In the neighbourhood of any point x^*,
$K^\mu(x)$ determined by $K_\mu(x^*)$, $D_\mu K_\nu(x^*)$:

$$[D_\rho, D_\sigma] K_\mu = -R^\lambda{}_{\mu\rho\sigma} K_\lambda$$

$$R^\lambda{}_{\mu\rho\sigma} + R^\lambda{}_{\sigma\mu\rho} + R^\lambda{}_{\rho\sigma\mu} = 0$$

$$[D_\rho, D_\sigma] K_\mu + [D_\mu, D_\rho] K_\sigma + [D_\sigma, D_\mu] K_\rho = 0$$

$$0 = D_\rho D_\sigma K_\mu - D_\sigma D_\rho K_\mu + D_\mu D_\rho K_\sigma - D_\rho D_\mu K_\sigma + D_\sigma D_\mu K_\rho - D_\mu D_\sigma K_\rho$$

(using the Killing eq.)

$$= 2\left(D_\rho D_\sigma K_\mu - D_\sigma D_\rho K_\mu - D_\mu D_\sigma K_\rho\right)$$

$$\Rightarrow [D_\rho, D_\sigma] K_\mu = D_\mu D_\sigma K_\rho$$

$$\Rightarrow \boxed{D_\mu D_\sigma K_\rho = -R^\lambda{}_{\mu\rho\sigma} K_\lambda}$$

hence, K_μ and $D_\nu K_\mu$ determine <u>all</u> derivatives, i.e. (Taylor expansion):

$$\Rightarrow K^{(n)}_\mu(x; x^*) = A_\mu{}^\rho(x; x^*) K^{(n)}_\rho(x^*)$$

$$+ B_\mu{}^{\rho\sigma}(x; x^*) D_\rho K^{(n)}_\sigma(x^*)$$

N

$\frac{N(N-1)}{2}$

$(*)$ $A_\mu{}^\rho(x^*, x^*) = \delta_\mu^\rho$

$B_\mu{}^{\rho\sigma}(x^*, x^*) = 0$

maximal # of Killing vectors in N dimensions:

$$N + \frac{N(N-1)}{2} = \frac{N(N+1)}{2}$$

<u>check</u>: Minkowski in 4D:

4 translations

3 rotations

3 boosts

(linearly independent)

Homogeneity: $\dim \mathcal{M} = N$

isometry: translations to all neighbouring points

$K_\mu^{(\lambda)}(x;x^*)$: $K_\mu^{(\lambda)}(x^*) = \delta_\mu^\lambda$

N independent Killing vectors

Isotropy:

isometry: all rotations around x^*

$$K_\mu^{(\lambda,\sigma)}(x;x^*) = -K_\mu^{(\sigma,\lambda)}(x;x^*) ;$$

$$K_\mu^{(\lambda,\sigma)}(x^*,x^*) \equiv 0$$

$$D_\nu K_\mu^{(\lambda,\sigma)}(x;x^*)\Big|_{x=x^*} = \frac{\partial}{\partial x^\nu} K_\mu^{(\lambda,\sigma)}(x;x^*)\Big|_{x=x^*} = \delta_\mu^\lambda \delta_\nu^\sigma - \delta_\nu^\lambda \delta_\mu^\sigma$$

$\rightsquigarrow \dfrac{N(N-1)}{2}$ independent Killing vectors

Isotropy around all points implies homogeneity:

$K_\mu^{(\lambda,\sigma)}(x,x^*)$ and $K_\mu^{(\lambda,\sigma)}(x,x^*+dx^*)$ must be both Killing vectors, and thus any linear combination as well

$\rightsquigarrow \dfrac{\partial}{\partial x^{*\rho}} K_\mu^{(\lambda,\sigma)}(x;x^*)$ is also Killing vector

$$\rightsquigarrow \quad 0 = \frac{\partial}{\partial x^{*\rho}} K_\mu^{(\lambda,\sigma)}(x^*; x^*)$$

$$= \frac{\partial}{\partial x^\rho} K_\mu^{(\lambda,\sigma)}(x; x^*)\Big|_{x = x^*} + \frac{\partial}{\partial x^{*\rho}} K_\mu^{(\lambda,\sigma)}(x; x^*)\Big|_{x = x^*}$$

$$\rightsquigarrow \quad \frac{\partial}{\partial x^{*\rho}} K_\mu^{(\lambda,\sigma)}(x, x^*)\Big|_{x=x^*} = -\delta_\mu^\lambda \delta_\rho^\sigma + \delta_\mu^\sigma \delta_\rho^\lambda$$

linear continuation:

$$\hat{K}_\mu^{(\sigma)}(x; x^*) = \frac{1}{N-1} \frac{\partial}{\partial x^{*\rho}} K_\mu^{(\rho,\sigma)}(x; x^*)$$

$$\rightsquigarrow \quad \hat{K}_\mu^{(\sigma)}(x^*; x^*) = \delta_\mu^\sigma$$

II.2 Maximally symmetric spaces

$$\#(\text{Killing vectors}) = \frac{N(N+1)}{2}$$

$$\dim \mathcal{M} = N$$

spaces are homogeneous and isotropic

$\rightsquigarrow$ a homogeneous & isotropic (about one point) = isotropic about every point
space is maximally symmetric (has $N(N+1)/2$ K's – shown above)

$\rightsquigarrow$ converse also true:
integrability condition for Killing vectors

$$D_\mu D_\sigma K_\rho = - R^\lambda{}_{\mu\rho\sigma} K_\lambda$$

$$\Leftrightarrow D_\nu D_\mu D_\sigma K_\rho - D_\mu D_\nu D_\sigma K_\rho =$$

$$= - R^\lambda{}_{\mu\rho\sigma} D_\nu K_\lambda + R^\lambda{}_{\nu\rho\sigma} D_\mu K_\lambda$$

$$- \left(D_\nu R^\lambda{}_{\mu\rho\sigma} - D_\mu R^\lambda{}_{\nu\rho\sigma} \right) K_\lambda$$

$$\overset{!}{=} - R^\lambda{}_{\sigma\nu\mu} D_\lambda K_\rho - R^\lambda{}_{\rho\nu\mu} D_\sigma K_\lambda$$

property of the curvature tensor from commutator of 2 covariant derivatives acting on 2-tensor ...

Killing vector for isotropy:

$$K_\lambda(x) = 0 \quad , \quad D_\nu K_\lambda = - D_\lambda K_\nu$$

(Killing eq.)

$$\Downarrow$$

$$\left(- R^\lambda{}_{\mu\rho\sigma} \delta^\tau_\nu + R^\lambda{}_{\nu\rho\sigma} \delta^\tau_\mu - R^\lambda{}_{\sigma\nu\mu} \delta^\tau_\rho + R^\lambda{}_{\rho\nu\mu} \delta^\tau_\sigma \right) \underbrace{D_\tau K_\lambda}_{\text{anti-symmetric}} =$$

$$= \left(D_\nu R^\lambda{}_{\mu\rho\sigma} - D_\mu R^\lambda{}_{\nu\rho\sigma} \right) \underbrace{K_\lambda}_{=0} = 0$$

$$\curvearrowright\ -R^{\lambda}{}_{\mu\rho\sigma}\,\delta^{\tau}_{\nu} + R^{\lambda}{}_{\nu\rho\sigma}\,\delta^{\tau}_{\mu} - R^{\lambda}{}_{\sigma\nu\mu}\,\delta^{\tau}_{\rho} + R^{\lambda}{}_{\rho\nu\mu}\,\delta^{\tau}_{\sigma} = -R^{\tau}{}_{\mu\rho\sigma}\,\delta^{\lambda}_{\nu} + R^{\tau}{}_{\nu\rho\sigma}\,\delta^{\lambda}_{\mu} - R^{\tau}{}_{\sigma\nu\mu}\,\delta^{\lambda}_{\rho} + R^{\tau}{}_{\rho\nu\mu}\,\delta^{\lambda}_{\sigma} \Big|\cdot\delta^{\nu}_{\tau}$$

$$\curvearrowright\ -(N-1)R^{\lambda}{}_{\mu\rho\sigma} - R^{\lambda}{}_{\sigma\rho\mu} + R^{\lambda}{}_{\rho\sigma\mu} = -R^{\lambda}{}_{\mu\rho\sigma} + \cancel{R^{\nu}{}_{\nu\rho\sigma}}\,\delta^{\lambda}_{\mu} - \underbrace{R^{\nu}{}_{\sigma\nu\mu}}_{=R_{\sigma\mu}}\delta^{\lambda}_{\rho} + \underbrace{R^{\nu}{}_{\rho\nu\mu}}_{=R_{\rho\mu}}\delta^{\lambda}_{\sigma}$$

$$\sim + \sim + \sim = 0$$

$$\Rightarrow\ (N-1)R_{\lambda\mu\rho\sigma} = R_{\sigma\mu}\,g_{\lambda\rho} - R_{\rho\mu}\,g_{\lambda\sigma}$$

antisymmetry in (μ,λ) as must be for $R_{\dots}$:

$$\Downarrow$$

$$R_{\sigma\mu}\,g_{\lambda\rho} - R_{\rho\mu}\,g_{\lambda\sigma} = -R_{\sigma\lambda}\,g_{\mu\rho} + R_{\rho\lambda}\,g_{\mu\sigma} \Big|\cdot g^{\lambda\rho}$$

$$\Rightarrow\ N\cdot R_{\sigma\mu} = g_{\mu\sigma}\cdot R$$

$$\Rightarrow\ \boxed{R_{\lambda\mu\rho\sigma} = \frac{R}{N(N-1)}\left(g_{\mu\sigma}\,g_{\lambda\rho} - g_{\mu\rho}\,g_{\lambda\sigma}\right)}$$

Bianchi identity:

$$0 = D^{\mu}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = \left(\frac{1}{N} - \frac{1}{2}\right) D_{\nu} R$$

$N > 2$: $R = -N(N-1)\cdot k = \text{const.}$

$$\rightsquigarrow \boxed{R_{\lambda\mu\rho\sigma} = -k\left(g_{\mu\sigma} g_{\lambda\rho} - g_{\mu\rho} g_{\lambda\sigma}\right)}$$

maximal symmetry: spaces of constant curvature