(VI) How to test inflation – density fluctuations from inflation

The gift of inflation – (near) scale-invariant quantum fluctuations in (near) de Sitter space!

Harrison – Zeldovich: a scale-invariant power spectrum of curvature perturbations = spatial variations of the gravitational potential of $\mathcal{O}(10^{-10})$ before CMB times can seed all large-scale cosmic structure.

$\downarrow$

How do we see that?

0. Black holes & holography – an analogy

Newtonian gravity in GR:

analogy: 4-velocity u in SR

$$u = (u^0, u^i) = (\sqrt{1-v^2}, \vec{v})$$

$$\xrightarrow[\vec{v} \to 0]{} (1,0,0,0)$$ "u^0 dominates in non-relativistic limit"

thus: gravitational potential

Newton's $\Delta \Phi = 4\pi G \rho$

must come from

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G \cdot T_{00} = 8\pi G \cdot \rho$$

metric that does that:

$$ds^2 = (1-2\Phi)dt^2 - (1+2\Phi)\left[dr^2 + r^2 d\Omega_2^2\right]$$

with: $\Phi = \frac{GM}{r}$ for mass M at $r = 0$

look at metric

$\rightsquigarrow$ at $r = R_S = 2GM$

time stops, dilated infinitely – like $v \to c$ in SR

$\longrightarrow$ <u>event horizon</u> of a black hole of mass M at:

$$R_S = 2GM$$

<u>'Schwarzschild radius'</u>

$$R_S = \frac{2M}{M_P^2} \quad \ldots$$

$$\Rightarrow A = 4\pi R_S^2 = 8\pi \cdot \frac{M^2}{M_P^4}$$

$$\Leftrightarrow M = \frac{M_P^2}{\sqrt{8\pi}} \sqrt{A}$$

$$\Rightarrow dM = \frac{M_P^2}{\sqrt{8\pi}} \frac{dA}{2\sqrt{A}}$$

$$= \frac{M_P^4}{16\pi} \cdot \frac{dA}{M}$$

$$= \frac{M_P^2}{8\pi} \cdot \frac{M_P^2}{2M} \cdot dA$$

$$\Rightarrow \frac{dM}{M_P^2} = \hbar \frac{M_P^2}{2M} \cdot \frac{dA}{8\pi\hbar}$$

$$\triangleq \quad dE = T \cdot dS$$

$\Rightarrow$ identify:

$$S = \frac{A}{8\pi\hbar}$$

$$T = \frac{\hbar}{2GM} \sim \frac{1}{R_S}$$

$\rightsquigarrow$ general rule:

a system with an event horizon of size R_H produces long-wave length quanta with temperature T:

$$T \sim \frac{1}{R_H}$$

now back to dS:

$\rightsquigarrow$ has an event horizon of size $R_{dS} \sim H^{-1}$

$\Rightarrow$ massless field quanta, e.g.:

- gravitons δg_{ij}
- inflaton field quanta $\delta\phi$

$\vdots$

are produced with temperature:

$$T_{dS} \sim H$$

$\rightsquigarrow$ thermal fluctuations:

$$\langle \delta g_{ij} \rangle \sim \langle \delta \phi \rangle \sim T_{dS} \sim H$$

Compute power spectrum of 2-point function of fluctuations:

$$\langle \delta g_{ij}^2 \rangle \sim \langle \delta \phi^2 \rangle \sim H^2$$

1. in more detail ...

quantum fluctuations in de Sitter space:

$\rightsquigarrow$ slow-roll inflation $\to$ long phase of quasi-exponential expansion $a \sim e^{H \cdot t}$ with slowly varying $H \sim \sqrt{V(\phi)}$

approximate space-time by exact exponential expansion – de Sitter (dS) space:

$$ds^2 = dt^2 - a^2(t) \cdot d\vec{x}_3^{\,2} = a^2(t) \cdot \left(d\eta^2 - d\vec{x}_3^{\,2}\right)$$

$$a(t) = e^{H \cdot t}, \quad H = \sqrt{\frac{\Lambda}{3}} = \text{const.}$$

$$d\eta = \frac{dt}{a} \iff \eta = -\frac{1}{aH}$$

'conformal time'

scalar field e.o.m. in conformal time for free scalar field:

$$S = \int d^4x \sqrt{-g} \cdot \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\sqrt{-g} = a^4$$

$$= \frac{1}{2} \int d^4x \frac{1}{\eta^2 H^2} \left[\phi'^2 - (\vec{\nabla}\phi)^2 \right]$$

$$(\;)' \equiv \frac{\partial}{\partial \eta} (\;)$$

$$\phi(\vec{r},\eta) = \int \frac{d^3k}{(2\pi)^3} \phi_k e^{i\vec{k}\vec{r}} \Rightarrow S = \int \frac{d^3k}{(2\pi)^3} S_k e^{i\vec{k}\vec{r}}$$

$$\Rightarrow S_k = \frac{1}{2} \int d\eta \frac{1}{\eta^2 H^2} \left(\phi_k'^2 - k^2 \cdot \phi_k^2 \right)$$

$\Rightarrow$ e.o.m. for scalar field fluctuations ϕ_k:

$$\frac{\partial}{\partial \eta} \left(\frac{1}{\eta^2 H^2} \frac{\partial}{\partial \eta} \phi_k \right) + \frac{k^2}{\eta^2 H^2} \phi_k = 0$$

$$\Rightarrow \phi_k'' - \frac{2}{\eta} \phi_k + k^2 \phi_k = 0$$

dS e.o.m.

solutions:

$$\phi_k = c_1 \frac{1 - ik\eta}{\sqrt{2k^3}} e^{ik\eta} + c_2 \cdot (h.c.)$$

now: the past is

$t \to 0,\ a \to 0 \Rightarrow \eta \to -\infty$

$\rightsquigarrow$ then the ϕ_k was short wavelength $\ll H^{-1}$

$\Rightarrow$ should be Minkowski mode

$$\phi_k \xrightarrow[\eta\to-\infty]{} \frac{i}{a\sqrt{2k}}\, e^{ik\eta}$$

$\Rightarrow$ choose: $c_1 = H$, $c_2 = 0$

'Bunch-Davies vacuum'

$$\Rightarrow \phi_k = H\cdot\frac{1-ik\eta}{\sqrt{2k^3}}\, e^{ik\eta}$$

quantize:

$$\phi = \int\frac{d^3k}{(2\pi)^3}\left(a_{\vec{k}}\,\phi_k\, e^{i\vec{k}\vec{r}} + a^+_{\vec{k}}\,\phi^*_k\, e^{-i\vec{k}\vec{r}}\right)$$

$$\left[a_{\vec{k}},\, a^+_{\vec{k}'}\right] = (2\pi)^3\,\delta^{(3)}(\vec{k}-\vec{k}')$$

compute 2-point function:

$$\langle\phi^2\rangle = \langle 0|\phi(x)\phi(x')|0\rangle$$

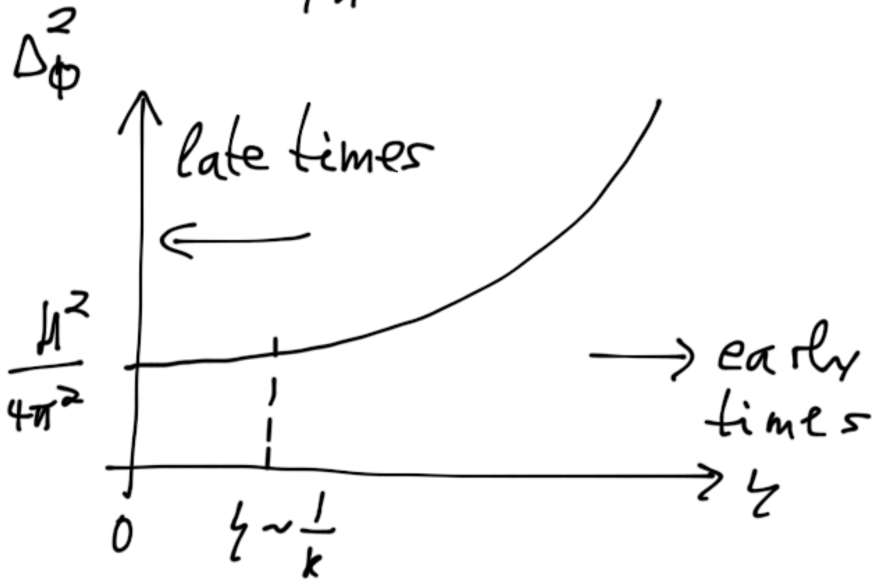
$$= \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\,\phi_k\phi^*_{k'}\,e^{i(\vec{k}\vec{r}-\vec{k}'\vec{r}')}\left[a_{\vec{k}},\, a^+_{\vec{k}'}\right]$$

$$= \int\frac{d^3k}{(2\pi)^3}\cdot|\phi_k|^2\cdot e^{i\vec{k}(\vec{r}-\vec{r}')}$$

$$= \int d\ln k \cdot \underbrace{\frac{|\phi_k|^2 \cdot k^3}{2\cdot\pi^2}}_{\equiv\, \Delta_\phi^2} \cdot \frac{\sin\left(|\vec{k}|\cdot|\vec{r}-\vec{r}'|\right)}{|\vec{k}|\cdot|\vec{r}-\vec{r}'|}$$

'power spectrum of 2-point function of ϕ'

$$\Rightarrow \Delta_\phi^2 = \frac{H^2}{4\pi^2}\cdot\left(1+k^2\eta^2\right)$$



crucial: Δ_ϕ^2 freezes at late times, when it becomes superhorizon
$\rightarrow$ and so does any other quantity which is quasi-scalar...

slow-roll vs. exact dS: corrections are $\mathcal{O}(\epsilon,\eta)$

$$\Rightarrow \Delta_\phi^2\Big|_{\underbrace{k=\frac{1}{\eta}=aH}_{\text{'at horizon crossing'}}} \simeq \frac{H^2}{4\pi^2}$$

compare to case of finite mass of ϕ:

- if $m^2 \ll H^2$ then $m^2 \ll k_{ph.}^2 = H^2$ at horizon crossing
- because of slow-roll: $\eta = \frac{m^2}{3H^2} \ll 1$ the inflaton automatically satisfies $m^2 \ll H^2$ during slow-roll inflation & will have dS quantum fluctuations with $\Delta_\phi^2 = (H/2\pi)^2 + \mathcal{O}(\epsilon, \eta)$

how does the metric fluctuate with ϕ? $\rightsquigarrow$ need fluctuations of gravitational potential to seed $\frac{\delta\rho}{\rho}$ 'density perturbations' ...

guidance: Schwarzschild metric of gravitating mass

$$ds^2 = (1-2\zeta)\,dt^2 - \frac{1}{1-2\zeta}\left(dr^2 + r^2 d\Omega_2^2\right)$$

becomes at weak fields:

$$ds^2 = (1-2\zeta)\,dt^2 - (1+2\zeta)\cdot\left(dr^2 + r^2 \cdot d\Omega_2^2\right)$$

form of metric perturbations after using various gauge transformations of GR:

$$ds^2 = (1-2\zeta)\,dt^2 - a^2 \cdot \left[(1+2\zeta)\delta_{ij} + h_{ij}\right] dx^i dx^j$$

scalar curvature perturbation

tensor perturbations

that this a good curvature perturbation variable & not a gauge artifact of GR, has to be checked $\longrightarrow$ long argument