6.6.

~ need $H(z)$ – use Friedmann eq.:

$$t_0: \; H_0^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}\rho_0 = H_0^2 \cdot \Omega_0 \qquad \Omega_0 = \frac{\rho_0}{\rho_{crit}}$$

$$t_z: \; H^2(z) + \frac{k}{a^2(t_z)} = \frac{8\pi G}{3}\cdot \rho(z) = H_0^2 \cdot \Omega_0 \cdot \frac{\rho(z)}{\rho_0}$$

and:

$$\frac{k}{a^2(t_z)} = \frac{k}{a_0^2}\cdot (1+z)^2$$

$$\Rightarrow H(z) = \sqrt{-\frac{k}{a_0^2}(1+z)^2 + H_0^2 \Omega_0 \frac{\rho(z)}{\rho_0}}$$

and $-\frac{k}{a_0^2} = H_0^2(1-\Omega_0)$

$$\Rightarrow H(z) = H_0 \sqrt{(1-\Omega_0)(1+z)^2 + \Omega_0 \cdot \frac{\rho(z)}{\rho_0}} \qquad (22)$$

$$1 - \Omega_0 = \Omega_k$$

and: $$\Omega_0 \frac{\rho(z)}{\rho_0} = \Omega_M^0 (1+z)^3 + \Omega_r^0 (1+z)^4 + \Omega_\Lambda$$

$\downarrow$

$m - M$ vs. z – graph

eq. (20) $\Rightarrow$ expansion age t_0 of the universe:

$$dt = -\frac{dz}{H(z)\cdot(1+z)}$$

$$\Downarrow$$

$$t_0 = \int_0^{z_{dec.}} \frac{dz'}{(1+z')\cdot H(z')} \qquad (22)$$

and use eq. (21) for $H(z)$.

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recall of the action principle in GR & Euler-Lagrange eq.s. of motion:

i) classical mechanics:

action $S = \int dt \cdot \mathcal{L}(x, \dot{x})$

Lagrangian $\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V(x)$

e.o.m.: $\frac{\delta S}{\delta x} = 0$ variational derivative

$\Rightarrow$ write putative solution $x = x_0(t)$ with variation δx:

$$x = x_0 + \delta x$$

$$\Rightarrow \dot{x}^2 = \dot{x}_0^2 + 2 \dot{x}_0 \delta \dot{x} + \mathcal{O}(\delta x^2)$$
$$= \dot{x}_0^2 + 2 \frac{d}{dt}(\dot{x}_0 \delta x) - \ddot{x}_0 \cdot \delta x + \mathcal{O}(\delta x^2)$$

$$V(x) = V(x_0) + \frac{\partial V}{\partial x} \delta x + \mathcal{O}(\delta x^2)$$

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$$\Rightarrow S = \int dt \, \mathcal{L}$$
$$= \int dt \left[\frac{1}{2} m \dot{x}_0^2 - V(x_0) - \delta x \left(\ddot{x}_0 + \frac{\partial V}{\partial x} \right) + \mathcal{O}(\delta x^2) \right]$$

$$\Rightarrow \frac{\delta S}{\delta x} = 0 \Leftrightarrow m \cdot \ddot{x}_0 = - \frac{\partial V}{\partial x}$$
$$(m \cdot a = F)$$

ii) now field theory:

scalar field ϕ

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi)$$

$$\phi = \phi_0(x^\mu) + \delta \phi$$

$$\Rightarrow \delta S = - \int d^4x \cdot \delta\phi \cdot \left[\partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} \right] + \mathcal{O}(\delta \phi^2)$$

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$$\frac{\delta S}{\delta \phi} = 0$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \phi = -\frac{\partial V}{\partial \phi}$$

iii) GR:

variation w.r.t. $g_{\mu\nu}$

$\rightsquigarrow \quad g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$

$\rightsquigarrow$ need to vary $g \equiv \det g_{\mu\nu}$

and $\sqrt{-g}$

result:

$$\delta g = g \cdot g^{\mu\nu} \delta g_{\mu\nu}$$

$$\Rightarrow \delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu}$$

plausibility argument:

$\rightsquigarrow$ assume $g_{\mu\nu}$ diagonal

($g_{\mu\nu}$ is real-symmetric, so can be always diagonalized)

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$$\Rightarrow g_{\mu\nu} = f_\mu \eta_{\mu\nu} = g_{\mu\mu} \eta_{\mu\nu}$$

$$\Rightarrow g = \prod_\mu g_{\mu\mu}$$

$$\Rightarrow \delta g = \sum_{\mu=0}^{3} \prod_{\nu\neq\mu} g_{\nu\nu}\, \delta g_{\mu\mu} = \underbrace{\left(\prod_\nu g_{\nu\nu}\right)}_{=g} \frac{1}{g_{\mu\mu}} \delta g_{\mu\mu}$$

if $g_{\mu\nu} = f_\mu \eta_{\mu\nu}$

$$\Rightarrow g^{\mu\nu} = \left(g^{-1}\right)^{\mu\nu} = \frac{1}{f_\mu} \eta^{\mu\nu} = \frac{1}{g_{\mu\mu}} \eta^{\mu\nu}$$

$$\Rightarrow \delta g = g \cdot \frac{1}{g_{\mu\mu}} \delta g_{\mu\mu} = g \cdot g^{\mu\mu} \delta g_{\mu\mu}$$

⋮

$$\rightsquigarrow \delta g = g \cdot g^{\mu\nu} \delta g_{\mu\nu}$$

The full gravitational + matter action is:

$$S = S_{E-H} + S_{matter}$$

$$= \underbrace{\int d^4x \sqrt{-g}\left[\frac{1}{16\pi G_N} R}_{S_{E-H}} + \mathcal{L}_{matter}\right]$$

e.g.: $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$

$$\frac{\delta S}{\delta\phi} = 0 : \text{ e.o.m. for } \phi.$$

we now need to vary w.r.t. $g_{\mu\nu}$:

a) $\delta(\sqrt{-g}R) = (\delta\sqrt{-g})\cdot R + \sqrt{-g}\cdot\delta \underbrace{R}_{= R_{\mu\nu}g^{\mu\nu}}$

$$= \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}R\cdot\delta g_{\mu\nu} + \sqrt{-g}\left(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)$$

$$= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}R\cdot\delta g^{\mu\nu} + \sqrt{-g}\cdot\left(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)$$

b) $\delta(\sqrt{-g}\mathcal{L}_{matter}) =$

$$= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\mathcal{L}_{matter}\cdot\delta g^{\mu\nu} + \sqrt{-g}\,\frac{\partial\mathcal{L}_{matter}}{\partial g^{\mu\nu}}\delta g^{\mu\nu}$$

$$= \delta g^{\mu\nu}\cdot\underbrace{\sqrt{-g}\left(\frac{\partial\mathcal{L}_{matter}}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}\mathcal{L}_{matter}\right)}_{\equiv T_{\mu\nu}}$$

$$\vdots$$

$$= \delta g_{\mu\nu}\cdot\underbrace{\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{matter})}{\delta g^{\mu\nu}}}_{\equiv T_{\mu\nu}}$$

finally: $\delta R_{\mu\nu}$ is a total derivative, so if space-time has no boundary, then:

$$\int d^4x \sqrt{-g}\, \delta R_{\mu\nu}\, g^{\mu\nu} = 0$$

$$\Rightarrow \delta S = - \int d^4x \sqrt{-g}\, \delta g^{\mu\nu} \cdot \left[\left(-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \right) \frac{1}{16\pi h} + T_{\mu\nu} \right]$$

$$+ \mathcal{O}((\delta g^{\mu\nu})^2)$$

$$\curvearrowright \frac{\delta S}{\delta g^{\mu\nu}} = 0$$

$$\Downarrow$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -16\pi h \cdot T_{\mu\nu}$$

<u>Einstein field equations</u>