IV Scalar fields & effective potential – symmetry breaking & thermal effects

Classical scalar field:

$$\phi = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k_0}} \left(a^+(\vec{k}) e^{ikx} + a^-(\vec{k}) e^{-ikx} \right)$$

$$k_0 = \sqrt{\vec{k}^2 + m^2}$$

promote field to an operator

→ creation & annihilation operators:

$$a^\pm(\vec{k}) \to a^\pm_{\vec{k}}$$

$$\left[a^+_{\vec{k}}, a^-_{\vec{k}'} \right] = (2\pi)^3 \cdot \delta^{(3)}(\vec{k} - \vec{k}')$$

$a^\pm_{\vec{k}}$ creates/annihilates particle with momentum $\vec{k}$:

$$a^+_{\vec{k}} |\Psi\rangle = |\Psi, \vec{k}\rangle$$

$$a^-_{\vec{k}} |\Psi, \vec{k}\rangle = |\Psi\rangle$$

Green's function:

$$G(x) = \langle 0 | T \phi(x) \phi(0) | 0 \rangle = \frac{i}{(2\pi)^4} \cdot \int \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon} d^4k$$

look at $G(0)$ and Wick rotate to Euclidean space ($k_0 \to -ik_4$):

$$G(0) = \langle 0 | \phi^2 | 0 \rangle = \langle \phi^2 \rangle = \frac{1}{(2\pi)^4} \int \frac{d^4k}{k^2 + m^2} = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\sqrt{\vec{k}^2 + m^2}}$$

if we average over state containing particles instead over vacuum $|0\rangle$:

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\sqrt{\vec{k}^2 + m^2}} \left(1 + 2\langle a^+_{\vec{k}} a^-_{\vec{k}} \rangle\right) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\sqrt{\vec{k}^2 + m^2}} \left(\frac{1}{2} + n_{\vec{k}}\right)$$

$n_{\vec{k}}$: number density of scalar particles with 3-momentum $\vec{k}$

examples:

- thermal Bose ensemble:

$$n_{\vec{k}} = \frac{1}{e^{\frac{\sqrt{\vec{k}^2 + m^2}}{T}} - 1}$$

- Bose condensate ϕ_0 of non-interacting particles of the field ϕ:

$$n_{\vec{k}} = (2\pi)^3 \phi_0^2 \cdot m \cdot \delta^{(3)}(\vec{k})$$

- coherent wave:

$$n_{\vec{k}} = (2\pi)^3 \phi_0^2 \sqrt{\vec{p}^2 + m^2} \cdot \delta^{(3)}(\vec{k} - \vec{p})$$

write Bose condensate case as:

$$\langle\phi^2\rangle = \underbrace{\frac{1}{(2\pi)^3}\int\frac{d^3k}{2\sqrt{\vec{k}^2+m^2}}}_{=\,\delta\phi_q^2} + \underbrace{\frac{1}{(2\pi)^3}\int\frac{d^3k}{\sqrt{\vec{k}^2}}\,n_{\vec{k}}}_{=\,\phi_0^2}$$

quantum $\to \delta\phi_q^2$; classical $\to \phi_0^2$

$$\begin{cases} n_{\vec{k}} = (2\pi)^3\,\phi_0^2\cdot k\cdot\delta^{(3)}(\vec{k}) \\ k = \sqrt{\vec{k}^2} \end{cases}$$

$\rightsquigarrow$ classical scalar field VEV of spontaneous symmetry breaking like a relativistically invariant Bose condensate of particles with zero momentum in the limit $m \to 0$.

i) <u>Quantum corrections to the effective potential at 1-loop</u>

- definition of free energy in thermo-dynamics:

state sum $Z = \mathrm{Tr}\, e^{iH}$

$\downarrow$

partition function $Z =: e^{-iF} \equiv e^{-iE}$

free energy $\to F$

- free energy in field theory:

action $S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$

$$Z = \int \mathcal{D}\phi\, e^{iS[\phi]} = e^{-iE}$$

with external source:

$$e^{-iE[J]} = \int \mathcal{D}\phi \cdot e^{iS[\phi] + i\int d^4x\, J\phi}$$

effective action $\Gamma[\phi]$ is Legendre transform of free energy $E[J]$, which gives functional as that of VEV ϕ_0 caused by J:

$$\phi_0 := \langle \phi \rangle_J = \frac{\int \mathcal{D}\phi \cdot \phi \cdot e^{iS[\phi] + i\int d^4x\, J\phi}}{\int \mathcal{D}\phi \cdot e^{iS[\phi] + i\int d^4x\, J\phi}}$$

$$\Rightarrow \Gamma[\phi_0] = -E[J] - J \cdot \frac{\delta E[J]}{\delta J} = -E[J] - \int d^4x\, J\phi_0$$

now expand around VEV:

$$\phi = \phi_0 + \eta$$

$$e^{-iE[J]} = \int \mathcal{D}\eta \cdot e^{iS[\phi_0] + i\int d^4x\, J\phi_0} \cdot e^{i\int d^4x \overset{0}{\cancel{\left(\frac{\delta \mathcal{L}}{\delta\phi} + J\right)\Big|_{\phi_0}}} \cdot \eta} \cdot e^{i\int d^4x\, d^4x' \left(\frac{1}{2}\cdot\frac{\delta^2\mathcal{L}}{\delta\phi^2}\right)\Big|_{\phi_0} \cdot \eta(x)\eta(x')} \cdot \ldots$$

Ehrenfest theorem: classical e.o.m. holds on expectation values

$$\Rightarrow E[J] = -S[\phi_0] - \int d^4x\, J\phi_0 + \frac{i}{2}\ln\det\left(-\frac{\delta^2\mathcal{L}}{\delta\phi^2}\right)$$

$$\Rightarrow \Gamma[\phi_0] = S[\phi_0] + \frac{i}{2} \ln \det\left(-\frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right)$$

$$\ln \det\left(-\frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right) = \mathrm{Tr} \ln\left(-\frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right) = \sum_k \ln\left(-\widetilde{\frac{\delta^2 \mathcal{L}}{\delta\phi^2}}\right) = \mathcal{V} \cdot T \cdot \int \frac{d^4k}{(2\pi)^4} \ln\left(-\widetilde{\frac{\delta^2 \mathcal{L}}{\delta\phi^2}}\right)$$

Fourier transform of $\frac{\delta^2 \mathcal{L}}{\delta\phi^2}$

Now look at scalar field theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \,, \quad \mu^2 > 0$$

$$\Rightarrow \frac{\delta^2 \mathcal{L}}{\delta\phi^2} = \partial_\mu^2 + \mu^2 - 3\lambda\phi^2 = \partial_\mu^2 - m^2(\phi)$$

$$m^2(\phi) = 3\lambda\phi^2 - \mu^2 = \left.\frac{\delta^2 \mathcal{L}}{\delta\phi^2}\right|_{\phi=\phi_0}$$

'<u>field-dependent mass</u>' and ϕ_0 renamed into ϕ

with: $S[\phi_0] = \int d^4x \cdot V(\phi_0) = \mathcal{V} \cdot T \cdot V(\phi_0)$

$$\Rightarrow \widetilde{\frac{\delta^2 \mathcal{L}}{\delta\phi^2}} = k^2 - m^2(\phi)$$

$\Rightarrow$ 1-loop effective action:

$$\frac{1}{\mathcal{V}\cdot T}\Gamma[\phi] = -V(\phi) - \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(k^2+m^2(\phi)\right)$$

with: $m^2(\phi) = \left.\frac{\delta^2\mathcal{L}}{\delta\phi^2}\right|_{\partial_\mu\phi=0}$

after Wick rotation $k_0 \to -ik_4$

$\Rightarrow$ 1-loop effective potential:

$$V_{eff.}(\phi) = V(\phi) + \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(k^2+m^2(\phi)\right)$$

for:

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4, \quad \mu^2 > 0$$

the classical VEV is:

$$\phi_0 = \langle\phi\rangle = \frac{\mu}{\sqrt{\lambda}}$$

can choose renormalization conditions:

$$\left.\frac{dV}{d\phi}\right|_{\phi=\mu/\sqrt{\lambda}} = 0$$

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=\mu/\sqrt{\lambda}} = 2\mu^2$$

with these we get:

$$V_{eff.}(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$+ \frac{3\lambda\phi^2-\mu^2}{64\pi^2}\ln\left(\frac{3\lambda\phi^2-\mu^2}{2\mu^2}\right)$$

$$+ \frac{21\lambda\mu^2}{64\pi^2}\phi^2 - \frac{27\lambda^2}{128\pi^2}\phi^4$$

consider now Abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{D_\mu \chi}\, D^\mu \chi$$

$$+\mu^2\bar{\chi}\chi - \lambda(\bar{\chi}\chi)^2$$

$$D_\mu = \partial_\mu - i\cdot e A_\mu$$

when $\mu^2 > 0$ again SSB, change to variables:

$$\chi \to \frac{1}{\sqrt{2}}(\phi + \phi_0)\cdot e^{i\frac{\zeta}{\phi_0}}$$

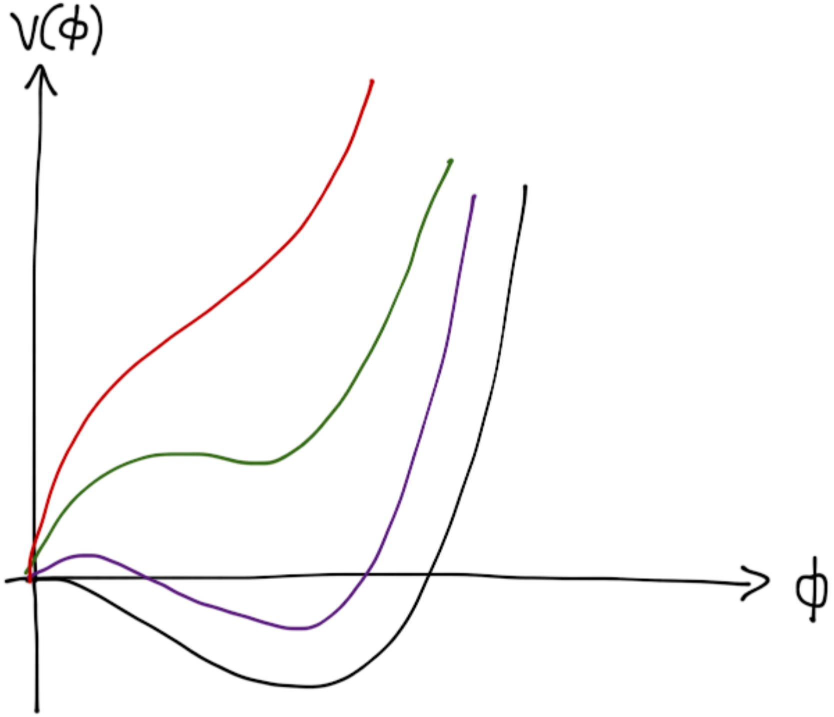
$$A_\mu \to A_\mu + \frac{1}{e\phi_0}\partial_\mu \zeta$$

and unitary gauge:

$$V_{eff.}(\phi) = -\frac{\mu^2}{2}\phi^2\cdot\left(1 - \frac{3e^4}{16\pi^2\lambda}\right)$$

$$+ \frac{\lambda}{4}\phi^4\cdot\left(1 - \frac{9e^4}{32\pi^2\lambda}\right)$$

$$+ \frac{3e^4}{64\pi^2}\phi^4\cdot\ln\left(\frac{\lambda\phi^2}{\mu^2}\right)$$



$\lambda > \frac{3e^4}{16\pi^2}$ $\qquad$ $\frac{3e^4}{32\pi^2} < \lambda < \frac{3e^4}{16\pi^2}$

$0 < \lambda < \frac{3e^4}{32\pi^2}$ $\qquad$ $\lambda = 0$

$\rightsquigarrow$ There can be phase transitions mediated by quantum tunneling (Coleman-de Luccia instantons) or thermal effects!