<u>the solution:</u>

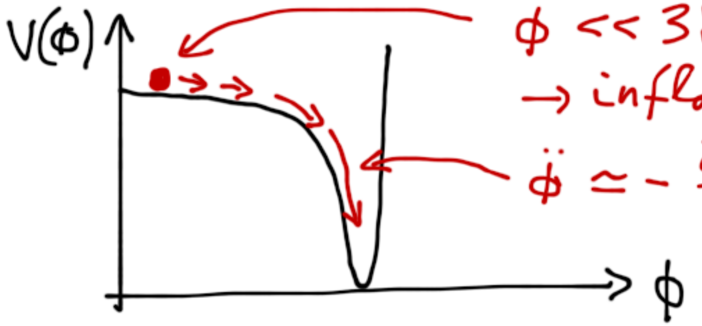
avoid trapping in a metastable minimum of the potential at $V>0$ and the subsequent tunneling ...

$\downarrow$

(I) Slow-roll inflation

(Albrecht & Steinhardt; Linde '82)

$\rightarrow$ slow-roll in potential $V(\phi)$:



$\ddot{\phi} \ll 3H\dot{\phi} = -\frac{\partial V}{\partial \phi}$

$\rightarrow$ inflation

$\ddot{\phi} \simeq -\frac{\partial V}{\partial \phi}$ $\rightarrow$ <u>no</u> inflation

consider: $\vec{\nabla}\phi = 0$, only $\dot{\phi}$

redshifts fast, if $a \sim e^{H \cdot t}$

then if: $\boxed{\dot{\phi}^2 \ll V}$ $\Rightarrow p = -\rho$

$\Downarrow$

motion dominated by potential energy

$$\begin{cases} a \sim e^{H \cdot t} \\ H \simeq \text{const.} \end{cases}$$

need this for $N_e \simeq H \cdot t \overset{!}{\simeq} 60$ efolds to solve the horizon etc. problems ...

can ensure this, if slow-roll:

e.o.m. for ϕ:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \equiv -V'$$

slow-roll: $|\ddot{\phi}| << |3H\dot{\phi}|, |V'|$

$$\Rightarrow 3H\dot{\phi} = -V' \quad \text{slow-roll e.o.m.} \quad (*)$$

then i): $p \simeq -\rho$

$$\Rightarrow 1 >> \frac{\dot{\phi}^2}{V} = \frac{V'^2}{9VH^2}, \quad H^2 \simeq \frac{V}{3}$$

$$= \frac{1}{3}\left(\frac{V'}{V}\right)^2 \equiv \frac{2}{3}\epsilon, \quad \boxed{\epsilon = \frac{1}{2}\left(\frac{V'}{V}\right)^2}$$

$\Rightarrow$ $\boxed{\epsilon << 1 \text{ ensures } p \simeq -\rho.}$

<u>1st slow-roll condition</u>

ensure slow-roll for long time:

↳ maintain: $\ddot{\varphi} << 3H\dot{\varphi}$

from $(*)$ $\Rightarrow$ $\dot{\varphi}^2 = \frac{V'^2}{3V}$

$$\Rightarrow \dot{\varphi}\ddot{\varphi} = \frac{1}{2}\left(\frac{V'^2}{3V}\right)' \cdot \dot{\varphi}$$

$$\Rightarrow \ddot{\varphi} = V' \cdot \left(\frac{1}{3}\frac{V''}{V} - \frac{V'^2}{V^2}\right)$$

define: $\boxed{\eta \equiv \frac{V''}{V}}$

$$\Rightarrow \frac{\ddot{\varphi}}{3H\dot{\varphi}} = 2\epsilon - \frac{1}{3}\eta << 1$$

implies: $\boxed{\eta << 1}$ if $\epsilon << 1$

<u>2nd slow-roll condition</u>

if:

$\epsilon, \eta << 1$

then:

$$\epsilon \simeq \epsilon_H = -\frac{\dot{H}}{H^2} = 1 - \frac{1}{H^2}\frac{\ddot{a}}{a}$$

$$\eta \simeq \eta_H = \frac{\dot{\epsilon}_H}{\epsilon_H H}$$

$\Rightarrow$ $\epsilon << 1$ implies $\epsilon_H << 1$

$\Rightarrow$ $\ddot{a} > 0$

consistent.

ϵ_H, η_H 'physical' Hubble slow-roll parameters

can do much more:

- multiple fields
- higher derivatives
- non-trivial kinetic terms

but here simple single-field slow-roll will suffice.

<u>General story here:</u>

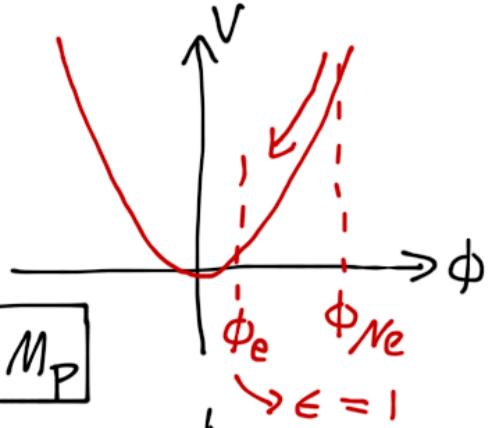
- need a $V(\phi)$ that satisfies $\epsilon, \eta << 1$ at some ϕ_{Ne},
- $\epsilon, \eta < 1$ for $N_e \simeq 60$ efolds at least, then $\epsilon > 1$ must be reached at some ϕ_e $\rightarrow$ slow-roll ends, ϕ oscillates $\rightarrow$ reheating, FRW

2 classes of $V(\phi)$:

i) large-field models

examples: $V(\phi) \sim \phi^p \,,\; p \geq 2$

e.g. $V(\phi) = \frac{m^2}{2}\phi^2$



$\phi_e : \epsilon(\phi_e) = 1$

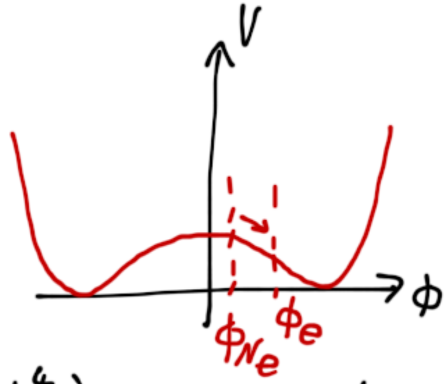
$$\epsilon = \frac{p^2}{2\phi^2} \Rightarrow \boxed{\phi_e \sim M_p}$$

$$N_e = \int_{t_e}^{t_{N_e}} H\,dt = \int_{\phi_e}^{\phi_{N_e}} \frac{H}{\dot\phi}\,d\phi = \int_{\phi_e}^{\phi_{N_e}} \frac{d\phi}{\sqrt{2\epsilon}}$$

$$\simeq \frac{\phi_{N_e}^2}{2p} \Rightarrow \boxed{\phi_{N_e} = \sqrt{2p\,N_e} \gg M_p}$$

ii) small-field models

example: hill-top in $V(\phi) = \lambda(\phi^2 - v^2)^2$ and $\phi_{N_e} \ll v$



$\downarrow$

$$V(\phi) = V_0\left(1 - \frac{2}{v^2}\phi^2 + \frac{\phi^4}{v^4}\right) , \quad V_0 = \lambda v^4$$

$$\Rightarrow \epsilon = \frac{8\phi^2}{v^4}\left(1 - \frac{\phi^2}{v^2}\right) , \quad \phi \ll v$$

$$\Rightarrow \phi_e \simeq \frac{1}{2\sqrt{2}}\left(\frac{v}{M_p}\right)^2 \cdot M_p$$

if $v \lesssim M_p \rightarrow$ small-field

$\hookrightarrow$ if a 2^{nd} field ends slow-roll by 'waterfall' into a minimum
$\rightarrow$ 'hybrid' inflation ...

(II) How to test inflation – density fluctuations from inflation

i) The gift of inflation – (near) scale-invariant quantum fluctuations in (near) de Sitter space!

$$\downarrow$$

How do we see that?

Newtonian gravity in GR:

analogy: 4-velocity u in SR

$$u = (u^0, u^i) = (\sqrt{1-v^2}, \vec{v}) \xrightarrow[\vec{v}\to 0]{} (1,0,0,0)$$

"u^0 dominates in non-relativistic limit"

thus:

gravitational potential

Newton's $\Delta \zeta = 4\pi G \rho$

must come from

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G \cdot T_{00} = 8\pi G \cdot \rho$$

metric that does that:

$$ds^2 = (1-2\zeta)\,dt^2 - (1+2\zeta)\left[dr^2 + r^2 d\Omega_2^2\right]$$

with: $\zeta = \frac{2GM}{r}$ for mass M at $r=0$

look at metric

$\rightsquigarrow$ at $r = R_S = 2GM$

time stops, dilated infinitely – like $v \to c$ in SR

$\longrightarrow$ event horizon of a black hole of mass M at:

$$R_S = 2GM$$

'schwarzschild radius'

$$R_S = \frac{2M}{M_P^2} \quad \ldots$$

$$\Rightarrow A = 4\pi R_S^2 = 8\pi \cdot \frac{M^2}{M_P^4}$$

$$\Leftrightarrow M = \frac{M_P^2}{\sqrt{8\pi}} \sqrt{A}$$

$$\Rightarrow dM = \frac{M_P^2}{\sqrt{8\pi}} \frac{dA}{2\sqrt{A}}$$

$$= \frac{M_P^4}{16\pi} \cdot \frac{dA}{M}$$

$$= \frac{M_P^2}{8\pi} \cdot \frac{M_P^2}{2M} \cdot dA$$

$$\Rightarrow \quad \frac{dM}{M_P^2} = \hbar \frac{M_P^2}{2M} \cdot \frac{dA}{8\pi\hbar}$$

$$\hat{=} \quad dE = T \cdot dS$$

$\Rightarrow$ identify:

$$S = \frac{A}{8\pi\hbar}$$

$$T = \frac{\hbar}{2GM} \sim \frac{1}{R_S}$$

$\rightsquigarrow$ general rule:

a system with an event horizon of size R_H produces long-wave length quanta with temperature T:

$$T \sim \frac{1}{R_H}$$

now back to dS:

$\rightsquigarrow$ has an event horizon of size $R_{dS} \sim H^{-1}$

$\Rightarrow$ massless field quanta, e.g.:

- gravitons δg_{ij}
- inflaton field quanta $\delta\phi$

$\vdots$

are produced with temperature:

$$T_{dS} \sim H$$

$\rightsquigarrow$ thermal fluctuations:

$$\langle \delta g_{ij} \rangle \sim \langle \delta\phi \rangle \sim T_{dS} \sim H$$

compute power spectrum of
2-point function of fluctuations:

$$\langle \delta g_{ij}^2 \rangle \sim \langle \delta\phi^2 \rangle \sim H^2$$

how does the metric fluctuate with ϕ? $\rightsquigarrow$ need fluctuations of gravitational potential to seed $\frac{\delta\rho}{\rho}$ 'density perturbations'

...

guidance: Schwarzschild metric of gravitating mass

$$ds^2 = (1-2\zeta)dt^2 - \frac{dr^2}{1-2\zeta} - d\vec{x}_2^2$$

at weak fields:

$$ds^2 = (1-2\zeta)dt^2 - (1+2\zeta)dr^2 - \ldots$$

(full argument long $\leftrightarrow$ gauge inv. of GR) now, in slow-roll:

inflaton jump $\delta\phi$

$\Rightarrow$ need $\delta N = H \cdot \delta t = \frac{H}{\dot{\phi}} \cdot \delta\phi$

more/less e-folds to reach reheating

$$\Rightarrow ds^2 = dt^2 - e^{2H \cdot t} \cdot d\vec{x}_3^{\,2}$$

$$\rightarrow ds^2 = dt^2 - e^{2H(t+\delta t)} \cdot d\vec{x}_3^{\,2}$$

$$= dt^2 - e^{2Ht} \cdot (1 + 2 \cdot \delta N) \cdot d\vec{x}_3^{\,2}$$

compare:

$$\zeta = \delta N = \frac{H}{\dot{\phi}} \cdot \delta\phi$$

'curvature perturbation' induced by inflaton fluctuation $\delta\phi$

$$\rightsquigarrow \quad \zeta^2 = \frac{H^2}{\dot{\phi}^2} \delta\phi^2$$

$$\Rightarrow \Delta_\zeta^2 = \frac{H^2}{\dot{\phi}^2} \Delta_\phi^2 = \frac{H^4}{4\pi^2 \dot{\phi}^2}$$

in slow-roll: $\dot{\phi} = -\frac{V'}{3H}$

$$\Rightarrow \Delta_\zeta^2 = \frac{1}{12\pi^2} \cdot \frac{V^3}{V'^2} = \frac{1}{24\pi^2} \cdot \frac{V}{\epsilon}$$

can show:

$$\frac{\delta\rho}{\rho} = \frac{2}{5}\sqrt{\Delta_\zeta^2} \to \frac{\Delta T}{T} \text{ of CMB}$$

$$\boxed{\begin{array}{l}\underline{\text{CMB}:} \\ \frac{\Delta T}{T} \sim 10^{-5} \\ \text{measures } \frac{V}{\epsilon}\end{array}}$$

in exact dS:

$$\Delta_\phi^2 = \text{const.}$$

in slow-roll can parametrize:

$$\Delta_\zeta^2(k) = \Delta_\zeta^2(k_0)\cdot\left(\frac{k}{k_0}\right)^{n_s-1}$$

$n_s - 1$: spectral tilt

expand:

$$\ln \Delta_\zeta^2(k) = \ln \Delta_\zeta^2(k_0) + \frac{d\ln\Delta_\zeta^2(k_0)}{d\ln k}\cdot\ln\frac{k}{k_0} + \ldots$$

$$\Rightarrow n_s - 1 = \left.\frac{d\ln\Delta_\zeta^2}{d\ln k}\right|_{k=k_0=aH}$$

relation between comoving wave number k and physical wave number $k_{phys.}$:

$$k = \frac{k_{phys.}}{a} = k_{phys.}\cdot e^{-N}$$

$$\Rightarrow d\ln k = -dN$$

$$\Rightarrow n_s - 1 = \left.\frac{d\ln\Delta_\zeta^2}{dN}\right|_{N\simeq 60}$$

$$\frac{d\ln\Delta_\zeta^2}{dN} = \frac{1}{\Delta_\zeta^2}\cdot\frac{d\Delta_\zeta^2}{dN}$$

$$= 12\pi^2\cdot\frac{V'^2}{V^3}\cdot\frac{d\phi}{dN}\cdot\frac{d}{d\phi}\left(\frac{1}{12\pi^2}\frac{V^3}{V'^2}\right)$$

$$\frac{d\phi}{dN} = \dot\phi\cdot\frac{dt}{dN} = \frac{\dot\phi}{H} = -\frac{V'}{3H^2}$$

$$= -\frac{V'}{V}$$

$$= -\frac{V'^3}{V^4}\cdot\left(3\frac{V^2}{V'}-2\frac{V^3}{V'^3}V''\right)$$

$$= -6\cdot\left(\frac{1}{2}\frac{V'^2}{V^2}\right)+2\cdot\frac{V''}{V} = -6\epsilon+2\eta$$

$$\Rightarrow n_\zeta = 1-6\epsilon+2\eta$$

WMAP: $n_\zeta = 0.963 \pm 0.013$, 1σ

PLANCK: $\Delta n_\zeta = 0.005$ at 2σ

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inflation also seeds primordial gravitational waves:

$\rightsquigarrow$ tensor perturbations of the dS metric

$$ds^2 = (1-2\zeta)dt^2 - \left[(1+2\zeta)\delta_{ij} + h_{ij}\right]dx^i dx^j$$

get a free wave e.o.m. like inflaton scalar field in the limit $V=\text{const}$.

no further 'translation factor' unlike Δ_ζ^2 – h_k is already a metric perturbations ...

define tensor-to-scalar ratio r:

$$r = \frac{\Delta_h^2}{\Delta_\zeta^2} = \frac{\dot\phi^2}{H^2} = 2\epsilon$$

... doing a better job on normalization:

$$\boxed{r = 16\epsilon}$$

example: $V \sim \phi^p$

$$n_s = 1 - \frac{2+p}{2N_e} \simeq 0.97$$

$$r = \frac{4p}{N_e} = 0.13 \quad \text{for } p = 2.$$

measuring r determines ϵ, and via $\delta\rho/\rho$ from $\Delta T/T$ the scale of inflation V!

2nd significance of r:

compute $N_e = \int H dt = \int \frac{d\varphi}{\sqrt{2\epsilon}}$

$$\Rightarrow N_e \simeq \frac{\Delta\phi}{M_p} \cdot \frac{1}{\sqrt{2\epsilon}}$$

$$\Leftrightarrow r = 16\epsilon \simeq \frac{8}{N_e^2} \cdot \left(\frac{\Delta\phi}{M_p}\right)^2$$

$$\Rightarrow \boxed{r \simeq 0.003 \cdot \left(\frac{50}{N_e}\right)^2 \cdot \left(\frac{\Delta\phi}{M_p}\right)^2}$$

'Lyth bound'

$\rightsquigarrow$ $r \sim 0.01$ corresponds to boundary between large-field and small-field inflation.

how to measure r ?

i) Δ_h^2 converts into $\frac{\Delta T}{T}$ at large angular scales $> 10^0$

$\rightarrow$ WMAP bound, $r \lesssim 0.2$

ii) B-mode polarization of CMB:

$\rightarrow$ look for curl-like pattern of polarization vectors around cold spot in CMB:

 instead of  or 

$\rightsquigarrow$ PLANCK : $r \lesssim 0.03 \ldots 0.05$
(extended run, 2.5 yr)

similar range for ground-based: QUIET, Keck array, Spider ...

by 2013-2014 : 3 yrs !

$\rightsquigarrow$ observational reach on r

$\hat{=}$ small-field / large-field boundary ...