$$\Rightarrow g_* = (\underbrace{8}_{\text{gluons}}+\underbrace{3}_{Z^0,W^\pm}+\underbrace{1}_{\gamma})\cdot\underbrace{2}_{\text{spin}} + \underbrace{4}_{\text{Higgs}} + \frac{7}{8}\Big(\underbrace{3}_{\text{rh } \nu\text{'s, 3 families}} + \underbrace{3}_{e,\mu,\tau}\cdot\underbrace{2}_{\text{spin}} + \underbrace{3}_{\text{colors}}\cdot\underbrace{3}_{\text{quarks, 3 families}}\cdot\underbrace{2}_{\text{particles \& antipart.}}\Big)\cdot\underbrace{2}_{\text{spin}}$$

$$= 106.75$$

early on: radiation domination

$$\leadsto \rho \simeq \rho_R = \frac{\pi^2}{30} g_* T^4$$

$$\Rightarrow H = \frac{\pi}{\sqrt{90}} g_*^{1/2} \cdot \frac{T^2}{M_P} \qquad (33)$$

with: $M_P = \sqrt{\frac{\hbar c^5}{8\pi G}} \simeq 2.4 \cdot 10^{18} \frac{\text{GeV}}{c^2}$

and: $H = \frac{1}{2t}$

$$\Rightarrow t = \frac{\sqrt{90}}{2\pi} \cdot g_*^{-1/2} \cdot \frac{M_P}{T^2} \sim \frac{1}{T^2} \qquad (34)$$

Thermodynamics of expansion:

2^{nd} law always holds.

$$\Rightarrow \quad T dS = dE + p\,dV \,, \quad S: \text{entropy}$$

$$= d(\rho V) + p\,dV$$

$$= d\left[(\rho + p)V\right] - V\,dp \qquad (35)$$

Maxwell relations:

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$$

and apply to 2^{nd} law ...

$$dS = \frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial T} dT$$

$$= \frac{1}{T} d\left[(\rho + p)V\right] - \frac{V}{T} dp$$

$$= \frac{\rho + p}{T} dV + \frac{V}{T} d\rho$$

We know: $\rho = \rho(T)$, $p = p(T)$

$$\Rightarrow \quad dS = \frac{\rho + p}{T} dV + \frac{V}{T} \frac{d\rho}{dT} \cdot dT$$

$$\Rightarrow \quad \frac{\partial S}{\partial V} = \frac{\rho + p}{T} \,, \quad \frac{\partial S}{\partial T} = \frac{V}{T} \frac{d\rho}{dT}$$

$$\Rightarrow \quad \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right) = \frac{1}{T} \frac{d\rho}{dT}$$

$$\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right) = -\frac{\rho+p}{T^2} + \frac{1}{T}\frac{dp}{dT} + \frac{1}{T}\cdot\frac{d\rho}{dT}$$

thus:

$$0 = \frac{\partial^2 S}{\partial T \partial V} - \frac{\partial^2 S}{\partial V \partial T} = -\frac{\rho+p}{T^2} + \frac{1}{T}\frac{dp}{dT}$$

$$\Leftrightarrow \quad \frac{dp}{dT} = \frac{\rho+p}{T} \qquad (36)$$

$$(35) \Rightarrow dS = \frac{1}{T} d\left[(\rho+p)V\right] - \frac{\rho+p}{T^2} V\,dT = d\left[\frac{(\rho+p)V}{T}\right]$$

$$\Rightarrow S = \frac{\rho+p}{T}\cdot a^3(t) \qquad (36).$$

first law:

$$dE = -p\,dV$$ in a closed systems

$\leftrightarrow$ total universe

$$\Rightarrow \boxed{dS = 0}$$

Entropy is conserved during expansion! (37)

$$\Rightarrow S = \frac{\rho+p}{T}\cdot a^3 = \text{const.}$$

$$\Rightarrow s = \frac{S}{V} = \frac{S}{a^3} = \frac{\rho+p}{T} \sim \frac{1}{a^3} \qquad (38)$$

for relativistic species thus:

$$p_R = \frac{1}{3}\rho_R$$

$$\Rightarrow s = \frac{\rho_R + p_R}{T} = \frac{4}{3}\cdot\frac{\rho_R}{T}$$

$$= \frac{2\pi^2}{45}\cdot g_{*s}\cdot T^3 \qquad (39)$$

$$g_{*s} = \sum_{i=\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{j=\text{fermions}} g_j\left(\frac{T_j}{T}\right)^3$$

(40)

note:

$$s \sim \frac{1}{a^3} \Rightarrow \boxed{N = V\cdot n = \frac{n}{s}}$$

particle number per comoving volume

(41)

and:

$$S = s a^3 = \text{const.} \Rightarrow \boxed{T \sim \frac{1}{a}} \quad (42)$$