To derive the dynamics of the expansion we will apply Einstein's equations to the FRW space-time. This yields, defining the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

with the FRW metric in the form of eq. (4):

$$G_{00} = 3 \cdot \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] \qquad (9)$$

$$G_{ij} = g_{ij} \cdot \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right]$$

where the metric (4) is written as:

$$ds^2 = dt^2 - a^2(t) \cdot g_{ij}\, dx^i dx^j .$$

For simplification and consistency with spatial homogeneity & isotropy energy and matter is often described by an energy-momentum tensor of a perfect fluid with energy density ρ, pressure p and 4-velocity u^α:

$$T^{\alpha\beta} = (\rho + p) \cdot u^\alpha u^\beta - p \cdot g^{\alpha\beta} . \qquad (10)$$

As the fluid is comoving with the expansion, it is at rest in the coordinates of the FRW metric (4):

$$u^\alpha = (1, 0, 0, 0)$$

$$\Rightarrow T^{\alpha\beta} = \begin{pmatrix} \rho & \\ & -p \cdot g^{ij} \end{pmatrix} \qquad (11)$$

and this, together with the LHS of the Einstein equations (9), gives us the Friedmann equations:

$$\begin{cases} 00: & \dfrac{\dot{a}^2}{a^2} \equiv H^2 = \dfrac{8\pi G}{3}\cdot\rho - \dfrac{k}{a^2} \\ ij: & 2\dfrac{\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2} + \dfrac{k}{a^2} = -8\pi G\cdot p \end{cases} \qquad (12)$$

$$\Rightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\cdot(\rho + 3p) \qquad (12')$$

"acceleration equation"

We see that the universe undergoes accelerated expansion, if:

$$p < -\frac{1}{3}\rho \quad .$$

The Friedmann equations are to be supplemented by energy conservation:

$$dE = d(\rho V) = -p\,dV \qquad (13)$$

where the cosmological volume V:

$$V \sim a^3(t)$$

$$\Rightarrow \quad d(\rho\cdot a^3) = -p\cdot d(a^3)$$

$$\Rightarrow \quad \frac{d}{dt}(\rho a^3) = a^3\left(\dot{\rho} + 3\rho\cdot\frac{\dot{a}}{a}\right) = -a^3\cdot 3p\cdot\frac{\dot{a}}{a}$$

$$\Leftrightarrow \quad \dot{\rho} + 3H\cdot(\rho + p) = 0 . \qquad (14)$$

Finally, we need an equation of state $p = p(\rho)$ for the description of matter to complete, and this is often well approximated by:

$$p = w \cdot \rho \quad . \qquad (15)$$

examples are:

$w = 0$ "dust" - nonrelativistic massive matter

$w = \frac{1}{3}$ radiation (photons...)

$w = -\frac{1}{3}$ the 3-curvature k

$w = -1$ cosmological constant or constant scalar potential

The case of a scalar field is particularly interesting concerning cosmological inflation. For a scalar field ϕ with potential $V(\phi)$ and Lagrangian:

$$\sqrt{-g}\,\mathcal{L} = \sqrt{-g} \cdot \left[\frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)\right]$$

one can show that:

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}\partial^\sigma\phi\,\partial_\sigma\phi - V(\phi)\right]$$

and from there:

$$\rho = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + V(\phi)$$

$$p = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) \quad .$$

If we can arrange for a situation where for a prolonged time:

$$\partial^\mu \phi \, \partial_\mu \phi \ll V(\phi)$$

$$\Rightarrow \quad p \simeq -\rho$$

and the scalar potential acts like a cosmological constant, which by definition is:

$$T_{\mu\nu} = \Lambda \cdot g_{\mu\nu} \qquad (16)$$

and thus has exactly:

$$p = -\rho \, .$$

We can now use the equation of state $p = w\rho$ together with energy conservation (14) to determine the scaling of the energy density ρ of a given form of matter or radiation with $a(t)$ due to the expansion:

$$\rho \sim a^{-3(1+w)} \, . \qquad (17)$$

For different sources this behaves like:

$$\rho \sim a^{-3} \quad \text{"dust"}$$
$$\rho \sim a^{-4} \quad \text{radiation}$$
$$\rho \sim a^{-2} \quad \text{curvature}$$

and

$$\rho \sim \text{const. for a c.c.}$$

↳ 2 important consequences:

i) A cosmological constant does not dilute under expansion.

ii) Any residual spatial curvature eventually comes to dominate over dust or radiation in an expanding universe, if we wait long enough.

Now look at the first Friedmann equation in (12):

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad .$$

We see that a spatially flat universe, having $k=0$, corresponds to a critical density $\rho_{cr.}$:

$$k=0 \quad \Rightarrow \quad \rho_{cr.} \equiv \frac{3H^2}{8\pi G} \qquad (18)$$

We can thus divide the first Friedmann equation by:

$$\frac{8\pi G}{3}\rho_{cr.}$$

define a cosmological density parameter:

$$\Omega = \frac{\rho}{\rho_{cr.}}$$

$$\Omega_k = \frac{\rho_k}{\rho_{cr.}}, \quad \rho_k = -\frac{3k}{8\pi G a^2}$$

$$= -\frac{k}{a^2 H^2}$$

$$\Rightarrow \quad \Omega + \Omega_k = 1 \quad . \qquad (19)$$

We see that:

$$\Omega_k \begin{cases} >0, & k=-1 \text{ 'open'} \\ <0, & k=+1 \text{ 'closed'} \end{cases}$$

implies that an open universe has:

$$\Omega < 1$$

and a closed one:

$$\Omega > 1 \ .$$

Here Ω contains all energy and matter which is not curvature, and if we divide up ρ into:

$$\rho = \rho_{rad.} + \rho_{matter} + \rho_\Lambda$$

then:

$$\Omega = \Omega_{rad.} + \Omega_{matter} + \Omega_{\Lambda} .$$

We can now use the results for scaling of different forms of energy density with a to determine $a(t)$ from the first Friedmann equation:

$$w = 0 : \quad \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^3} \quad \Rightarrow a(t) \sim t^{2/3}$$
$$\Rightarrow H(t) = \frac{2}{3t}$$

$$w = \frac{1}{3} : \quad \frac{\dot{a}^2}{a^2} \sim \frac{1}{a^4} \quad \Rightarrow a(t) \sim \sqrt{t}$$
$$\Rightarrow H(t) = \frac{1}{2t}$$

$$w = -1 : \quad \frac{\dot{a}}{a} = \text{const.} \Rightarrow a(t) = e^{H \cdot t}$$
$$\Rightarrow H = \text{const.}$$

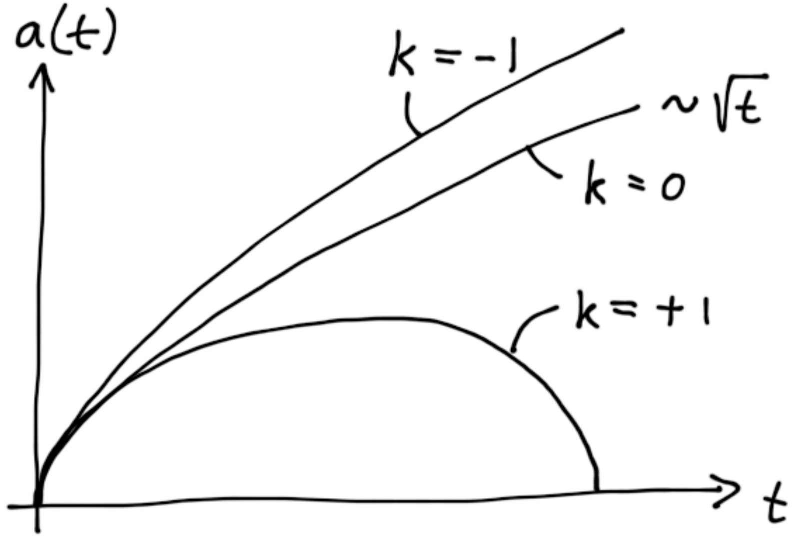
This describes a de Sitter space time dS_4.

all in the case of a flat universe $k = 0$.

In the presence of spatial curvature these results change.

In particular, a universe whose

expansion is driven by dust or radiation, re-collapses if its closed ($k=+1$), and expands forever if flat or open:



radiation-dominated universe

We see, that in a matter or radiation dominated universe the Hubble parameter:

$$H(t) \sim \frac{1}{t} \xrightarrow[t \to 0]{} \infty$$

diverges at a finite time in the past.

Now as the Hubble parameter:

$$H^2 = \frac{\dot{a}^2}{a^2} \subset G_{00}, G_{ij} \hat{=} \text{curvature}$$

is linked to the curvature of the universe, the diverging $H(t)$ with

$t \to 0$ entails diverging curvature of space-time. Thus, a curvature singularity – a point of infinite curvature, energy density, and pressure – ensues at $t \to 0$, which is a point where space-time itself ceases to exist.

The Hawking-Penrose singularity theorems state for this situation, that for an expanding universe driven by energy or matter with:

$$p > -\frac{1}{3}\rho$$

Such a curvature singularity to the past (and for a $k = +1$ closed universe also for the future) cannot be avoided.

$$\downarrow$$

Inflation with a scalar field, or a cosmological constant, with:

$$p \simeq -\rho$$

thus may provide a way out from the inevitability of a past curvature singularity.