Theoretical Cosmology

literature:

Kolb & Turner: The Early Universe (1990)

Mukhanov: Physical Foundations of Cosmology (2005)

Weinberg: Gravitation & Cosmology (1972)
Cosmology (2008)

Dodelson: Modern Cosmology (2003)

Rubakov: Introduction to the theory of the early Universe, 2 vols

String Cosmology, edited by: Johanna Erdmenger (2009) Wiley-VCH

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Google/arXiv: TASI Lectures:

Introduction to Cosmology (2004) Trodden, Carroll astro-ph/0401547

Tasi lectures on Inflation (2009), Daniel Baumann arXiv: 0907.5424

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Aspects of Inflationary Cosmology

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Abstract:

The past 15 years have led us into the era of observational precision cosmology. The redshift of distant supernovae, large-scale structure data, and in particular the results from the WMAP satellite, and more recent ground based experiments - and in the near future the PLANCK satellite - on the cosmic microwave background (CMB) have led to a new cosmological concordance model of about 73% dark energy, 23% cold dark matter, and only about 4% baryonic matter, as well as spectacular confirmation of the most generic predictions of cosmological inflation. Inflation denotes a very early phase of extremely rapid exponential expansion of the early Universe which was driven by a large amount of vacuum energy which later dissipated. The inflationary early phase naturally solves various initial condition problems of the hot big bang.

In these lectures we will start from a discussion of inflationary cosmology, and then proceed to explain why models of inflationary expansion of the very early Universe need a short-distance completion by a theory of quantum gravity. Using string theory as a candidate theory of quantum gravity, we will then look at the challenges of embedding inflation, and describe the basics of promising constructions which realize string inflation. The necessary string theoretic concepts - such as basics of string compactifications and moduli stabilization - will be introduced as part of the lectures without assuming detailed knowledge of string theory. We will also discuss possible testing against observables of the cosmic microwave background radiation data as provided e.g. by the WMAP satellite.

Topics will include:

1) An introduction to FRW cosmology and scalar field driven cosmological inflation
2) A discussion of inflation in effective field theory, and the arising need for UV completion
3) Introduction to string cosmology, focussing on the challenges for embedding inflation into string theory, in particular moduli stabilization
4) Inflationary models in string theory, of both small-field and large-field type, which may be discriminated by the primordial gravitational wave signal

Knowledge of General Relativity, basic FRW cosmology and QFT will be quite helpful. Basic knowledge of string theory and string compactification are helpful, of course, but not required to follow the course.

Topics in Particle Cosmology

large scale structure
CMB
} origin & growth of density perturb.

dark matter
baryon asymmetry
nucleosynthesis
EW phase transition
\# of 'neutrinos'
} hot early universe, non-equilibrium processes

cosmological const. 'dark energy' $\Big\}$ structure of space-time, horizons, inflation

(I) The expanding universe

1. Dynamics of expansion

The expansion of the universe as a whole is governed by Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \cdot T_{\mu\nu} \tag{6}$$

applied to the FRW metric of a spatially homogeneous and isotropic expanding space-time

Here we denote:

$$\boxed{R_{\mu\nu} = g^{\rho\sigma} R^{\rho}{}_{\mu\sigma\nu}}$$ Ricci tensor

$$\boxed{R = g^{\mu\nu} R_{\mu\nu}}$$ Ricci scalar

$$\boxed{R^{\rho}{}_{\mu\sigma\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}}$$ Riemann curvature tensor

$$\boxed{\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\delta} \left(\frac{\partial g_{\mu\delta}}{\partial x^\nu} + \frac{\partial g_{\nu\delta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\delta} \right)}$$ Christoffel symbols

The field equations follow from the Einstein-Hilbert action:

$$S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - 2\Lambda \right)$$

$$+ \int d^4x \sqrt{-g}\, \mathcal{L}_{\mathrm{matter}}$$

give $T_{\mu\nu}$

if we define:

$$\sqrt{-g}\,\mathcal{L}_{EH} = \sqrt{-g}\cdot R$$

$$\sqrt{-g}\,\mathcal{L}_{\Lambda+\text{matter}} = \sqrt{-g}\left(-\frac{\Lambda}{16\pi G} + \mathcal{L}_{\text{matter}}\right)$$

$$\Rightarrow \left\{ \begin{array}{l} \dfrac{\delta\left(\sqrt{-g}\,\mathcal{L}_{EH}\right)}{\delta g^{\mu\nu}} \to R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\[2em] T_{\mu\nu} = \dfrac{2}{\sqrt{-g}} \cdot \dfrac{\delta\left(\sqrt{-g}\,\mathcal{L}_{\Lambda+\text{matter}}\right)}{\delta g^{\mu\nu}} \end{array} \right. \tag{6a}$$

via the Euler-Lagrange equations:

$$\frac{\delta\left(\sqrt{-g}\,\mathcal{L}\right)}{\delta g^{\mu\nu}} = 0\,. \tag{7}$$

note: the partial derivative $\dfrac{\partial}{\partial x^\mu} = \Lambda_\mu{}^{\nu'} \dfrac{\partial}{\partial x^{\nu'}}$ transforms as a vector (covariant 1-tensor) under Lorentz transformations on Minkowski space-time.

On curved space-time there is a generalized covariant derivative D_μ transforming as a covariant 1-tensor under general coordinate transformations.

Acting on a covariant vector A_ν it reads:

$$D_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} - \Gamma^\rho_{\mu\nu} A_\rho$$

and on a contravariant 2-tensor $C^{\mu\nu}$, for example:

$$D_\mu C^{\rho\sigma} = \frac{\partial C^{\rho\sigma}}{\partial x^\mu} + \Gamma^\rho_{\mu\delta} C^{\delta\sigma} + \Gamma^\sigma_{\mu\delta} C^{\delta\rho}$$

One can show that:

$$D_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0$$

where as usual:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = g^{\mu\rho} g^{\nu\sigma} \left(R_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} R \right)$$

and by Einstein's equations this implies:

$$D_\mu T^{\mu\nu} = 0$$

which is the covariant conservation of energy and momentum.

Thus, energy-momentum conservation is a consequence of the gravitational field equations in GR.

$\Rightarrow$ The space-time of a spatially homogeneous & isotropic universe necessarily has a metric of FRW form:

$$ds^2 = dt^2 - a^2(t)\cdot\left[d\rho^2 + f^2(\rho)\cdot d\Omega_2^2\right]$$

$$\text{with } f(\rho) = \begin{cases} \sin\rho \\ \rho \\ \sinh\rho \end{cases} \qquad (3)$$

$a(t)$: scale factor of the universe – "scale of its spatial size"

2 other useful coordinate choices:

$\rightsquigarrow$ change of radial variable

$$ds^2 = dt^2 - a^2(t)\cdot\left[\frac{dr^2}{1-k\cdot r^2} + r^2\cdot d\Omega_2^2\right] \qquad (4)$$

calculate 3-curvature R_{ij} for $[\ldots]$ in (4):

$$R_{ij} \sim k\cdot g_{ij} \Rightarrow {}^3R = R^i{}_i \sim k$$

$\Rightarrow$ 3-slices of constant time are Einstein spaces of constant curvature k.

$$k = \begin{cases} 1 & , f(\rho) = \sin\rho \,, \text{"closed universe"} \\ 0 & , f(\rho) = \rho \,, \text{"flat universe"} \\ -1 & , f(\rho) = \sinh\rho \,, \text{"open universe"} \end{cases} \qquad (4')$$

sometimes, conformal time η is useful instead of comoving time t:

$$\eta(t) = \int^t \frac{dt'}{a(t')}$$

$$ds^2 = a^2(t(\eta)) \cdot \left[d\eta^2 - \frac{dr^2}{1 - k \cdot r^2} - r^2 \cdot d\Omega_2^2 \right] \qquad (5)$$

the Hubble parameter H then is:

$$H \equiv \frac{\dot{a}}{a} \,, \quad \dot{(\ )} = \frac{d}{dt}$$

$$\Rightarrow \quad v(r) \simeq \dot{a} = H \cdot a \simeq H \cdot r$$

for distances not too large.

<u>comoving coordinates:</u>

a coordinate system, in which all massive matter eventually comes to rest by the dilution of its kinetic energy with the expansion as $\sim a^{-3}(t)$, and this means: to rest with the expanding frame of reference given by the comoving coordinates.

Comoving coordinates <u>co-move</u> with the matter receding by just the expansion itself.

Thus, in comoving coordinates matter stays at rest, if not sped otherwise, because the coordinate frame stretches with the expansion itself, and thus is 'comoving' with the locally-at-rest matter stretching the same way with the expansion.

Thus comoving time t measures cosmological age as seen by an observer swept along by the expansion, and agrees with 'red shift-time' as inferred via distances from red shifts.

To derive the dynamics of the expansion we will apply Einstein's equations to the FRW space-time. This yields, defining the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

with the FRW metric in the form of eq. (4):

$$G_{00} = 3 \cdot \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right] \qquad (9)$$

$$G_{ij} = g_{ij} \cdot \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right]$$

where the metric (4) is written as:

$$ds^2 = dt^2 - a^2(t) \cdot g_{ij}\, dx^i dx^j .$$

For simplification and consistency with spatial homogeneity & isotropy energy and matter is often described by an energy-momentum tensor of a perfect fluid with energy density ρ, pressure p and 4-velocity u^α:

$$T^{\alpha\beta} = (\rho + p) \cdot u^\alpha u^\beta - p \cdot g^{\alpha\beta} . \quad (10)$$

As the fluid is comoving with the expansion, it is at rest in the coordinates of the FRW metric (4):

$$u^\alpha = (1, 0, 0, 0)$$

$$\Rightarrow T^{\alpha\beta} = \begin{pmatrix} \rho & \\ & -p \cdot g^{ij} \end{pmatrix} \quad (11)$$

and this, together with the LHS of the Einstein equations (9), gives us the Friedmann equations:

$$\begin{cases} 00: & \dfrac{\dot{a}^2}{a^2} \equiv H^2 = \dfrac{8\pi G}{3} \cdot \rho - \dfrac{k}{a^2} \\ ij: & 2\dfrac{\ddot{a}}{a} + \dfrac{\dot{a}^2}{a^2} + \dfrac{k}{a^2} = -8\pi G \cdot p \end{cases} \quad (12)$$

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi}{3} G \cdot (\rho + 3p) \quad (12')$$

"acceleration equation"

We see that the universe undergoes accelerated expansion, if:

$$p < -\frac{1}{3}\rho .$$

The Friedmann equations are to be supplemented by energy conservation:

$$dE = d(\rho V) = -p\,dV \qquad (13)$$

where the cosmological volume V:

$$V \sim a^3(t)$$

$$\Rightarrow \quad d(\rho \cdot a^3) = -p \cdot d(a^3)$$

$$\Rightarrow \quad \frac{d}{dt}(\rho a^3) = a^3\left(\dot\rho + 3\rho \cdot \frac{\dot a}{a}\right) = -a^3 \cdot 3p \cdot \frac{\dot a}{a}$$

$$\Leftrightarrow \quad \dot\rho + 3H \cdot (\rho + p) = 0 . \qquad (14)$$

Finally, we need an equation of state $p = p(\rho)$ for the description of matter to complete, and this is often well approximated by:

$$p = w \cdot \rho \quad . \qquad (15)$$

examples are:

$w = 0$	"dust" - nonrelativistic massive matter
$w = \frac{1}{3}$	radiation (photons ...)
$w = -\frac{1}{3}$	the 3-curvature k
$w = -1$	cosmological constant or constant scalar potential