2. inflationary curvature perturbation

now, in slow-roll :

inflaton jump $\delta\phi$

$$\Rightarrow \text{ need } \delta N = H \cdot \delta t = \frac{H}{\dot{\phi}} \cdot \delta\phi$$

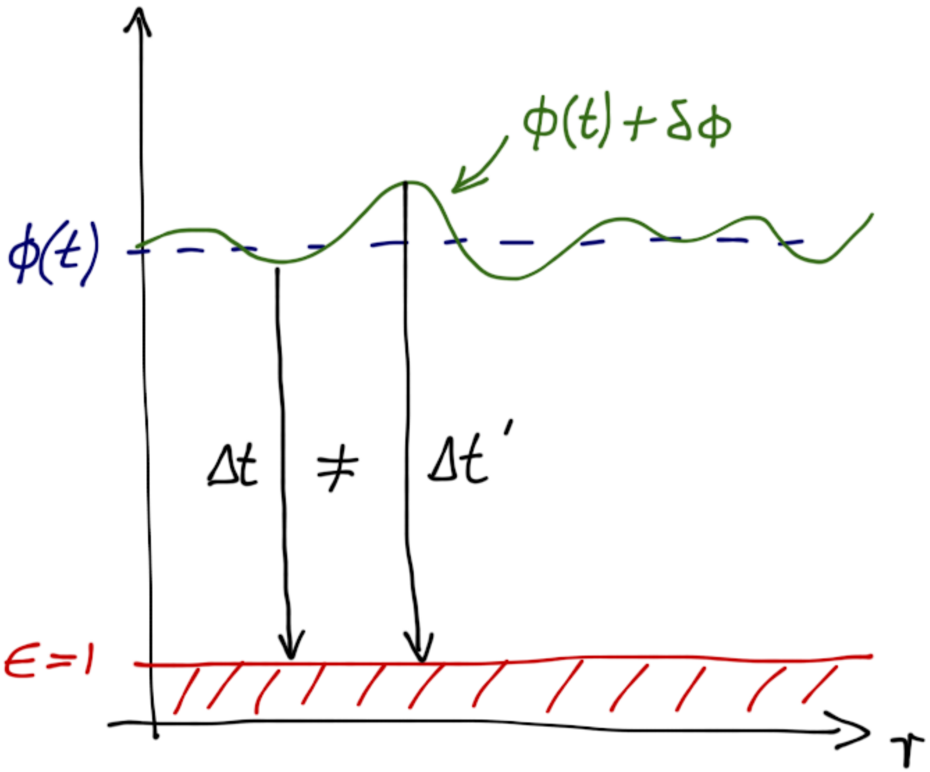
more/less e-folds to reach reheating

$$\Rightarrow ds^2 = dt^2 - e^{2H\cdot t} \cdot d\vec{x}_3^{\,2}$$

$$\longrightarrow ds^2 = dt^2 - e^{2H(t+\delta t)} \cdot d\vec{x}_3^{\,2}$$

$$= dt^2 - e^{2Ht} \cdot (1 + 2 \cdot \delta N) \cdot d\vec{x}_3^{\,2}$$

we can display this graphically :



compare: $\zeta = \delta N = \frac{H}{\dot\phi} \cdot \delta\phi$

gauge-invariant
'curvature perturbation' induced
by inflaton fluctuation $\delta\phi$

$$\curvearrowright \quad \zeta^2 = \frac{H^2}{\dot\phi^2} \delta\phi^2$$

$$\Rightarrow \Delta^2_\zeta = \frac{H^2}{\dot\phi^2} \Delta^2_\phi = \frac{H^4}{4\pi^2 \dot\phi^2}$$

in slow-roll: $\dot\phi = -\frac{V'}{3H}$

$$\Rightarrow \Delta^2_\zeta = \frac{1}{12\pi^2} \cdot \frac{V^3}{V'^2} = \frac{1}{24\pi^2} \cdot \frac{V}{\epsilon}$$

can show:

$$\frac{\delta\rho}{\rho} = \frac{2}{5}\sqrt{\Delta^2_\zeta} \rightarrow \frac{\Delta T}{T} \text{ of CMB}$$

$$\downarrow$$

$$\boxed{\begin{array}{l}\text{CMB:} \\ \frac{\Delta T}{T} \sim 10^{-5} \\ \text{measures } \frac{V}{\epsilon}\end{array}}$$

in exact dS:

$$\Delta^2_\phi = \text{const.}$$

in slow-roll can parametrize:

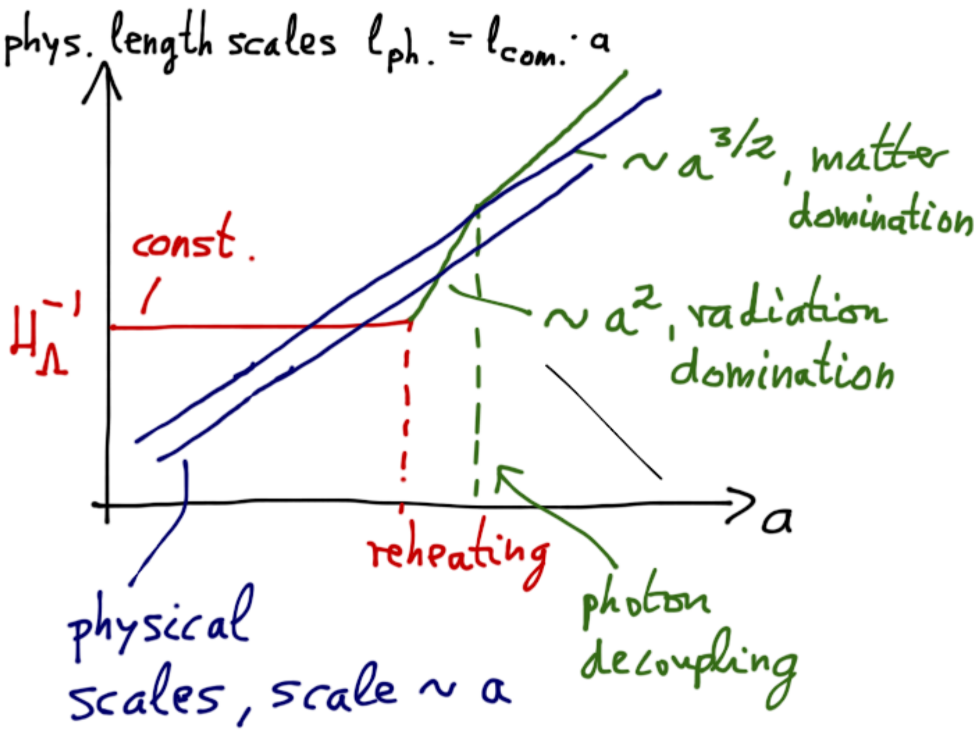
$$\Delta^2_\zeta(k) = \Delta^2_\zeta(k_0) \cdot \left(\frac{k}{k_0}\right)^{n_s - 1}$$

$\nwarrow$ spectral tilt

expand:

$$\ln \Delta^2_\zeta(k) = \ln \Delta^2_\zeta(k_0) + \frac{d \ln \Delta^2_\zeta(k_0)}{d \ln k} \cdot \ln \frac{k}{k_0}$$

$$+ \ldots$$



physical wavelengths $\lambda_{ph} \sim \frac{a}{k}$ of a given comoving wavenumber k leave horizon during inflation at different times $t_i = t(k_i)$ for different comoving wavenumbers k_i:

$$\lambda_{ph.} = H^{-1} \Rightarrow t(k_i) \sim H^{-1} \cdot \ln\left(\frac{k_i}{H}\right)$$

$\Rightarrow$ comoving wavenumber k is a time parameter

can compute evolution of Δ_R^2 with time as:

$$\frac{d\Delta_\zeta^2}{dt} = \frac{d\Delta_\zeta^2}{d\ln k}$$

$$\Rightarrow n_s - 1 = \left.\frac{d\ln\Delta_\zeta^2}{d\ln k}\right|_{k=k_0=aH}$$

relation between comoving wave number k and physical wave number $k_{phys.}$:

$$k_{phys.} = \frac{k}{a} = k \cdot e^{-N_e}$$

$$\Rightarrow d\ln k = dN_e$$

$$\Rightarrow n_s - 1 = \left.\frac{d\ln\Delta_\zeta^2}{dN_e}\right|_{N_e \simeq 60}$$

$$\frac{d\ln \Delta_s^2}{dN_e} = \frac{1}{\Delta_s^2} \cdot \frac{d\Delta_s^2}{dN_e}$$

$$= 12\pi^2 \cdot \frac{V'^2}{V^3} \cdot \frac{d\phi}{dN_e} \cdot \frac{d}{d\phi}\left(\frac{1}{12\pi^2} \frac{V^3}{V'^2}\right)$$

$$\frac{d\phi}{dN_e} = \dot{\phi} \cdot \frac{dt}{dN_e} = \frac{\dot{\phi}}{H} = -\frac{V'}{3H^2} = -\frac{V'}{V}$$

$$= -\frac{V'^3}{V^4} \cdot \left(3\frac{V^2}{V'} - 2\frac{V^3}{V'^3}V''\right)$$

$$= -6 \cdot \left(\frac{1}{2} \frac{V'^2}{V^2}\right) + 2 \cdot \frac{V''}{V} = -6\epsilon + 2\eta$$

$$\Rightarrow n_s = 1 - 6\epsilon + 2\eta$$

WMAP: $n_s = 0.963 \pm 0.013$, 1σ

PLANCK: $\Delta n_s = 0.005$ at 2σ

SPT (South Pole Telescope) 2012 & WMAP + BAO (baryon acoustic oscillations) + HST (Hubble space telescope measurement of H today):

$$n_s = 0.954 \pm 0.0081, \ 1\sigma$$

(red-tilt at 5σ!)

3. inflation also seeds primordial gravitational waves

$\sim$ tensor perturbations of the dS metric

$$ds^2 = (1-2\zeta)dt^2 - a^2\left[(1+2\zeta)\delta_{ij} + h_{ij}\right]dx^i dx^j$$

tensor perturbations

$\Downarrow$ expanding out Einstein-Hilbert action

$$S = \frac{1}{8}\int d^4x \, \frac{1}{\eta^2 H^2}\left(h_{ij}' h^{ij\,\prime} - (\vec{\nabla} h_{ij})^2\right)$$

$\Rightarrow$ $\tilde{h}_{ij} = \frac{1}{2} h_{ij}$ like massless scalar field in de Sitter

Fourier decompose:

$$\tilde{h}_{ij} = \int \frac{d^3k}{(2\pi)^3} e_{ij} \, \tilde{h}_k \, e^{i\vec{k}\vec{r}}$$

polarization tensor

$\Downarrow$

e.o.m.:

$$\tilde{h}_k'' - \frac{2}{\eta}\tilde{h}_k' + k^2 \cdot \tilde{h}_k = 0$$

$$\Rightarrow \tilde{h}_k = H \cdot \frac{1 - ik\eta}{\sqrt{2k^3}} e^{ik\eta}$$

$\Rightarrow$ quantization of $\tilde{h}_k$ as for ϕ_k

$$\Rightarrow \Delta_t^2 = 2 \cdot \Delta_h^2 = \frac{|h_k^2| \cdot k^3}{\pi^2} = \frac{2}{\pi^2} H^2$$

2 polarizations

no further 'translation factor' unlike Δ_ζ^2 – h_k is already a metric perturbation ...

define tensor-to-scalar ratio r:

$$\boxed{r = \frac{\Delta_t^2}{\Delta_\zeta^2} = 8\,\frac{\dot\phi^2}{H^2} = 16\,\epsilon}$$

example: $V \sim \phi^p$

$n_s = 1 - \frac{2+p}{2N_e} \simeq 0.97$

$r = \frac{4p}{N_e} = 0.13$ for $p = 2$.

measuring r determines ϵ, and via $\delta\rho/\rho$ from $\Delta T/T$ the scale of inflation V!

2nd significance of r:

compute $N_e = \int H\,dt = \int \frac{d\varphi}{\sqrt{2\epsilon}}$

$$\Rightarrow \quad N_e \simeq \frac{\Delta\phi}{M_p} \cdot \frac{1}{\sqrt{2\epsilon}}$$

$$\Leftrightarrow \quad r = 16\,\epsilon \simeq \frac{8}{N_e^2} \cdot \left(\frac{\Delta\phi}{M_p}\right)^2$$

$$\Rightarrow \quad \boxed{r \simeq 0.003 \cdot \left(\frac{50}{N_e}\right)^2 \cdot \left(\frac{\Delta\phi}{M_p}\right)^2}$$

'Lyth bound'

$\rightsquigarrow$ $r \sim 0.01$ corresponds to boundary between large-field and small-field inflation.

how to measure r ?

i) Δ_h^2 converts into $\frac{\Delta T}{T}$ at large angular scales $> 10^0$

$\rightarrow$ WMAP + SPT bound: $r \lesssim 0.11$

ii) B-mode polarization of CMB:

$\rightarrow$ look for curl-like pattern of polarization vectors around cold spot in CMB:

 instead of  or 

$\curvearrowright$ PLANCK : $r \lesssim 0.03 \ldots 0.05$ (extended run, 2.5 yr)

similar range for ground-based: QUIET, Keck array, Spider, SPT

by 2013-2014 : 3 yrs!

$\curvearrowright$ observational reach on r $\hat{=}$ small-field / large-field boundary ...

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2 examples

i) $V(\phi) = \frac{1}{2} m^2 \phi^2$

$$\epsilon(\phi) = \frac{2}{\phi^2}, \quad \eta(\phi) = \frac{2}{\phi^2} \Rightarrow \phi_{end} = \sqrt{2}$$

$$N_e = \int_0^{\phi} \frac{d\phi}{\sqrt{2\epsilon}} = \frac{\phi^2}{4} \Rightarrow \phi_{N_e} = 2\sqrt{N_e} \gg 1$$

"large-field"

$$\Rightarrow \quad \epsilon = \frac{1}{2N_e}, \quad \eta = \frac{1}{2N_e}$$

$$\Rightarrow \begin{cases} n_s = 1 - 6\epsilon + 2\eta = \frac{2}{N_e} \\ r = 16\epsilon = \frac{8}{N_e} \end{cases}$$

$$N_e = 60 \Rightarrow \begin{cases} n_s \simeq 0.967 \\ r \simeq 0.13 \end{cases}$$

ii) $V(\phi) = V_0\left(1 - \frac{\lambda_3}{3}\phi^3\right), \quad \lambda_3 \gg 1$

$$\epsilon(\phi) = \frac{1}{2}\lambda_3^2\phi^4, \quad \eta(\phi) = -2\lambda_3\phi$$

$$\Rightarrow \phi_{end} = \frac{2^{1/4}}{\sqrt{\lambda_3}} \ll 1 \text{ for } \lambda_3 \gg 1$$

"small-field"

$$N_e = \frac{1}{\lambda_3\phi} \Leftrightarrow \phi_{N_e} = \frac{1}{\lambda_3 N_e} \ll \phi_{end}$$

$$\Rightarrow \quad \epsilon = \frac{1}{2\lambda_3^2 N_e^4}, \quad \eta = -\frac{2}{N_e}$$

$$\Rightarrow \begin{cases} n_s = 1 - 6\epsilon + 2\eta \simeq 1 + 2\eta = 1 - \frac{4}{N_e} \\ r = 16\epsilon = \frac{8}{\lambda_3^2 N_e^4} \end{cases}$$

$$N_e = 60 : \begin{cases} n_s \simeq 0.93 \\ r \simeq \frac{1}{\lambda_3^2} \cdot 7 \cdot 10^{-6} \ll 7 \cdot 10^{-6} \end{cases}$$

4. CMB & curvature perturbation

Cosmic Microwave Background (CMB)

→ Planck spectrum with:

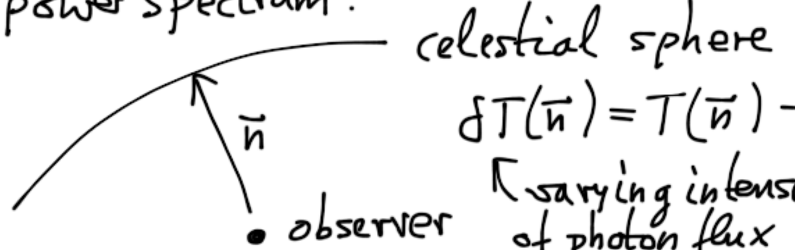
$$T = (2.726 \pm 0.001)\,K$$

anisotropies: $\frac{\Delta T}{T} \sim 10^{-3}$ 'dipole'

$\frac{\Delta T}{T} \sim 10^{-5}$ 'primordial'

needed for galaxy formation

power spectrum:



$$\delta T(\vec{n}) = T(\vec{n}) - \underbrace{T_0}_{\text{average}}$$

↖ varying intensity of photon flux

$$\theta(\vec{n}) \equiv \frac{\delta T(\vec{n})}{T_0} = \sum_{\ell} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\vec{n})$$

$$Y^*_{\ell,m} = (-1)^m Y_{\ell,-m} \quad , \quad \delta T = \delta T^*$$

$$\Rightarrow \quad a^*_{\ell m} = (-1)^m \cdot a_{\ell,-m}$$

$$a_{\ell m} = \int d\vec{n} \cdot \theta(\vec{n})\, Y^*_{\ell m}(\vec{n})$$

fluctuations represent departure from isotropy; for random fluctuations & on average isotropy:

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = C_\ell \cdot \delta_{\ell\ell'} \delta_{mm'}$$

i.e. on average, no direction singled out.

$$C_\ell = \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2$$

(assume) average over sky

$\simeq$ average over H^{-1}-sized universes:

$$\langle \delta T(\vec{n}_1)\,\delta T(\vec{n}_2)\rangle = T_0^2 \sum_\ell C_\ell \underbrace{\sum_m Y_{\ell m}(\vec{n}_1) Y^*_{\ell m}(\vec{n}_2)}$$

$$= T_0^2 \cdot \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \cdot P_\ell(\vec{n}_1 \cdot \vec{n}_2)$$

contribution of large-ℓ multipoles to variance:

$$\langle \delta T(\vec{n})^2 \rangle = \langle T(\vec{n})^2 \rangle - \langle T(\vec{n}) \rangle^2$$

$$= T_0^2 \cdot \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \quad P_\ell(1) = 1$$

$$\sum_{\ell'=\ell-\frac{\Delta}{2}}^{\ell+\frac{\Delta}{2}} (2\ell'+1) C_{\ell'} \simeq \sum_{\ell'=\ell-\frac{\Delta}{2}}^{\ell+\frac{\Delta}{2}} (2\ell'+1) \cdot C_\ell$$

$$= \left[\left(\ell+\frac{\Delta}{2}\right)\left(\ell+\frac{\Delta}{2}+1\right) - \left(\ell-\frac{\Delta}{2}\right)\left(\ell-\frac{\Delta}{2}+1\right) + \ell + \frac{\Delta}{2} - \ell + \frac{\Delta}{2}\right] \cdot C_\ell$$

$$= \left(\cancel{\ell^2} + 2\ell\frac{\Delta}{2} + \cancel{\frac{\Delta^2}{4}} + \cancel{\ell} + \frac{\Delta}{2} - \cancel{\ell^2} + 2\ell\frac{\Delta}{2} - \cancel{\frac{\Delta^2}{4}} - \cancel{\ell} + \frac{\Delta}{2} + \cancel{\ell} + \frac{\Delta}{2} - \cancel{\ell} + \frac{\Delta}{2}\right) \cdot C_\ell$$

$$= 2\Delta \cdot (\ell+1) C_\ell \rightarrow d\ell \cdot 2(\ell+1) C_\ell$$

assumption:

$$\langle \delta T(\vec{n}_1)\,\delta T(\vec{n}_2)\rangle_{\text{universes}} = \langle \delta T(\vec{n}_1)\,\delta T(\vec{n}_2)\rangle_{\substack{\text{patches}\\ \text{in sky}}}$$

for each patch: a set of $a_{\ell m}$

observation: hypothesis

$a_{\ell m}$ are independent Gaussian random variables:

$$P(a_{\ell m})\,da_{\ell m} = \frac{1}{\sqrt{2\pi C_\ell}}\, e^{-\frac{a_{\ell m}^2}{2C_\ell}} \cdot da_{\ell m}$$

i.e. variance depends on ℓ, but not on m.