here: $\lambda = \frac{m^2 \langle \sigma v \rangle}{H(m)}$

$\leadsto$ will turn out that WIMPS with weak scale masses give $\Omega_{DM} \simeq 0.3$

note: at high T the reaction is in kinetic equilibrium, but not always in chemical equilibrium...

This implies:

$$f(E) \sim e^{-\frac{E-\mu}{T}}$$

for $T \ll E-\mu$

$\leadsto$ put differently, μ need not be at its equilibrium value.

$\Rightarrow$ Can rewrite the bracket $\{\dots\}$ of eq. (B) as:

$$\{\dots\} = e^{-\frac{E_A+E_B}{T}} \cdot \left(e^{\frac{\mu_C+\mu_D}{T}} - e^{\frac{\mu_A+\mu_B}{T}} \right)$$

$$(E_A + E_B = E_C + E_D)$$

use the n_i for $T \ll m_i$:

$$\Rightarrow \{\dots\} = e^{-\frac{E_A+E_B}{T}} \cdot \left[\frac{n_C n_D}{n_C^{(0)} n_D^{(0)}} - \frac{n_A n_B}{n_A^{(0)} n_B^{(0)}} \right]$$

with: $\frac{n_i}{n_i^{(0)}} = e^{\mu_i/T}$ equil. n_i at $\mu_i = 0$.

now define the thermally averaged cross section:

$$\langle \sigma v \rangle = \frac{1}{n_A^{(0)} n_B^{(0)}} \cdot \int \frac{d^3 p_i}{(2\pi)^3 E_i} e^{-\frac{E_A + E_B}{T}} \cdot |\mathcal{M}|^2 \cdot (2\pi)^4 \times \delta^{(4)}(p)$$

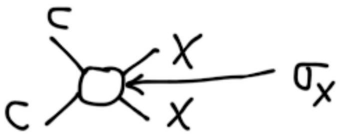
Simplifies the Boltzmann eq. (B):

$$\frac{1}{a^3} \frac{d(n_A a^3)}{dt} = n_A^{(0)} n_B^{(0)} \langle \sigma v \rangle \cdot \left[\frac{n_C n_D}{n_C^{(0)} n_D^{(0)}} - \frac{n_A n_B}{n_A^{(0)} n_B^{(0)}} \right]$$

$$\Downarrow$$

$$\frac{1}{a^3} \frac{d(n_X a^3)}{dt} = \left(n_X^{(0)}\right)^2 \langle \sigma_X v \rangle \cdot \left[1 - \left(\frac{n_X}{n_X^{(0)}} \right)^2 \right] \qquad (B')$$

for X-production: ($\mu_C = 0$ early on)



to obtain freeze-out value of n_X, go to limit:

$$\frac{dn_X}{dt} \to 0 \quad \text{for } t \to \infty$$

We have:

$$\frac{1}{a^3} \frac{d(n_X a^3)}{dt} = \frac{dn_X}{dt} + 3 \frac{\dot{a}}{a} \cdot n_X = \frac{dn_X}{dt} + 3H \cdot n_X$$

thus we demand:

$$\frac{dn_X}{dt} = 0 \quad \text{at late times}$$

furthermore, at late times:

$$\frac{m_X}{T} \to \infty$$

$$\Rightarrow \quad n_X^{(0)} \sim e^{-\frac{m_X}{T}} \to 0$$

exponentially fast at late times

$\Rightarrow$ (B') becomes:

$$3H \cdot n_X \simeq \langle \sigma_X v \rangle \cdot n_X^2 \quad (B'')$$

if we realize:

$$\langle \sigma_X v \rangle n_X = \langle \Gamma_X \rangle$$

thermally averaged reaction rate

this is nothing else than:

$$H \simeq \langle \Gamma_X \rangle$$

our old freeze-out criterion!

We get:

$$(B'') \Leftrightarrow \quad n_X \sim \frac{H}{\langle \sigma_X v \rangle}$$

thus we get for ρ_X:

$$\rho_X = m_X n_X \sim \frac{m_X H}{\langle \sigma_X v \rangle} \quad (**)$$

and at freeze-out:

$$H \sim \frac{T_{f.o.}^2}{M_P}, \quad T_{f.o.} \sim m_X$$

$$(**) \Rightarrow \rho_X \sim \frac{m_X T_{f.o.}^2}{M_P \langle \sigma_X v \rangle} \sim \frac{T_{f.o.}^3}{M_P \langle \sigma_X v \rangle}$$

now, ρ_X is ρ_X at freeze-out, dilutes until now $\rightarrow$ non-relat. matter:

$$\rho_{X,0} = \rho_X \left(\frac{a_{f.o.}}{a_0} \right)^3 \sim \frac{T_0^3}{M_P \langle \sigma_X v \rangle} \left(\frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \right)^3$$

now, because:

$$S = \text{const.} = s \cdot a^3 \Rightarrow T \sim \frac{1}{a}$$

$$\Rightarrow \frac{T_{f.o.}}{T_0} \cdot \frac{a_{f.o.}}{a_0} \sim \mathcal{O}(1)$$

$$\Rightarrow \rho_{X,0} \sim \frac{T_0^3}{M_P \langle \sigma_X v \rangle}$$

we can now calculate to-day's density parameter $\Omega_{X,0}$ of the X particles:

$$\Omega_{X,0} = \frac{\rho_{X,0}}{\rho_{cr,0}} \sim \frac{T_0^3}{M_P^3} \cdot \frac{1}{H_0^2 \langle \sigma_X v \rangle}$$

now we have:

$$T_0 \sim 10^{-4}\,\mathrm{eV}\,, \quad M_P \sim 2\cdot 10^{18}\,\mathrm{GeV}$$

$$\frac{H_0}{c} \simeq 3\cdot 10^{-9}\,\frac{s}{m}\cdot\frac{72\cdot 10^{3}\,m}{s\cdot \mathrm{Mpc}} \simeq \frac{2\cdot 10^{-4}}{\mathrm{Mpc}} \simeq \frac{2\cdot 10^{-4}}{3\cdot 10^{6}\cdot 10\cdot 10^{12}\,\mathrm{km}} \simeq 6\cdot 10^{-29}\,\frac{1}{\mathrm{cm}}$$

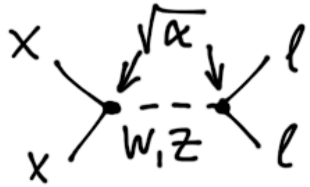
plug into $\Omega_{X,0}$:

$$\Omega_{X,0} \sim 10^{-94}\cdot\frac{10^{58}}{36}\cdot\frac{\mathrm{cm}^2}{\langle\sigma_X v\rangle}$$

$$\Rightarrow \Omega_{X,0} \sim \frac{3\cdot 10^{-38}\,\mathrm{cm}^2}{\langle\sigma_X v\rangle} \quad (\Omega)$$

for $\langle\sigma_X v\rangle$ we get:

σ_X :



$\alpha \sim 0.01$
(EW interaction!)

$$\Rightarrow \mathcal{M} \sim \frac{\alpha}{M_W^2} \sim \alpha G_F$$

$$\Rightarrow \langle\sigma_X v\rangle \sim \alpha^2 G_F^2 m_X^2 \simeq 10^{-4}\cdot\left(\frac{1}{(300\,\mathrm{GeV})^2}\right)^2\cdot(100\,\mathrm{GeV})^2\cdot\left(\frac{m_X}{100\,\mathrm{GeV}}\right)^2 \simeq 10^{-10}\,\mathrm{GeV}^{-2}\cdot\left(\frac{m_X}{100\,\mathrm{GeV}}\right)^2$$

little conversion help:

$$\mathrm{GeV}^{-1} \sim 0.2\,\mathrm{fm} = 2\cdot 10^{-14}\,\mathrm{cm}$$

$$\Rightarrow \langle \sigma_X v \rangle \sim \left(\frac{m_X}{100\,\mathrm{GeV}}\right)^2 \cdot 10^{-37}\,\mathrm{cm}^2$$

plug this into (Ω):

$$\Rightarrow \Omega_{X,0} \sim 0.3 \cdot \left(\frac{100\,\mathrm{GeV}}{m_X}\right)^2 \tag{83}$$

$\rightsquigarrow$ A WIMP with mass of $\sim 100\,\mathrm{GeV}$, natural in EW-scale SUSY, is a perfect DM candidate!